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# EXISTENCE AND PROPERTIES OF $h$-SETS

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**Abstract**. In this note we shall consider the following problem: which conditions should satisfy a function $h\colon (0,1) \to \mathbb{R}$ in order to guarantee the existence of a (regular) measure $\mu$ in $\mathbb{R}^n$ with compact support $\Gamma \subset \mathbb{R}^n$ and

$$c_1 h(r) \leq \mu(B(\gamma, r)) \leq c_2 h(r), \tag{$\heartsuit$}$$

for some positive constants $c_2$, and $c_2$ independent of $\gamma \in \Gamma$ and $r \in (0,1)$? The theory of self-similar fractals provides outstanding examples of sets fulfilling ($\heartsuit$) with $h(r) = r^d$, $0 \leq d \leq n$, and a suitable measure $\mu$. Analogously, we shall rely on some recent techniques for the construction of pseudo self-similar fractals in order to deal with our more general task.

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## 1. Introduction

We try to generalise the idea of $(d, \Psi)$-sets: these sets have been introduced by D. Edmunds and H. Triebel in [4] and [5] as a perturbation of $d$-sets. Their definition is given in terms of qualitative behaviour of some Radon measures in $\mathbb{R}^n$. Roughly speaking, a $(d, \Psi)$-set $\Gamma$ $(0 < d < n)$ is the support of a Radon measure $\mu$, such that

$$\mu(B(\gamma, r)) \sim r^d \Psi(r), \qquad r \in (0,1) \quad \text{and} \quad \gamma \in \Gamma, \tag{1.1}$$

where $\Psi(r)$ is a perturbation at most of logarithmic growth (or decay) near 0. We point out that throughout this paper we use the equivalence "$\sim$" in

$$a_k \sim b_k \qquad \text{or} \qquad \varphi(r) \sim \psi(r)$$

always to mean that there are two positive numbers $c_1$ and $c_2$ such that

$$c_1 a_k \leq b_k \leq c_2 a_k \qquad \text{or} \qquad c_1 \varphi(r) \leq \psi(r) \leq c_2 \varphi(r)$$

for all admitted values of the discrete variable $k$ or the continuous variable $r$. Here $a_k$, $b_k$ are positive numbers and $\varphi$, $\psi$ are positive functions.

The initial problem concerning (1.1) was to prove that, given any admissible $d$ and $\Psi$, *there actually exist* a compact set $\Gamma$ and a Radon measure $\mu$ with $\operatorname{supp} \mu = \Gamma$, satisfying the required properties. The proof of the existence of

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these sets can be found in the quoted papers and, in a more detailed form, in [12, 22.8].

It is also important to know how different can be all the measures $\mu$ satisfying (1.1), *for a fixed* $(d,\Psi)$*-set* $\Gamma$. The situation turns out to be similar to the case of $d$-sets: if $\Gamma$ is a $(d,\Psi)$-set, then all finite Radon measures $\mu$ with $\operatorname{supp}\mu = \Gamma$ and (1.1) are equivalent to each other and there is a canonical representative $\mathcal{H}^{d,\Psi}|\Gamma$ constructed analogously to the $s$-dimensional Hausdorff measure, $s > 0$ (see Definition 3.3 below) and then restricted to $\Gamma$. Proofs and further comments can be found in [1, 2] and [3].

Now, in a more general context, we ask for which functions $h : [0,1] \to \mathbb{R}$ there exist a Radon measure $\mu$ and compact set $\Gamma = \operatorname{supp}\mu \subset \mathbb{R}^n$ such that

$$\mu(B(\gamma,r)) \sim h(r), \qquad \gamma \in \Gamma, \quad r \in [0,1]. \tag{1.2}$$

In order to deal with this general approach we need basically two major ingredients: a refined theory of Hausdorff measures and densities and some knowledge of sets generated by infinitely many contractions. General Hausdorff measures and densities of measures help us to reduce the problem of the generic measure $\mu$ appearing in (1.2) to a (good) standard representative, whereas sets defined in terms of (possibly) infinitely many contractions provide a nice class of examples.

## 2. A Counterexample

Let $\varepsilon > 0$ and consider the function $h(r) = r^{n+\varepsilon}$. We claim that in $\mathbb{R}^n$ there cannot exist a compact set $\Gamma$ and a Radon measure $\mu$ with $\operatorname{supp}\mu = \Gamma$ such that

$$\mu(B(\gamma,r)) \sim r^{n+\varepsilon}, \qquad \gamma \in \Gamma,\ r \in [0,1].$$

As a matter of fact any measure with this property should be equivalent to $\mathcal{H}^{n+\varepsilon}|\Gamma$ (see Theorem 3.6 below). But this measure is identically zero in $\mathbb{R}^n$ and hence our claim is proved.

This example shows that for an appropriate measure $\mu$ the class of functions $h$ for which $\mu(B(\gamma,r)) \sim h(r)$ depends on the dimension $n$ of $\mathbb{R}^n$.

## 3. Hausdorff Measures and Upper Densities

The definition of Hausdorff measures follows the so-called Carathéodory construction procedure. We collect here the main definitions and results and refer to [10] and [14] for a complete survey on these topics, including all proofs of this section and further references.

**Definition 3.1.** Let $\mathbb{H}$ denote the class of all right continuous monotone increasing functions $h\colon [0,+\infty] \to [0,+\infty]$ such that $h(u) > 0$ if $u > 0$. We refer to $\mathbb{H}$ as to the set of all *gauge functions*.

**Definition 3.2.** If $A \subset \mathbb{R}^n$ and $\delta > 0$, we denote by $\delta(A)$ the family of all open $\delta$-coverings of $A$, i.e., the collection of all sequences $\{A_i\}_{i\in\mathbb{N}}$ of sets $A_i$ such that $A_i$ is open, $\operatorname{diam}(A_i) < \delta$ and $\cup_{i=1}^{\infty} A_i \supset A$. Of course, if $A \neq \varnothing$, then

$\operatorname{diam}(A) = \sup_{x,y\in A} |x-y|$ is the diameter of $A$ rigorously complemented with $\operatorname{diam}(\varnothing) = 0$.

**Definition 3.3.** Let $h \in \mathbb{H}$ and define, for $A \subset \mathbb{R}^n$, that $h(A) = h(\operatorname{diam}(A))$ if $A \neq \varnothing$ and $h(\varnothing) = 0$. Then the set function

$$\mathcal{H}^h(A) = \lim_{\delta\to 0}\Big\{\inf\Big\{\sum_{i=1}^{\infty} h(A_i) : \{A_i\}_{i\in\mathbb{N}} \in \delta(A)\Big\}\Big\}, \qquad A \subset \mathbb{R}^n,$$

is called the *Hausdorff measure corresponding to the gauge function h.*

Of course, for $h(r) = r^s$ one simply writes $\mathcal{H}^s$.

*Remark* 3.4. The $h$-measure defined above is actually well defined: for any $A \subset \mathbb{R}^n$, the approximate measure

$$\mathcal{H}^h_\delta(A) = \inf\Big\{\sum_{i=1}^{\infty} h(A_i) : \{A_i\}_{i\in\mathbb{N}} \in \delta(A)\Big\}$$

is monotone in $\delta$, i.e., $\mathcal{H}^h_{\delta_1}(A) \geq \mathcal{H}^h_{\delta_2}(A)$ if $\delta_1 \leq \delta_2$ and hence $\mathcal{H}^h(A) = \sup_{\delta>0} \mathcal{H}^h_\delta(A)$ (of course, one allows $+\infty$ as a possible outcome). Moreover, $\mathcal{H}^h$ is an (outer) Borel regular measure.

It can also be noted that only values of $h(r)$ with $r$ near zero are really important. Hence one can think of $h(r)$ as a function, say, from $[0,1) \to \mathbb{R}_+$.

Extremely useful local characteristics of Borel measures are their upper and lower densities. Let $h$ be a gauge function with the property that $h(0) = 0$.

**Definition 3.5.** Let $\mu$ be a locally finite Borel measure. Then

$$\overline{\mathcal{D}}^h\mu(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{h(2r)}$$

and

$$\underline{\mathcal{D}}^h\mu(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{h(2r)}$$

are called the *upper*, respectively the *lower, h-density of $\mu$ at $x$.* For a special choice $h(r) = r^s$ one writes of course $\overline{\mathcal{D}}^s$, and $\underline{\mathcal{D}}^s$.

The following theorem will turn out to be of great importance later on. We briefly denote by $\mathfrak{X}_\mu$ the class of all $\mu$-measurable sets and by $\mathfrak{B}(\mathbb{R}^n)$ the class of all Borel sets in $\mathbb{R}^n$.

**Theorem 3.6.** *Let $\mu$ be a locally finite Borel measure and let $h$ be a gauge function such that $h(0) = 0$ and $h(2r) \leq Kh(r)$ for some positive constant $K$. Then*

(i) $\mu(A) \leq K \sup_{x\in A} \overline{\mathcal{D}}^h\mu(x)\mathcal{H}^h(A), \qquad A \in \mathfrak{X}_\mu,$

*and*

(ii) $\mu(A) \geq \inf_{x\in A} \overline{\mathcal{D}}^h\mu(x)\mathcal{H}^h(A), \qquad A \in \mathfrak{B}(\mathbb{R}^n).$

## 4. Alphabets and Words

Following [7], we collect here some useful notation.

Let $N \geq 2$ be a natural number, then by the *alphabet* we mean the set of *letters* $\mathcal{A} = \{1, \dots, N\}$. $\mathcal{A}$ is endowed with the discrete topology and with the equally distributed Radon measure $\tau^1$ defined on atoms by $\tau^1(i) = N^{-1}$ for $i \in \mathcal{A}$.

A finite word $\alpha$ of length $k = |\alpha|$ is any element of $\mathcal{A}^k$ and will be denoted by $\alpha = \alpha_1\alpha_2 \dots \alpha_k$. $\mathcal{A}^k$ is endowed with the product measure $\tau^k = \times_{i=1}^k \tau^1$.

Given any two finite words $\alpha = \alpha_1 \dots \alpha_k$ and $\beta = \beta_1 \dots \beta_h$ we denote by $\alpha\beta$ the finite word $\alpha_1 \dots \alpha_k\beta_1 \dots \beta_h$. The set of finite words is endowed with the following relation: $\alpha \prec \beta$ if $|\alpha| \leq |\beta|$ and $\alpha_i = \beta_i$, $i = 1, \dots, |\alpha|$.

An *infinite word* (or simply a *word*) is any member of $\mathfrak{A} = \mathcal{A}^{\mathbb{N}}$ and will be denoted by $I = i_1i_2\dots$. We extend the relation introduced above: if $\alpha$ is a finite word and $I \in \mathfrak{A}$, then $\alpha \prec I$ means that $\alpha_j = i_j$, $j = 1, \dots, |\alpha|$.

If $\alpha$ is a finite word and $I \in \mathfrak{A}$, then $\alpha I$ denotes the shifted word obtained by chaining $\alpha$ and $I$, i.e., by $\alpha_1, \dots \alpha_{|\alpha|}i_1i_2\dots$ and $\alpha^*$ is the set $\{\alpha I, I \in \mathfrak{A}\}$. The $k$-th stop of a word $I$ is the finite word $i_1 \dots i_k$, and will be indicated with $I_{|k}$.

Let $n_1, \dots, n_m \in \mathbb{N}$; then we define the projection $\pi_{n_1\dots n_m} \colon \mathfrak{A} \to \mathcal{A}^m$ by $\pi_{n_1\dots n_m}(i_1i_2,\dots) = i_{n_1}\dots i_{n_m}$.

Since $\mathfrak{A} = \prod_{i=1}^\infty \mathcal{A}$, we endow the set of all words with the product topology $\mathcal{T}$ and the product $\sigma$-algebra $\mathfrak{M}$. We collect in the following proposition some very well known facts.

**Proposition 4.1.** *$\mathfrak{A}$ is a compact (complete and separable) metric space. For any choice of the numbers $n_i \in \mathbb{N}$ the projection operator $\pi_{n_1\dots n_m}$ is uniformly continuous.*

*Moreover, if we let $\tau = \times_{i=1}^\infty \tau^1$, we get a Radon finite measure on $\mathfrak{A}$ such that*

$$\tau \circ \pi^{-1}_{n_1\dots n_m} = \tau^m, \tag{4.1}$$

*for any choice of the numbers $n_i$.*

Here we have adopted the notation for image (or distribution) measures with respect to a function $f$: $(\tau \circ f^{-1})(A) = \tau(f^{-1}(A))$. Sometimes the same measure is denoted by $\tau_f$, $\tau_\sharp f$ or $f(\tau)$.

## 5. Cantor-Type Sets

In this section we quote the results obtained by G. Follo in [6] about sets generated by infinitely many contractions. The situation presented in that paper is rather general. Here we need a special case where the ambient space is $\mathbb{R}^n$, the contractions are even similarities whose number at every step is the same constant. Hence the following results hold true and are proved in a much more general setting.

We recall that if $x \in \mathbb{R}^n$ and $A$ is a nonempty set of $\mathbb{R}^n$, then we let $d(x, A) = \inf_{a \in A} |x - a|$ be the distance from $x$ to $A$. Remember also that on the class $\mathfrak{K}$

of all nonempty compact sets of $\mathbb{R}^n$ one defines the so-called *Hausdorff metric* given by

$$d_{\mathcal{H}}(H, K) = \sup\{d(k, H), d(h, K) : k \in K, h \in H\}.$$

This is actually a distance and $d_{\mathcal{H}}(\{x\}, \{y\}) = |x - y|$ for any $x, y \in \mathbb{R}^n$. It is known that the metric space $(\mathfrak{K}, d_{\mathcal{H}})$ is complete.

Let $N \geq 2$ be a fixed number and let $\mathfrak{F} = \{\mathfrak{F}_1, \mathfrak{F}_2, \dots\}$ be a sequence of finite sets (with cardinality $N$) of contractive similarities, i.e., for every $k \in \mathbb{N}$, $\mathfrak{F}_k = \{f_{k,1}, \dots, f_{k,N}\}$ is a set of $N$ different contractive similarities in $\mathbb{R}^n$, each one with the same similarity ratio $\varrho_k$. We call (improperly) $\mathfrak{F}$ a *system of similarities (of order $N$)*.

Then define, for $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, $\mathfrak{F}_k(A) = \cup_{j=1}^N f_{k,j}(A)$ and let

$$\mathfrak{F}_k \circ \mathfrak{F}_{k+1}(A) = \mathfrak{F}_k(\mathfrak{F}_{k+1}(A)), \qquad k = 1, 2, \dots .$$

**Theorem 5.1.** *Consider the sequence $\{\mathfrak{F}_k\}_{k\in\mathbb{N}}$ described above and suppose that these two conditions hold true:*

(i) *there exists a nonempty compact set $Q \subset \mathbb{R}^n$ such that $\mathfrak{F}_k(Q) \subset Q$ for every $k \in \mathbb{N}$;*
(ii) $\lim_{k\to\infty} \prod_{j=1}^k \varrho_k = 0$.

*Then the sequence of sets*

$$\mathfrak{F}_1 \circ \dots \circ \mathfrak{F}_k(A) = \bigcup_{i_1,\dots,i_k \in \{1,\dots,N\}} f_{1,i_1} \circ \dots \circ f_{k,i_k}(A)$$

*is convergent in the Hausdorff metric for any non-empty compact set $A$. Moreover, the limit (which is a compact set of $\mathbb{R}^n$) is independent of $A$ and will be denoted by $K$.*

The set $K$ can be described in a more direct (and useful) way: choose a starting point $x_0$ in $\mathbb{R}^n$ and consider a word $I = i_1 i_2 \dots \in \mathfrak{A}$ (the number $N$ of letters of the alphabet $\mathcal{A}$ defined in the previous section is the same number $N$ of similarities involved here). For any $k \in \mathbb{N}$ let

$$f_{I_{|k}}(x_0) = f_{1,i_1} \circ f_{2,i_2} \circ \dots \circ f_{k,i_k}(x_0). \tag{5.1}$$

**Theorem 5.2.** *For any $I \in \mathfrak{A}$, the sequence $\{f_{I_{|k}}(x_0)\}_k$ defined above converges to a point $p(I) \in K$ independent of $x_0$. The application $p \colon \mathfrak{A} \to K$ defined in this way is surjective and uniformly continuous.*

Here we make a short digression and add an observation: the map $p$ is not generally injective. However, under the following assumption, $p$ turns out to be a one-to-one map. We got an inspiration, when reading an analogous statement in the new book of J. Kigami ([9, Proposition 1.2.5]).

**Proposition 5.3.** *Suppose that for every $k \in \mathbb{N}$, $f_{k,i}(Q) \cap f_{k,j}(Q) = \varnothing$, $i \neq j$, then $p$ is injective.*

*Proof.* Let us introduce some notation: if $I = i_1 i_2 \cdots \in \mathfrak{A}$ and $m \in \mathbb{N}_0$, then $\sigma^m I = i_{m+1} i_{m+2} \ldots$ and $p^{m+1}(I) = \lim_{k\to\infty} f_{m+1,i_{m+1}} \circ \cdots \circ f_{m+k,i_{m+k}}(x_0)$. Finally, if $I \neq J$, then $\delta(I,J) = \min\{k : i_k \neq j_k\} - 1$.

To prove our assertion, first observe that if $I \neq J$, then $p(I) = p(J)$ if and only if $p^{m+1}(\sigma^m I) = p^{m+1}(\sigma^m J)$, where $m = \delta(I,J)$ (reduction). Indeed, let $I, J \in \alpha^*$, for $|\alpha| = m$, then $p(I) = f_\alpha(p^{m+1}\sigma^m I) = f_\alpha(p^{m+1}\sigma^m J) = p(J)$. Since $f_\alpha$ is injective the "if" part is proved. The other direction is obvious.

So, if $I \neq J$ and $p(I) = p(J)$, up to a shift of $m$ symbols, the first letters of $I$ and $J$ (now related to the system $\{\mathfrak{F}_{k+m}\}_{k=1}^\infty$ are different and $f_{m+1,i}(Q) \cap f_{m+1,j}(Q) \neq \varnothing$ for some $i \neq j$. $\square$

This rather easy assertion will hold true in our setting thereafter when we shall construct a pseudo self-similar fractal using the techniques described above. Hence, in our case $p$ is even a homeomorphism and the topological nature of the fractal $K = p(\mathfrak{A})$ is then completely described by that of $\mathfrak{A}$.

Let us come back to our path and let us define a suitable measure on $K$: let

$$\mu = \tau \circ p^{-1}, \tag{5.2}$$

where $\tau$ is the Radon measure defined on $\mathfrak{A}$ in Proposition 4.1.

**Theorem 5.4.** *The measure $\mu = \tau \circ p^{-1}$ is a finite Radon measure with $\operatorname{supp}\mu = K$. Moreover, for any Radon measure $\nu$ with $\nu(\mathbb{R}^n) = 1$ and compact support, the sequence*

$$\nu_k = N^{-k} \sum_{|\sigma|=k} \nu \circ f_\sigma^{-1}$$

*converges weakly to $\mu$ as $k$ tends to infinity.*

Of course, $f_\sigma$ is defined as in (5.1): if $|\sigma| = k$, then $f_\sigma(x) = f_{1,\sigma_1} \circ \cdots \circ f_{k,\sigma_k}(x)$ for $x \in \mathbb{R}^n$.

The first part of the theorem is an immediate consequence of the properties of image measures under continuous functions. The second (and more important) part can be regarded as an extension of the existence theorem of invariant measures on self-similar fractals.

*Remark* 5.5. As we have pointed out all these results hold in a much more general form. In any case the theory developed in the previous sections is widely sufficient for us. See [6] for a complete survey.

## 6. Construction

Let us come to the main part of this note, i.e., the construction of an $h$-set. Below follows a more precise definition.

**Definition 6.1.** Let, for $n \in \mathbb{N}$, $\mathfrak{H}_n$ be the set of all continuous monotone increasing functions $h\colon [0,\infty) \to [0,\infty)$ such that there exist $m \in \mathbb{N}$ with $m \geq 2$ and a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with

(D$_1$) $h(\lambda_1 \cdots \lambda_k) \sim m^{-nk}$, $k \in \mathbb{N}$;
(D$_2$) $0 < \inf_k \lambda_k \leq \sup_k \lambda_k < m^{-1}$.

The class $\mathfrak{H}_n$ is a suitable set of functions providing examples of compact sets $\Gamma$ able to carry a measure $\mu$ with $\mu(B(\gamma,r)) \sim h(r)$, for $\gamma \in \Gamma$ and all $r$ sufficiently small: as a matter of fact the number $m^n$ and the coefficients $\lambda_k$ will be used for the construction of a pseudo self-similar fractal (as shown in Section 5) with the desired properties.

*Remark* 6.2. Notice that any $h \in \mathfrak{H}_n$ has the so-called doubling condition, i.e, there exists a positive constant $C = C(h)$ such that

$$h(2r) \leq Ch(r), \qquad 0 < r < 1.$$

In (8.9) we shall considerably improve this assertion.

*Remark* 6.3. The assumption $\sup_k \lambda_k < m^{-1}$ is technically useful, but strong: the function $h(r) = r^n$ is not contained in $\mathfrak{H}_n$. Analogously, functions of type $r^n|\log r|^{\varkappa}$, for $\varkappa > 0$ (related to $(n,\Psi)$-sets), are not to be considered in the above schematisation. For those functions it may happen that $\sup_k \lambda_k = m^{-1}$. We can of course incorporate these particular examples into $\mathfrak{H}_n$: we already know that $n$-sets and $(n,\Psi)$-sets exist. Analogously, the case $\inf_k \lambda_k = 0$ is a limiting situation. An interesting example where this strong decay occurs is given by $h(r) = |\log r|^{-\varkappa}$ with $\varkappa > 0$, which, consequently, does not belong to $\mathfrak{H}_n$. However, one could prove that there exists a compact set $\Gamma$ with $\mathcal{H}^{|\log(\cdot)|^{-\varkappa}}|\Gamma(B(\gamma,r)) \sim |\log r|^{-\varkappa}$, for $\gamma \in \Gamma$ and $0 < r < 1$.

By these examples it should be clear that our approach does not take into account the existing limiting cases.

If a function $h$ verifies the following stronger condition, then $h \in \mathfrak{H}_n$. Sometimes this is easier to check than Conditions ($D_1$) and ($D_2$) of Definition 6.1.

**Proposition 6.4.** *Let $h$ be a strictly increasing continuous function and denote by $h^{-1}$ its inverse. If there exist two real numbers $\delta > 1$ and $K > 0$ with*

(i) $h(2^\delta r) \leq 2^n h(r)$ *and*
(ii) $h^{-1}(2s) \leq K h^{-1}(s)$

*for all $r$ and $s$ sufficiently small, then $h \in \mathfrak{H}_n$.*

The proof of this proposition is rather simple and we omit it.

Here we give the main definition we have in mind.

**Definition 6.5.** Let $\Gamma$ be a non-empty compact set in $\mathbb{R}^n$ and $h \in \mathfrak{H}_n$. Then $\Gamma$ is calledan *$h$-set* if there exists a finite Radon measure $\mu$ in $\mathbb{R}^n$ whose support is $\Gamma$, such that

$$\mu(B(\gamma,r)) \sim h(r), \qquad \gamma \in \Gamma,\ r \in (0,1).$$

Such a measure $\mu$ will be called *related to $h$* or, simply, an *$h$-measure.*

**Theorem 6.6.** *For every $h \in \mathfrak{H}_n$ there exists an $h$-set, i.e., there exist a compact set $\Gamma \subset \mathbb{R}^n$ and a finite Radon measure $\mu$ such that*

(i) $\operatorname{supp} \mu = \Gamma$;
(ii) $\mu(B(\gamma,r)) \sim h(r)$, *for $\gamma \in \Gamma$ and $r \in (0,1)$.*

*Moreover, all measures $\mu$ satisfying* (i) *and* (ii) *are equivalent to $\mathcal{H}^h|\Gamma$.*

*Proof.* Let $\{\lambda_k\}_{k\in\mathbb{N}}$ and $m$ be, respectively, the sequence and the number related to $h$ by Definition 6.1. Let $Q = [-1/2, 1/2]^n$ be the unit cube in $\mathbb{R}^n$. Let us divide it naturally into $m^n$ sub-cubes with sides of length $m^{-1}$ parallel to the coordinate axes and denote their centers by $v_j$, $j = 1, \dots, m^n$. Then define

$$f_{k,j}(x) = \lambda_k x + v_j, \qquad k \in \mathbb{N},\ j = 1, \dots, m^n.$$

Now, we let $\mathfrak{F} = \{\mathfrak{F}_k\}_{k\in\mathbb{N}}$ be the system of similarities defined by

$$\mathfrak{F}_k = \{f_{k,1}, \dots, f_{k,m^n}\}.$$

We can apply Theorem 5.1 to the family $\{\mathfrak{F}_k\}_{k\in\mathbb{N}}$. In fact, condition 5.1–(i) is fulfilled, taking precisely our cube $Q$. 5.1–(ii) is also true since in this case the similarity ratios $\varrho_k = \lambda_k < m^{-1} \leq 1/2$. Therefore there exists a compact set $\Gamma \subset Q$ which is the limit of $\mathfrak{F}_1 \circ \cdots \circ \mathfrak{F}_k(Q)$ as $k \to \infty$.

Our aim is to prove that this $\Gamma$ satisfies conditions (i) and (ii) of the theorem, with respect to the measure $\mu$ defined in (5.2) and described in Theorem 5.4 adapted to the present situation. We remark that the number $N$ appearing there is the same number $N = m^n$ of contractions in each $\mathfrak{F}_k$.

Now, fix $x_0 \in \mathbb{R}^n$ and let $B = B(\gamma, r)$ be a ball of radius $r \in (0,1)$ and center $\gamma \in \Gamma$.

*Step* 1. By Theorem 5.2 there exists a word $I \in \mathfrak{A}$ such that

$$p(I) = \lim_{k\to\infty} f_{1,i_1} \circ \cdots \circ f_{k,i_k}(x_0) = \gamma.$$

Therefore, for all $k$ greater than some $k_0$

$$f_{I_{|k}}(Q) = f_{1,i_1} \circ \cdots \circ f_{k,i_k}(Q) \subset B. \tag{6.1}$$

In order to see this, choose $q \in Q$. Then $|f_{I_{|k}}(q) - p(I)| = \lambda_1 \cdots \lambda_k |q - q'|$, where $q'$ belongs to the pseudo self-similar fractal $\widetilde{\Gamma}$ generated by the system of similarities $\widetilde{\mathfrak{F}} = \{\mathfrak{F}_{k+1}, \mathfrak{F}_{k+2}, \dots\}$, which is in the same (or possibly better) conditions as $\mathfrak{F}$. Hence also $\widetilde{\Gamma} \subset Q$ and consequently $\lambda_1 \cdots \lambda_k |q - q'| \leq \lambda_1 \cdots \lambda_k \operatorname{diam}(Q)$.

Thus, we choose a minimal index $k$ (sometimes called *Markov stop*) such that condition (6.1) is fulfilled.

Equivalently, $k$ is a minimal index for which $\lambda_1 \cdots \lambda_k \operatorname{diam}(Q) \leq r$.

Then, with this choice of $k = k(r)$, one can easily check that

$$\lambda_0 r < \lambda_1 \cdots \lambda_k \operatorname{diam}(Q) \leq r,$$

where $\lambda_0 = \inf_k \lambda_k > 0$. This inequality and the doubling condition for $h$ imply that

$$h(r) \leq ch(\lambda_1 \cdots \lambda_k) \leq c' m^{-kn}, \tag{6.2}$$

for some unimportant constants $c$ and $c'$ depending only on $h$.

Thanks to the definition of $\mu$ and (4.1), the measure $\mu(f_{I_{|k}}(Q))$ can be easily estimated:

$$\begin{aligned}\mu(f_{I_{|k}}(Q)) &= \tau\circ p^{-1}(f_{I_{|k}}(Q)) = \tau\Big(p^{-1}(f_{I_{|k}}(Q))\Big)\\ &= \tau\Big(\{J\in\mathfrak{A} : p(J)\in f_{I_{|k}}(Q)\}\Big) \geq \tau\Big(\{J\in\mathfrak{A} : J\in I^*_{|k}\}\Big)\\ &= \tau(I^*_{|k}) = \tau(\pi^{-1}_{1,\dots,k}(I_{|k})) = \tau^k(I_{|k}) = m^{-kn}. \qquad (6.3)\end{aligned}$$

Hence, by (6.2) and (6.3), we have

$$\frac{\mu(B(\gamma,r))}{h(r)} \geq \frac{\mu(f_{I_{|k}}(Q))}{h(r)} \geq \frac{m^{-kn}}{cm^{-kn}} = C.$$

Therefore $\mu(B(\gamma,r)) \geq Ch(r)$ for some constant $C>0$ independent of $\gamma\in\Gamma$ and $r\in(0,1)$. This is half of the desired estimation.

*Step* 2. Let again $k=k(r)$ be the Markov stop for $\{\lambda_j\}_{j\in\mathbb{N}}$. Consider the set of all words of length $k$ and define $Q_\alpha = f_\alpha(Q)$, for every $\alpha$ with $|\alpha|=k$. Of course, the sets $Q_\alpha$, $|\alpha|=k$, are pairwise disjoint.

Now we need the condition $\lambda^0=\sup_j\lambda_j<m^{-1}$ which we have not yet used. By simple calculations one gets that if $\gamma\in Q_\alpha$ and $\gamma'\in Q_{\alpha'}$ for two different words $\alpha$ and $\alpha'$ with length $k$, then $|\gamma-\gamma'| > (m^{-1}-\lambda^0)\lambda_1\cdots\lambda_{k-1}$. Let $c=(m^{-1}-\lambda^0)(\mathrm{diam}(Q))^{-1}$. Of course, we have

$$\mu(B(\gamma,cr)) = \tau(\{I\in\mathfrak{A} : |p(I)-\gamma|\leq cr\}).$$

Let $\gamma=p(\alpha_\gamma J)$, for $|\alpha_\gamma|=k$ and $J\in\mathfrak{A}$. If $I\notin\alpha^*_\gamma$, then $p(I)\notin Q_{\alpha_\gamma}$. Hence, as we have said above, $|p(I)-\gamma| > (m^{-1}-\lambda^0)\lambda_1\cdots\lambda_{k-1} > cr$ and therefore $p(I)\notin B(\gamma,cr)$. Consequently,

$$\mu(B(\gamma,cr)) \leq \tau(\alpha^*_\gamma) = m^{-nk}.$$

This shows that

$$\mu(B(\gamma,r)) \leq Ch(r)$$

for some unimportant positive constant $C$.

Combining together this result and what we have obtained in Step 1, we conclude that there are two positive constants $c_1$ and $c_2$ such that

$$c_1h(r) \leq \mu(B(\gamma,r)) \leq c_2h(r), \qquad r\in(0,1), \quad \gamma\in\Gamma. \qquad (6.4)$$

*Step* 3. It remains to show that $\mu$ is essentially unique. In order to get this result note that (6.4) immediately implies that

$$0 < c \leq \overline{\mathcal{D}}^h\mu(\gamma) \leq C < \infty$$

for some constants $c$ and $C$ independent of $\gamma\in\Gamma$. Thanks to Theorem 3.6 we can conclude that

$$\begin{aligned}\mu(A) = \mu(A\cap\Gamma) &\sim \sup_{x\in A\cap\Gamma}\overline{\mathcal{D}}^h\mu(x)\mathcal{H}^h(A\cap\Gamma)\\ &\sim \mathcal{H}^h(A\cap\Gamma) = \mathcal{H}^h|\Gamma(A), \qquad A\in\mathfrak{B}(\mathbb{R}^n).\end{aligned}$$

This concludes the proof of the theorem. □

## 7. Examples

As we have remarked in the introduction, our aim was to give a unified and general approach to the study of (compact) sets $\Gamma$ whose geometrical structure is defined by

$$\mu(B(\gamma, r)) \sim h(r), \qquad \gamma \in \Gamma,\ r \in (0,1),$$

where $\mu$ is a finite Radon measure supported by $\Gamma$. Apart from the limiting cases discussed in Remark 6.3, the outcome is satisfactory: as we will see, all $(d, \Psi)$-sets ($0 < d < n$) introduced by D. Edmunds and H. Triebel in [4, 5] (including the case of $d$-sets) are particular $h$-sets; we shall also show some examples of $h$-sets which are not $(d, \Psi)$-sets. Hence, after showing that this approach is consistent and produces something new, in Section 8 we will study some geometrical properties of $h$-sets.

We briefly recall the definition of $(d, \Psi)$-sets.

**Definition 7.1.** A positive monotone function $\Psi$ defined on the interval $(0, 1)$ is said *admissible* if

$$\Psi(2^{-2j}) \sim \Psi(2^{-j}), \qquad j \in \mathbb{N}. \tag{7.1}$$

**Example 7.2.** Let $b \in \mathbb{R}$ and $0 < c < 1$. Then the function

$$\Psi(r) = |\log cr|^b$$

is an admissible function.

We shall use some properties of admissible functions in the sequel: we refer to [1, 3, 4, 5] and [12, Section 22] for their proofs and further observations. For the sake of simplicity one might always think of $\Psi(r) = |\log cr|^b$ for some $b \in \mathbb{R}$ and $0 < c < 1$, though not all admissible functions can be described in this way.

**Definition 7.3.** Let $\Gamma$ be a compact subset of $\mathbb{R}^n$, $\Psi$ be an admissible function and $d$ be a real number with $0 < d < n$. Then $\Gamma$ is said to be a $(d, \Psi)$*-set* if there exists a finite Radon measure $\mu$ such that $\operatorname{supp} \mu = \Gamma$ and

$$\mu(B(\gamma, r)) \sim r^d \Psi(r), \qquad \gamma \in \Gamma, \quad 0 < r < 1.$$

The following theorem essentially asserts that $h$-sets extend the definition of $(d, \Psi)$-sets.

**Theorem 7.4.** *Any $(d, \Psi)$-set, with $0 < d < n$ is an $h$-set, with $h(r) \sim r^d \Psi(r)$.*

*Proof.* Up to equivalent functions, we may assume $h(r) = r^d \Psi(r)$ to be continuous and monotone decreasing, with $\Psi(1) = 1$, but this is not the point.

We have to find a sufficiently large positive integer $m$ and a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that Conditions ($D_1$) and ($D_2$) of Definition 6.1 be satisfied. Let

$$\lambda_k = m^{-\frac{n}{d}} \Psi^{-\frac{1}{d}}(m^{-\frac{nk}{d}}) \Psi^{\frac{1}{d}}(m^{-\frac{n(k-1)}{d}}), \qquad k = 1, 2, \dots . \tag{7.2}$$

With this choice we have, for $k = 1, 2, \dots,$

$$\lambda_1 \cdots \lambda_k = m^{-\frac{nk}{d}} \Psi^{-\frac{1}{d}}(m^{-\frac{nk}{d}}). \tag{7.3}$$

Let us estimate $h(\lambda_1 \cdots \lambda_k)$. From (7.3) we get

$$h(\lambda_1 \cdots \lambda_k) = m^{-nk} \Psi^{-1}(m^{-\frac{nk}{d}}) \Psi(m^{-\frac{nk}{d}} \Psi^{-\frac{1}{d}}(m^{-\frac{nk}{d}})).$$

Both for an increasing and for a decreasing admissible function $\Psi$ the last expression is greater than $m^{-nk}$; hence it remains to show the upper estimation. We consider the case of decreasing $\Psi$. Then, by the properties of admissible functions,

$$m^{-\frac{2nk}{d}} = o(m^{-\frac{nk}{d}} \Psi^{-\frac{1}{d}}(m^{-\frac{nk}{d}})), \qquad k \to \infty,$$

and hence there exists a positive constant $C$ independent of $k$ such that

$$m^{-\frac{nk}{d}} \Psi^{-\frac{1}{d}}(m^{-\frac{nk}{d}}) \geq C m^{-\frac{2nk}{d}}.$$

Therefore

$$h(\lambda_1 \cdots \lambda_k) \leq m^{-nk} \Psi^{-1}(m^{-\frac{nk}{d}}) \Psi(C m^{-\frac{2nk}{d}}).$$

By (7.1) we finally get $h(\lambda_1 \cdots \lambda_k) \leq c m^{-nk}$ which is the upper estimate we wished. If $\Psi$ is increasing, one proceeds analogously.

Let us consider now the coefficients $\lambda_k$. By monotonicity we have always $\lambda_k \leq m^{-\frac{n}{d}} < m^{-1}$ ($\Psi$ decreasing) or $\lambda_k \geq m^{-\frac{n}{d}}$ ($\Psi$ increasing). We consider now the case with $\Psi$ increasing. The techniques we use to prove the upper estimation for $\lambda_k$ in this case are analogous to those needed for the lower estimation in the other one.

By the properties of admissible functions we have

$$\Psi^{\frac{1}{d}}(m^{-\frac{n(k-1)}{d}}) \leq c(\log m)^b \Psi^{\frac{1}{d}}(m^{-\frac{nk}{d}})$$

for some positive constants $c$ and $b$ independent of $m$ and $k$. Hence, inserting this estimate into (7.2) and choosing $m$ large enough (remember that $d < n$) we have, for some $\varepsilon > 0$,

$$0 < m^{-\frac{n}{d}} \leq \lambda_k \leq m^{-1-\varepsilon}, \qquad k = 1, 2, \dots,$$

which is the desired assertion. $\square$

Another interesting class of $h$-sets can be obtained with the choice

$$h(r) = r^d e^{\pm|\log r|^{\varkappa}}, \qquad 0 < r < 1,$$

for $0 < \varkappa \leq 1/2$ and $0 < d < n$. Notice that $\exp\{\pm|\log r|^{\varkappa}\}$ is not an admissible function in the sense of Definition 7.1, as an easy calculation immediately shows.

To compare more precisely $\exp\{\pm|\log r|^{\varkappa}\}$ and an admissible function $\Psi(r)$ we pass to the logarithms (and exploit some properties of admissible functions):

$$|\log \exp\{\pm|\log r|^{\varkappa}\}| \sim |\log r|^{\varkappa},$$

and

$$|\log |\Psi(r)|| \leq c \log |\log r|,$$

where $r > 0$ is small enough. These estimations show that $\exp\{\pm|\log r|^{\varkappa}\}$ and $\Psi(r)$ are not comparable.

**Proposition 7.5.** *Let $0 < \varkappa \leq 1/2$ and $0 < d < n$. Then the function $h(r) = r^d \exp\{\pm|\log r|^{\varkappa}\}$ for $0 < r < 1$ belongs to $\mathfrak{H}_n$.*

*Proof.* Let us consider the case $h(r) = r^d \exp(-|\log r|^{\varkappa})$, the other one being analogous. Let

$$\lambda_k = m^{-\frac{n}{d}} \exp\Big\{c(k^{\varkappa} - (k-1)^{\varkappa})\Big\}$$

with $c = \frac{1}{d}(\frac{n}{d})^{\varkappa}(\log m)^{\varkappa}$. With this choice we have

$$\lambda_1 \cdots \lambda_k = m^{-\frac{nk}{d}} \exp\{ck^{\varkappa}\}.$$

Hence

$$\begin{aligned} h(\lambda_1 \cdots \lambda_k) = m^{-nk} \exp\{dck^{\varkappa}\} \exp\Big\{-|\log m^{-\frac{nk}{d}} \exp ck^{\varkappa}|^{\varkappa}\Big\} \\ = m^{-nk} \exp\Big\{dck^{\varkappa} - \Big|\frac{nk}{d}\log m - ck^{\varkappa}\Big|^{\varkappa}\Big\} \\ = m^{-nk} \exp\Big\{dck^{\varkappa}\big(\underbrace{1 - (1 - c'k^{\varkappa-1})^{\varkappa}}_{\sim c''k^{\varkappa-1}}\big)\Big\} \\ \sim m^{-nk} \exp\{dck^{2\varkappa-1}\} \sim m^{-nk} \end{aligned}$$

since $2\varkappa - 1 \leq 0$.

For the estimation of the coefficients $\lambda_k$ we have, by monotonicity arguments and taking into account $n/d > 1$, that

$$m^{-\frac{n}{d}} \leq \lambda_k \leq m^{-\frac{n}{d}} e^{c(\log m)^{\varkappa}} < m^{-1}$$

(with $c$ independent of $m$) provided $m$ large enough.

As we have said, the case $h(r) = r^d \exp\{|\log r|^{\varkappa}\}$ can be treated analogously and therefore the proof is concluded. □

## 8. Properties of the Measure

In this section we derive some geometric properties of an $h$-set. For example, in the simplest case with $h(r) = r^d$ it is known that the (local) Hausdorff dimension is equal to $d$. With some more effort one gets that also in a slightly more general case of $(d, \Psi)$-sets the dimension is still $d$ (see [1, 3]). But the special structure of the measure $\mu$ related to an $h$-set allows us not only to derive the Hausdorff dimension but also other significant measure-theoretic properties. In the sequel we shall restrict our attention to the Minkowski content, to the Hausdorff dimension of an $h$-set $\Gamma$ and to the Hausdorff dimension of the measure $\mu \sim \mathcal{H}^h|\Gamma$ supported by $\Gamma$. Finally, we shall consider two pure geometric properties of an $h$-set: the so-called Markov inequality related to a set and the ball condition. We shall give a sufficient condition for which $h$-sets preserve Markov's inequality, generalising an analogous theorem stated by A. Jonsson and H. Wallin for $d$-sets, and we shall show that every $h$-set verifies the so-called ball condition.

We begin by recalling the definition of the Hausdorff dimension of a set (let in the following $\inf \varnothing = \sup \varnothing = 0$).

**Definition 8.1.** Let $A \subset \mathbb{R}^n$. Then the number

$$\dim_{\mathcal{H}} A = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} \quad \Big(= \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}\Big)$$

is called the *Hausdorff dimension* of $A$.

**Definition 8.2.** For $s \geq 0$ the *upper* and the *lower $s$-dimensional Minkowski content* of a bounded subset $A$ of $\mathbb{R}^n$ are defined by

$$\overline{\mathcal{M}}^s(A) = \limsup_{\delta \to 0} \frac{\mathcal{L}^n(A_\delta)}{\delta^{n-s}}$$

and

$$\underline{\mathcal{M}}^s(A) = \liminf_{\delta \to 0} \frac{\mathcal{L}^n(A_\delta)}{\delta^{n-s}},$$

respectively. If the limits coincide, then the common value $\mathcal{M}^s(A)$ is called *$s$-dimensional Minkowski content of the set $A$.*

Recall that $A_\delta := \{x \in \mathbb{R}^n : d(x, A) \leq \delta\}$ is the parallel set of $A$ of amount $\delta$ and $\mathcal{L}^n$ denotes the Lebesgue $n$-dimensional outer measure. Accordingly, the upper and the lower Minkowski dimension are defined as follows.

**Definition 8.3.** Let $A$ be a bounded subset of $\mathbb{R}^n$; then

$$\overline{\dim}_{\mathcal{M}} A = \inf\{s \geq 0 : \overline{\mathcal{M}}^s(A) = 0\} \quad \Big(= \sup\{s \geq 0 : \overline{\mathcal{M}}^s(A) = \infty\}\Big)$$

and

$$\underline{\dim}_{\mathcal{M}} A = \inf\{s \geq 0 : \underline{\mathcal{M}}^s(A) = 0\} \quad \Big(= \sup\{s \geq 0 : \underline{\mathcal{M}}^s(A) = \infty\}\Big)$$

are called the *upper* and the *lower Minkowski dimension of* $A$, respectively. Should the limits coincide, then the common value $\dim_{\mathcal{M}} A$ is called the *Minkowski dimension of $A$.*

There is an immediate characterisation of these Minkowski dimensions, which is easier to handle.

**Proposition 8.4.** *Let $A$ be a bounded subset of $\mathbb{R}^n$. Then*

$$\overline{\dim}_{\mathcal{M}} A = n - \liminf_{\delta \to 0} \frac{\log \mathcal{L}^n(A_\delta)}{\log \delta}$$

*and*

$$\underline{\dim}_{\mathcal{M}} A = n - \limsup_{\delta \to 0} \frac{\log \mathcal{L}^n(A_\delta)}{\log \delta}.$$

Finally, we introduce the local and the global Hausdorff dimension of a Borel measure.

**Definition 8.5.** Let $\mu$ be a locally finite Borel measure and $x \in \mathbb{R}^n$. Then

$$\underline{\dim}_{\mathcal{H}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

is called the *lower pointwise dimension of* $\mu$ *at* $x$.

**Definition 8.6.** Let $\mu$ be a locally finite Borel measure. Then

$$\dim_{\mathcal{H}}\mu = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \underline{\dim}_{\mathcal{H}}\mu(x)$$

is called the *Hausdorff dimension of* $\mu$.

Now, let $\Gamma$ be an $h$-set, and let $\mu$ be a related measure (by this we always mean that $\mu$ is a finite Radon measure with $\mu(B(\gamma,r)) \sim h(r)$, for small values of $r$). It is easy to show that

$$\liminf_{r\to 0} \frac{\log \mu(B(\gamma,r))}{\log r} = \liminf_{r\to 0} \frac{\log h(r)}{\log r}$$

(and analogously for the lim sup), since $\log \mu(B(\gamma,r))/\log r = \log h(r)/\log r + o(1)$ as $r \to 0$. This proves the following theorem.

**Theorem 8.7.** *If* $\mu$ *is a measure related to an* $h$*-set, then*

$$\dim_{\mathcal{H}} \mu = \liminf_{r\to 0} \frac{\log h(r)}{\log r}.$$

In order to evaluate the Minkowski content of an $h$-set, let us consider the following lemmas.

**Lemma 8.8.** *Let, for* $0 < r < 1$, $\mathcal{B}^{(r)} = \{B(\gamma_i,r)\}_{i=1}^{N_r}$, $\gamma_i \in \Gamma$, *be a finite covering of an* $h$*-set* $\Gamma$ *such that* $B(\gamma_i,r/3) \cap B(\gamma_j,r/3) = \varnothing$ *for* $i \neq j$ *(such a covering will be called optimal). Then*

$$N_r \sim \frac{1}{h(r)}, \qquad 0 < r < 1,$$

*where the equivalence constants do not depend on* $r$.

*Proof.* The proof is simple:

$$\begin{aligned}
\mu(\Gamma) = \mu\Big(\bigcup_{i=1}^{N_r} B(\gamma_i,r)\Big) &\le N_r \max_{1\le i\le N_r} \mu(B(\gamma_i,r)) \le cN_r h(r) \\
&\le c' N_r h(r/3) \le c'' N_r \min_{1\le i\le N_r} \mu(B(\gamma_i,r/3)) \le c'' \sum_{i=1}^{N_r} \mu(B(\gamma_i,r/3)) \\
&= c''\mu\Big(\bigcup_{i=1}^{N_r} B(\gamma_i,r/3)\Big) \le c''\mu(\Gamma).
\end{aligned}$$

Since $0 < \mu(\Gamma) < \infty$, we get $N_r \sim 1/h(r)$, which is the desired assertion. □

**Lemma 8.9.** *Let $\Gamma$ be an $h$-set. Then*
$$\mathcal{L}^n(\Gamma_\delta) \sim \frac{\delta^n}{h(\delta)}.$$

*Proof.* Of course we can find positive constants $c_1$ and $c_2$ (sufficiently small and large, respectively) independently of $\delta$ and two optimal coverings $\mathcal{B}^{(c_1\delta)} = \{B(\gamma_i, c_1\delta)\}_{i=1}^{N_{c_1\delta}}$ and $\mathcal{B}^{(c_2\delta)} = \{B(\tilde{\gamma}_j, c_2\delta)\}_{j=1}^{N_{c_2\delta}}$ such that
$$\bigcup_{i=1}^{N_{c_1\delta}} B(\gamma_i, c_1\delta) \subset \Gamma_\delta \subset \bigcup_{j=1}^{N_{c_2\delta}} B(\tilde{\gamma}_j, c_2\delta).$$
By the above Lemma 8.8 we finally get
$$\mathcal{L}^n(\Gamma_\delta) \sim \frac{\delta^n}{h(\delta)}$$
(with equivalence constants independent of $\delta$) which is the desired assertion. □

We can now state the main assertion for the Minkowski dimensions of $h$-sets.

**Theorem 8.10.** *Let $\Gamma$ be an $h$-set. Then*
$$\underline{\dim}_{\mathcal{M}}\Gamma = \liminf_{\delta\to 0} \frac{\log h(r)}{\log r}$$
*and*
$$\overline{\dim}_{\mathcal{M}}\Gamma = \limsup_{\delta\to 0} \frac{\log h(r)}{\log r}.$$

*Proof.* Thanks to Lemma 8.9 and Proposition 8.4 we get
$$\begin{aligned}\overline{\dim}_{\mathcal{M}}(\Gamma) &= n - \liminf_{\delta\to 0} \frac{\log(\delta^n/h(\delta))}{\log\delta} \\ &= n - \liminf_{\delta\to 0} \frac{\log\delta^n - \log h(\delta)}{\log\delta} = \limsup_{\delta\to 0} \frac{\log h(\delta)}{\log\delta}.\end{aligned}$$
Considering analogously $\underline{\dim}_{\mathcal{M}}(\Gamma)$ we conclude the proof. □

*Remark* 8.11. We observe that it may really happen that
$$\liminf_{\delta\to 0} \frac{\log h(\delta)}{\log\delta} < \limsup_{\delta\to 0} \frac{\log h(\delta)}{\log\delta}, \tag{8.1}$$
for some function $h \in \mathfrak{H}_n$. To see this, first observe that if $h$ and $g$ are two positive equivalent functions from, say, $(0,1) \to \mathbb{R}$, then
$$\liminf_{r\to 0} \frac{\log h(r)}{\log r} = \liminf_{r\to 0} \frac{\log g(r)}{\log r}$$
and analogously for the lim sup.

If $h \in \mathfrak{H}_n$, then one can easily construct an equivalent function $g$ such that
$$\liminf_{r\to 0} \frac{\log g(r)}{\log r} = \liminf_{k\to\infty} \frac{\log g(\lambda_1\cdots\lambda_k)}{\log \lambda_1\cdots\lambda_k}$$

(analogously for the $\limsup$), where $m$ and $\lambda_k$, $k \in \mathbb{N}_0$, come from the definition of $h \in \mathfrak{H}_n$. Since the above expressions, as we pointed out, are independent of the particular representative of $h \in \mathfrak{H}_n$ we have that

$$\liminf_{r\to 0} \frac{\log h(r)}{\log r} = n \liminf_{k\to\infty} \left( \frac{|\log_m \lambda_1 \cdots \lambda_k|}{k} \right)^{-1} \tag{8.2}$$

(analogously for the $\limsup$), for any $h \in \mathfrak{H}_n$.

Therefore, the existence of a function $h \in \mathfrak{H}_n$ with (8.1) is simply reduced to the choice of an appropriate sequence $\{\lambda_k\}_{k\in\mathbb{N}_0}$, such that the right hand-side of (8.2) oscillates (which is possible), complemented by the observation that for any given sequence $\{\lambda_k\}_{k\in\mathbb{N}_0}$ with $0 < \inf_k \lambda_k \le \sup_k \lambda_k < m^{-1}$, for some integer $m \ge 2$, one can easily construct a function $h \in \mathfrak{H}_n$ related to that sequence.

Finally, let us come to the most interesting property (for us) of an $h$-set.

**Theorem 8.12.** *Let* $\Gamma$ *be an* $h$*-set. Then*

$$\dim_{\mathcal{H}} \Gamma \cap B(\gamma, r) = \liminf_{\delta\to 0} \frac{\log h(\delta)}{\log \delta} \qquad \gamma \in \Gamma, \quad r \in (0,1). \tag{8.3}$$

The expression on the left-hand side is sometimes called the *local Hausdorff dimension of the set* $\Gamma$ and in general it is a function depending on $\gamma$ and $r$. The theorem says that in the case of $h$-sets it is always constant. This implies the weaker assertion that the Hausdorff dimension of $\Gamma$ is $\liminf_{\delta\to 0} \log h(\delta)/\log\delta$.

*Proof.* Let us denote the inferior limit in (8.3) by $D$. Let us consider the sequence $\{s_k = \lambda_1 \cdots \lambda_k\}_{k=1}^{\infty}$ exploited in the proof of the Main Theorem 6.6. Recall that $h(s_k) \sim m^{-nk}$ and $0 < \alpha < \lambda_k < \beta < 1$, $k = 1, 2 \ldots$, (we shall later return to the exact value of these constants now it is only important that $0 < \alpha < \beta < 1$). By what we said in Remark 8.11, $D = \limsup_{k\to\infty} \log h(s_k)/\log s_k$, and the latter is easily estimated as follows:

$$\frac{n \log m}{|\log \alpha|} \le \liminf_{k\to\infty} \frac{\log h(s_k)}{\log s_k} \le \frac{n \log m}{|\log \beta|}.$$

Now that we know that $0 < D < +\infty$, we can apply the so-called *mass distribution principle* for estimating the Hausdorff dimension of a set. Let $0 < s < D$ and $\gamma \in \Gamma$, then we have

$$\begin{aligned} 2^s \overline{\mathcal{D}}^s \mu(\gamma) = \limsup_{r\to 0} \frac{\mu(B(\gamma,r))}{r^s} &= \limsup_{r\to 0} \exp\{\log \mu(B(\gamma,r)) - s \log r\} \\ &= \exp\{\limsup_{r\to 0} \log r (\log h(r)/\log r - s)\} = 0. \end{aligned} \tag{8.4}$$

Then, thanks to Theorem 3.6–(i) and (8.4),

$$0 < \mu(\Gamma) \le c \overbrace{\sup_{\gamma\in\Gamma} \overline{\mathcal{D}}^s \mu(\gamma)}^{=0} \mathcal{H}^s(\Gamma).$$

The only noncontradictory assertion we can derive is that $\mathcal{H}^s(\Gamma) = \infty$. This implies $s \le \dim_{\mathcal{H}} \Gamma$. Since $s$ was arbitrarily chosen in $(0, D)$, we get $D \le \dim_{\mathcal{H}} \Gamma$. Analogously, taking $s > D$, one gets

$$\inf_{\gamma\in\Gamma} \overline{\mathcal{D}}^s \mu(\gamma) = \infty. \tag{8.5}$$

Again, thanks to Theorem 3.6–(ii) and (8.5), we infer that

$$\infty > \mu(\Gamma) \ge \overbrace{\inf_{\gamma\in\Gamma} \overline{\mathcal{D}}^s \mu(\gamma)}^{=\infty} \mathcal{H}^s(\Gamma).$$

The only possibility to avoid a contradiction is that $\mathcal{H}^s(\Gamma) = 0$. This shows that $\dim_{\mathcal{H}} \Gamma \le s$. Since $s$ was arbitrarily chosen in $(D, +\infty)$, we get that $\dim_{\mathcal{H}} \Gamma \le D$. Therefore we have that the Hausdorff dimension of $\Gamma$ is $D$.

It remains to show that this notion is even local, i.e., if we choose $\Gamma(\gamma, r) = \Gamma \cap B(\gamma, r)$, $\gamma \in \Gamma$, $0 < r < 1$, we have to show that $\dim_{\mathcal{H}} \Gamma(\gamma, r) = D$. But this is straightforward: one can reconsider the above proof replacing $\Gamma(\gamma, r)$ is still an $h$-set with respect to $\widetilde{\mu} = \mathcal{H}^h|\Gamma(\gamma, r)$, maybe with different equivalence constants, but this does not matter. Hence the dimension of $\Gamma(\gamma, r)$ is still $D$. But this is what we have claimed and the proof is concluded. □

*Remark* 8.13. Using the version of Theorem 3.6 for lower densities instead of upper densities and using the same techniques as in the above proof, one could show that if $\Gamma$ is an $h$-set, then

$$\dim_{\mathcal{P}} \Gamma \cap B(\gamma, r) = \limsup_{\delta\to 0} \frac{\log h(\delta)}{\log \delta}, \qquad \gamma \in \Gamma,\ r \in (0, 1),$$

where $\dim_{\mathcal{P}} A$ denotes the *packing dimension* of a set $A \subset \mathbb{R}^n$.

*Remark* 8.14. As a by-product of the above proof we have

$$0 < \frac{n \log m}{|\log \lambda_0|} \le \dim_{\mathcal{H}} \Gamma \le \frac{n \log m}{|\log \lambda^0|} < n,$$

where $\lambda_0 = \inf_k \lambda_k$ and $\lambda^0 = \sup_k \lambda_k$ come from condition ($D_2$) of Definition 6.1.

Now we would like to investigate the so-called local Markov inequality related to an $h$-set. This property turns out to be rather helpful in problems concerning the linear extension to $\mathbb{R}^n$ of functions belonging to Besov-type spaces suitably defined on $h$-sets.

We give the definition of a set preserving the local Markov inequality and a useful geometrical characterisation. Sets preserving this inequality are studied in detail in [8] (see also [13]): we refer to this work for a general survey.

**Definition 8.15.** We say that a closed non-empty set $F \subset \mathbb{R}^n$ *preserves the local Markov's inequality* if the following condition holds for all positive integers $k$:

for all polynomials $P$ of degree at most $k$ and all balls $B = B(f, r)$ with $f \in F$ and $0 < r \leq 1$, we have

$$\max_{F\cap B} |\nabla P| \leq cr^{-1} \max_{F\cap B} |P| \tag{8.6}$$

with a constant $c$ depending only on $F$, $n$ and $k$. We then refer to (8.6) as *Markov's inequality on $F$*.

For the sake of brevity a set preserving the local Markov's inequality is simply said *Markovian*.

The definition imposes, in some sense, a certain "thickness" on the set $F$. However, also very "dusty" and sparse sets (hence, low-dimensional sets) can also be Markovian. What really matters is a non-local alignment condition, as Proposition 8.16 points out. For instance, *for any admissible function $h \in \mathfrak{H}_n$*, the pseudo self-similar fractal constructed in the proof of the Main Theorem 6.6 preserves the local Markov inequality, as one easily verifies with the help of the following proposition (see [8, Theorem 2, p. 39]).

**Proposition 8.16.** *A closed non-empty subset $F$ of $\mathbb{R}^n$ is a Markovian set if and only if the following geometric condition does not hold: for every $\varepsilon > 0$ there exists a ball $B = B(f, r)$, with $f \in F$ and $0 < r \leq 1$, so that $B \cap F$ is contained in some band of the form $\{x \in \mathbb{R}^n : |b \cdot (x - f)| < \varepsilon r\}$, where the dot stands for the scalar product and $|b| = 1$.*

The following proposition gives a sufficient condition for a function $h \in \mathfrak{H}_n$ so that the resulting $h$-set preserves the local Markov's inequality. Essentially, we extend [8, Theorem 3, p. 39].

**Theorem 8.17.** *Let $n \geq 2$ and let $\Gamma$ be an $h$-set in $\mathbb{R}^n$. Suppose that for every $c > 0$ there exists $\varepsilon > 0$ such that $h(\varepsilon r)/h(r) \leq c\varepsilon^{n-1}$, for $0 < r \leq 1$. Then $\Gamma$ is a Markovian set.*

*Proof.* We prove that the geometric characterisation expressed in the above Proposition 8.16 does not hold. Let $\mu$ be an $h$-measure on $\Gamma$. Let $\varepsilon > 0$ to be chosen appropriately later on and let $B = B(\gamma, r)$ be a ball with $\gamma \in \Gamma$ and $0 < r \leq 1$. Let $D = B \cap \{x : |b \cdot (\gamma - x)| < \varepsilon r\}$ be the intersection of $B$ with a chosen band.

Then at most $(2r)^{n-1}\varepsilon r/(\varepsilon r)^n = 2^{n-1}\varepsilon^{1-n}$ cubes of side $\varepsilon r$ cover $D$. If one of these cubes intersects $\Gamma$, then it is contained in a ball of radius $\sqrt{n}\varepsilon r$ and center $\gamma' \in \Gamma$. Then we have

$$\mu(B) = \mu(B(\gamma, r)) \geq c_1 h(r)$$

and

$$\mu(D) \leq 2^{n-1}\varepsilon^{1-n}\mu(B(\gamma', \sqrt{n}\varepsilon r)) \leq c_2\varepsilon^{1-n}h(r\varepsilon),$$

where $c_1$ and $c_2$ are positive constants depending only on $h$.

Since $\mu(B) - \mu(D) \geq h(r)(c_1 - c_2\varepsilon^{1-n}h(\varepsilon r)/h(r))$, by assumption we can choose $\varepsilon$ so that $\mu(B) - \mu(D) > 0$. But this means that $B \cap \Gamma$ is not entirely contained in the chosen band. □

Now we discuss another purely geometric property of a set, which is helpful in some problems concerning traces of Besov spaces $B^s_{pq}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$ on $h$-sets.

**Definition 8.18.** A non-empty Borel set $\Gamma$ satisfies the *ball condition* if there exists a number $0<\eta<1$ with the following property:

for any ball $B(\gamma,r)$ centered at $\gamma\in\Gamma$ and of radius $0<r\le 1$ there is a ball $B(x,\eta r)$ centered at some $x\in\mathbb{R}^n$, such that

$$B(\gamma,r)\supset B(x,\eta r)\quad\text{and}\quad B(x,\eta r)\cap\overline{\Gamma}=\varnothing.$$

This definition coincides essentially with [11, Definition 18.10, p. 142].

Any set satisfying the ball condition has Lebesgue measure zero, and we might intuitively imagine that it is "self-similarly porous".

Here, we are interested in the interplay between finite Radon measures $\mu$ and their support. More precisely, let us suppose that $\mu$ is a finite Radon measure with compact support $\Gamma$ such that

$$\mu(B(\gamma,r))\sim\Phi(r),\qquad \gamma\in\Gamma,\ r\in(0,1),\tag{8.7}$$

for some suitable continuous function $\Phi$ in $[0,1)$.

In [12, Proposition 9.18, p. 139] one finds a necessary and sufficient condition on $\Phi$ under which $\Gamma=\operatorname{supp}\mu$ satisfies the ball condition:

**Theorem 8.19.** *Let $\mu$ be a finite Radon measure in $\mathbb{R}^n$, with compact support $\Gamma$, satisfying* (8.7)*, where $\Phi$ is supposed to be a continuous monotonically increasing function on $[0,1)$ with $\Phi(r)>0$ if $0<r<1$ and $\Phi(0)=0$. Then $\Gamma$ satisfies the ball condition if and only if there are two positive numbers $c$ and $\lambda$ such that*

$$\Phi(2^{-\nu})\le c2^{(n-\lambda)\varkappa}\Phi(2^{-\nu-\varkappa})\quad\textit{for all }\nu,\varkappa\in\mathbb{N}_0.\tag{8.8}$$

Here is the main assertion concerning $h$-sets and the ball condition.

**Theorem 8.20.** *Any $h$-set satisfies the ball condition.*

*Proof.* We shall prove that any admissible $h$ satisfies the conditions required in the above Theorem 8.19. Of course, the only assumption to be checked is (8.8).

Let $m$ and $\{\lambda_k\}$ be, respectively, the number and the sequence from Definition 6.1. Let $\varepsilon>0$ be small so that $\lambda_k\le m^{-1-\varepsilon}$ for $k=1,2\dots$ (by Condition ($D_2$) it is possible to find such $\varepsilon$). It is not difficult to realise that

$$h(2^{-\nu+\varkappa})\le cm^{np_\varkappa}h(2^{-\nu}),\qquad \nu,\ \varkappa\in\mathbb{N},\tag{8.9}$$

where $c$ is a positive constant depending only on $h$ and

$$p_\varkappa=\min\{k\in\mathbb{N}:m^{-(1-\varepsilon)k}\le 2^{-\varkappa}\}.$$

With this refined doubling condition we immediately get

$$\begin{aligned}h(2^{-\nu})\le cm^{p_\varkappa n}h(2^{-\nu-\varkappa})&\le c'm^{(p_\varkappa-1)n}h(2^{-\nu-\varkappa})\\&\le c'm^{\frac{\varkappa n}{(1+\varepsilon)\log m}}h(2^{-\nu-\varkappa})=c'2^{(n-\lambda)\varkappa}h(2^{-\nu-\varkappa}),\end{aligned}$$

with $\lambda = n(1 - 1/(1+\varepsilon)) > 0$.

This is what we claimed and the proof is concluded. $\square$

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