Georgian Mathematical Journal
Volume 9 (2002), Number 1, 33–52

ON THE THEORY OF FUNCTIONAL-DIFFERENTIAL INCLUSION OF NEUTRAL TYPE

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Abstract. A Cauchy problem for a functional-differential inclusion of neutral type with a nonconvex right-hand side is investigated. Questions of the solvability of such a problem are considered, estimates analogous to the Filippov's estimates are obtained and the density principle is proved.

2000 Mathematics Subject Classification: 34A60, 34G20, 34K35, 49J30, 49K25.
Key words and phrases: Multi-valued mappings, functional-differential inclusion of neutral type, Cauchy problem, fixed point, differential inclusions.

A functional-differential inclusion of neutral type with a non-convex right hand side was investigated in [1, 2]. In these papers problems of the existence of solutions and the structure of a set of solutions were considered. The solution of the inclusion is understood as some extension of an absolutely continuous function, given on a segment, to a "wider" segment. In paper [2] the assumption about the so-called L-selector was one of the main conditions which allowed one to investigate the structure of a set of solutions of neutral type differential inclusion. In our opinion this assumption is very restrictive. Even in the example given in [2] it is not always fulfilled. Therefore we replace it with τ-Volterra property. Furthermore, we mean that the solution of a differential inclusion is not an extension of a given function but is this solution in itself. More exactly, we consider the classical solution of the Cauchy problem (see [3]) whose the definition is analogous to the definition of a solution of the Cauchy problem for functional-differential equation in monograph [4]. Such a definition of the Cauchy problem for a functional-differential inclusion of neutral type does not reject the traditional definition [1, 2, 5] but includes this problem as a particular case. For the problem considered here we obtain estimates of solutions of neutral type functional-differential inclusions analogous to the Filippov's estimates (see [6, 7, 8, 9, 10]), and the density principle for such inclusions has been proved [6, 7, 8]. Note that the density principle is the fundamental property in the theory of differential inclusions (see Remark 4 below, also [9]). Besides, it must be mentioned that the following investigations are based on the method of continuous selectors of multi-valued mappings with non-convex images is suggested in [11, 12, 13, 14, 15].

Let Y be a Banach space with norm $\|\cdot\|$ and let $U \subset Y$. Denote by $\overline{\text{co}}(U)$ a

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

convex closed hull of the set U, $\|U\| = \sup\{\|u\|, u \in U\}$. Let $\Phi_1, \Phi_2 \subset Y$. Denote $h^+_Y[\Phi_1, \Phi_2] = \sup\{\rho_Y[y, \Phi_2], y \in \Phi_1\}$, where $\rho[\cdot,\cdot]$ is a distance between a point and a set, $h_Y[\Phi_1, \Phi_2] = \max\{h^+_Y[\Phi_1, \Phi_2], h^+_Y[\Phi_2, \Phi_1]\}$ is a Hausdorff distance between the sets Φ_1 and Φ_2.

Let $\mathbb{R}^n$ be the space of n-dimensional vector-columns with the norm $|\cdot|$; comp$[\mathbb{R}^n]$ be the set of all nonempty compacts of the space $\mathbb{R}^n$; $\mathbb{R}^{n\times n}$ be the space of $n\times n$ matrices with norm $|\cdot|$ conformed to the space $\mathbb{R}^n$. Let the set $\mathcal{U} \subset [a, b]$ be Lebesgue measurable, $\mu(\mathcal{U}) > 0$ (μ is the Lebesgue measure). Denote $L^n(\mathcal{U})$ ($L^n_\infty(\mathcal{U})$) the space of functions $x : \mathcal{U} \to \mathbb{R}^n$ with Lebesgue integrable (measurable and essentially bounded) coordinates and norm $\|x\|_{L(\mathcal{U})} = \int\limits_{\mathcal{U}} |x(s)|\, ds$ ($\|x\|_{L_\infty(\mathcal{U})} = \operatorname{vraisup}_{t\in\mathcal{U}} |x(t)|$), and $C^n[a, b]$ ($D^n[a, b]$) the space of continuous (absolutely continuous) functions $x : [a, b] \to \mathbb{R}^n$ with norm $\|x\|_{C[a,b]} = \max\{|x(t)| : t \in [a, b]\}$ ($\|x\|_D = |x(a)| + \|\dot{x}\|_L$); $C^1_+[a, b]$ ($L^1_+[a, b]$) the cone of nonnegative functions of the space $C^1[a, b]$ ($L^1[a, b]$). The continuous operator $Z : C^n[a, b] \to C^1_+[a, b]$ is defined by the equality $(Zx)(t) = |x(t)|$ (for integrable functions Z is defined analogously).

We call the set $\Phi \subset L^n[a, b]$ *decomposable* if for all Lebesgue measurable sets $\mathcal{U}_1, \mathcal{U}_2 \subset [a, b]$ so that $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$, $\mathcal{U}_1 \cup \mathcal{U}_2 = [a, b]$, and for any $x, y \in \Phi$, the inclusion $\chi(\mathcal{U}_1)x + \chi(\mathcal{U}_2)y \in \Phi$ is valid, where $\chi(\cdot)$ is the characteristic function of a set.

Denote $\Pi[L^n[a, b]]$ the class of all nonempty bounded closed and decomposable sets of $L^n[a, b]$.

All the sets considered below are Lebesgue measurable and the multi-valued mappings are measurable as in [16]. All multi-valued mappings are Hausdorff continuous. To be brief we do not write the index $\mathbb{R}^n$ in notation of the Hausdorff metric.

We say that a mapping $\Phi : C^n[a, b] \times L^n[a, b] \to \Pi[L^n[a, b]]$ *possesses property* $\mathcal{A}$, if the following conditions are fulfilled: for all $z \in L^n[a, b]$ and all $x, y \in C^n[a, b]$ satisfying the relation $x = y$ on $[a, \nu]$, $\nu \in (a, b]$, the equality $\Phi(x, z) = \Phi(y, z)$ is valid on $[a, \nu]$ (the Volterra property in the first argument); there exists $\tau \in (0, b - a]$ such that for all $x \in C^n[a, b]$ and all $y, z \in L^n[a, b]$, for which the equality $y = z$ is valid on $[a, \nu]$, $\nu \in (a, b]$, the relation $\Phi(x, y) = \Phi(x, z)$ is fulfilled on $[a, \nu + \tau] \cap [a, b]$ (τ-Volterra property in the second argument); for all $(x, y) \in C^n[a, b] \times L^n[a, b]$ the relation $\Phi(x, y) = \Phi(x, 0)$ is valid on $[a, a + \tau]$. Without loss of generality we assume that there exists a natural number m such that the equality $\frac{b-a}{m} = \tau$ is valid.

Let us consider a Cauchy problem

$$\dot{x} \in \Phi(x, \dot{x}), \quad x(a) = x_0, \quad (x_0 \in \mathbb{R}^n), \tag{1}$$

where the continuous mapping $\Phi : C^n[a, b] \times L^n[a, b] \to \Pi[L^n[a, b]]$ possesses property $\mathcal{A}$.

Let $\tau \in (a, b]$ and let $\Phi \subset L^n[a, b]$. Denote by $\Phi|_{[a,\tau]}$ a set of restrictions of functions from the set Φ on the segment $[a, \tau]$. Let us define the continuous operators $P_\tau : C^n[a, \tau] \to C^n[a, b]$, $Q_\tau : L^n[a, \tau] \to L^n[a, b]$, $\Phi_\tau : C^n[a, b] \times$

$L^n[a,b] \to \Pi[L^n[a,\tau]]$ by the equalities

$$
\begin{aligned}
(P_\tau x)(t) &= \begin{cases} x(t), & \text{if } \ t \in [a,\tau], \\ x(\tau), & \text{if } \ t \in (\tau,b], \end{cases} \\
(Q_\tau x)(t) &= \begin{cases} x(t), & \text{if } \ t \in [a,\tau], \\ 0, & \text{if } \ t \in (\tau,b], \end{cases}
\end{aligned}
\tag{2}
$$

$$
\Phi_\tau(x,y) = \Phi(x,y)|_{[a,\tau]}. \tag{3}
$$

We say that an absolutely continuous function $x : [a,\tau] \to \mathbb{R}^n$ is a local solution of problem (1), defined on the segment $[a,\tau]$ ($\tau \in (a,b]$) if the relations

$$
\dot{x} \in \Phi_\tau(P_\tau(x), Q_\tau(\dot{x})), \quad x(a) = x_0 \tag{4}
$$

are fulfilled. If $\tau = b$, then we call such a local solution simply a solution. We call problem (4) the local problem (1) on the segment $[a,\tau]$ ($\tau \in (a,b]$).

We say that the problem

$$
\dot{x} = T(x,\dot{x}), \quad x(a) = |x_0|, \quad x_0 \in \mathbb{R}^n, \tag{5}
$$

where the mapping $T : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ is continuous, has an upper solution $u \in D^1[a,b]$ if for an arbitrary solution y of problem (5) the inequality $y \leqslant u$ is fulfilled and an arbitrary local problem (5) on the segment $[a,\tau]$ ($\tau \in (a,b)$) has an upper solution which is a restriction of the function u on the segment $[a,\tau]$.

We say that a continuous mapping $T : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ *possesses property* $\mathcal{B}$ if it is isotonic, possesses property $\mathcal{A}$, and problem (5) has an upper solution.

We say that a continuous mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ *possesses property* $\mathcal{C}$ if it possesses property $\mathcal{A}$ and there exists a continuous mapping $T : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ possessing property $\mathcal{B}$ such that for all $(x,y) \in C^n[a,b] \times L^n[a,b]$ and all measurable sets $\mathcal{U} \subset [a,b]$ the inequality

$$
\|\Phi(x,y)\|_{L(\mathcal{U})} \leqslant \|T(Zx,Zy)\|_{L(\mathcal{U})} \tag{6}
$$

is fulfilled.

Further, we consider problem (5) with the operator $T : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ satisfying inequality (6).

Let the operator $\Delta : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ be continuous. For every $i = 2,3,\dots,m$ we define the continuous mapping $\Delta^i : C^1_+[a,b] \to L^1_+[a,b]$ by the equalities

$$
\Delta^i(x,0) = \Delta(x,\Delta^{i-1}(x,0)), \quad \Delta^m(x) = \Delta^m(x,0).
$$

Lemma 1. *Let the operator* $\Delta : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ *possess property* $\mathcal{B}$ *and let* $y \in D^1[a,b]$ *satisfy the inequality*

$$
\dot{y} \leqslant \Delta(y,\dot{y}), \quad y(a) \leqslant |x_0|. \tag{7}
$$

Further, let u be the upper solution of the problem

$$\dot{z} = \Delta^m(z), \quad z(a) = |x_0|. \tag{8}$$

Then the inequalities $y \leqslant u$ and $\dot{y} \leqslant \dot{u}$ are fulfilled.

Indeed, let the absolutely continuous function $y : [a,b] \to \mathbb{R}^1$ satisfy the conditions of the theorem. Then the inequality

$$\dot{y} \leqslant \Delta(y, 0). \tag{9}$$

holds on the segment $[a, a+\tau]$. Further, on the segment $[a, a+2\tau]$ the estimate

$$\dot{y} \leqslant \Delta(y, \Delta(y,0)) = \Delta^2(y,0)$$

results from inequalities (7) and (9).

Carrying on with the procedure, we obtain the inequality $\dot{y} \leqslant \Delta^m(y)$. As the operator $\Delta^m : C^n_+[a,b] \to L^1_+[a,b]$ is isotonic, from [17] and the last inequality we obtain the estimates $y \leqslant u$ and $\dot{y} \leqslant \dot{u}$.

Corollary 1. *Let the mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possess property $\mathcal{C}$. Then for every local solution x of the problem* (1) *($\tau \in (a,b]$) defined on the segment $[a,\tau]$, there is an estimate $\|x\|_{C[a,\tau]} \leqslant \|u\|_{C[a,b]}$ and for almost every $t \in [a,\tau]$ the relation $|\dot{x}(t)| \leqslant \dot{u}(t)$ is correct, where u is an upper solution of problem* (5).

Indeed, let x be a local solution of problem (1) which is defined on the segment $[a,\tau]$. Then for all measurable sets $\mathcal{U} \subset [a,\tau]$ the relation $\|\dot{x}\|_{L(\mathcal{U})} \leqslant \|T(Zx, Z\dot{x})\|_{L(\mathcal{U})}$ is correct. Hence, as the set $\mathcal{U}$ is arbitrary, the inequality $Z\dot{x} \leqslant T(Zx, Z\dot{x})$ exists in the cone $L^1_+[a,\tau]$.

Let us define the absolutely continuous function $p : [a,\tau] \to \mathbb{R}^1$ by the equality

$$p(t) = |x_0| + \int_a^t |\dot{x}(t)|\, dt.$$

It follows from the definition of the function p that the relation $\dot{p} \leqslant T(p, \dot{p})$ holds on the segment $[a,\tau]$. Therefore according to the lemma for every $t \in [a,\tau]$ we obtain the inequality $|x(t)| \leqslant u(t)$ and for almost every $t \in [a,\tau]$ the estimate $|\dot{x}(t)| \leqslant \dot{u}(t)$.

Theorem 1. *Let the continuous mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possess property $\mathcal{C}$. Then for an arbitrary function $q \in D^n[a,b]$ and an arbitrary $\varepsilon > 0$ there exists a solution $x \in D^n[a,b]$ of problem* (1) *such that for all measurable sets $\mathcal{U} \subset [a,b]$ the inequality*

$$\|\dot{q} - \dot{x}\|_{L(\mathcal{U})} \leqslant \rho_{L(\mathcal{U})}[\dot{q}; \Phi(x,\dot{x})] + \varepsilon\mu(\mathcal{U}) \tag{10}$$

is fulfilled.

Proof. Let $q \in D^n[a,b]$ and $\varepsilon > 0$. Further, for brevity, let us denote $P_i(\cdot) \equiv P_{a+i\tau}(\cdot)$, $Q_i(\cdot) \equiv Q_{a+i\tau}(\cdot)$, where the mappings $P_{a+i\tau}(\cdot)$; $Q_{a+i\tau}(\cdot)$ are defined by equalities (2) if $\tau = a + i\tau$. Let $z \in L^n[a, a+i\tau]$, $i = 1, 2, \dots, m-1$. Define the continuous mapping $Q_i^z : L^n[a+i\tau, a+(i+1)\tau] \to L^n[a,b]$ by the relations

$$(Q_i^z(x))(t) = \begin{cases} z(t), & \text{if} \quad t \in [a, a+i\tau], \\ x(t), & \text{if} \quad t \in (a+i\tau, a+(i+1)\tau], \\ 0, & \text{if} \quad t \in (a+(i+1)\tau, b]. \end{cases}$$

Let $d > 0$. Define the continuous operator $W^d : C^n[a,b] \to C^n[a,b]$ by the equalities

$$(W^d(x))(t) = \begin{cases} x(t), & \text{if} \quad |x(t)| \leqslant d, \\ \dfrac{d}{|x(t)|} x(t), & \text{if} \quad |x(t)| > d. \end{cases}$$

Consider the local problem (1) on the segment $[a, a+\tau]$. As the mapping $\Phi(\cdot,\cdot)$ possesses property $\mathcal{A}$, on the segment $[a, a+\tau]$ problem (1) has the form

$$\dot{x} \in \Phi_1((P_1(x)), 0), \quad x(a) = x_0. \tag{11}$$

Here $\Phi_1(\cdot,\cdot) \equiv \Phi_{a+\tau}(\cdot,\cdot)$, where the mapping $\Phi_{a+\tau}(\cdot,\cdot)$ is defined by equality (3) when $\tau = a + \tau$.

Let $d = \|u\|_{C[a,b]} + 1$, where u is an upper solution of problem (5). Consider the problem

$$\dot{x} \in \Phi_1(W^d(P_1(x)), 0), \quad x(a) = x_0. \tag{12}$$

Let H_1 and H_2 be the sets of solutions of problems (11) and (12), respectively. We shall prove that $H_1 = H_2$. Let us assume the contrary. Then because of the definition of the operator $W^d(\cdot)$ there exists $y \in H_2$ such that $\|y\|_{C[a,a+\tau]} > d$. As $|x_0| < d$, there exists a number $\alpha \in [a, a+\tau)$ such that $\|y\|_{C[a,\alpha]} = d$. Hence the restriction of the function y on the segment $[a,\alpha]$ is a local solution of problem (1), defined on the segment $[a,\alpha]$, but it contradicts the corollary of the lemma. Thus $H_1 = H_2$.

As the mapping $\Phi_1(W^d(P_1(\cdot)), 0) : C^n[a, a+\tau] \to \Pi[L^n[a, a+\tau]]$ is continuous, by [15] for $\varepsilon > 0$ and restriction of q to the segment $[a, a+\tau]$ (which we also denote by q) there exists a continuous mapping $g_1 : C^n[a, a+\tau] \to L^n[a, a+\tau]$ which possesses the following properties: the inclusion

$$g_1(y) \in \Phi_1(W^d(P_1(y)), 0) \tag{13}$$

is fulfilled for all $y \in C^n[a, a+\tau]$ and the estimate

$$\|\dot{q} - g_1(y)\|_{L(\mathcal{U})} \leqslant \rho_{L(\mathcal{U})}[\dot{q}, \Phi_1(W^d(P_1(y)), 0)] + \varepsilon\mu(\mathcal{U}) \tag{14}$$

holds for all measurable $\mathcal{U} \subset [a, a+\tau]$ and all $y \in C^n[a, a+\tau]$.

Consider an equation

$$x = x_0 + V_1(g_1(x)), \tag{15}$$

in the space $C^n[a, a+\tau]$, where the mapping $V_1 : L^n[a,b] \to C^n[a,b]$ is defined by the equality

$$(V_1 z)(t) = \int_a^t z(s)\,ds, \quad t \in [a,b].$$

From inclusion (13) and the definition of the operator $W^d(\cdot)$ we deduce that $V_1(g_1(C^n[a, a+\tau]))$ is a precompact set of the space $C^n[a, a+\tau]$. Therefore according to Schauder theorem equation (15) is solvable. Let x_1 be a solution of equation (15). It is evident that x_1 is a solution of problem (12). Moreover, it follows from equality (14) that the estimate

$$\|\dot{q} - \dot{x}_1\|_{L(\mathcal{U})} \leqslant \rho_{L(\mathcal{U})}[\dot{q}, \Phi_1(P_1(x_1), 0)] + \varepsilon\mu(\mathcal{U})$$

is fulfilled for every measurable set $\mathcal{U} \subset [a, a+\tau]$.

Let

$$\Omega_{x_1} = \{z \in C^n[a, a+2\tau] : \forall t \in [a, a+\tau]\ z(t) = x_1(t)\}. \tag{16}$$

Let us define a mapping $\Phi_2(P_2(\cdot), Q_2^{\dot{x}_1}(\cdot)) : \Omega_{x_1} \times L^n[a+\tau, a+2\tau] \to \Pi[L^n[a, a+2\tau]]$ by the equality

$$\begin{aligned}\Phi_2(P_2(y), Q_2^{\dot{x}_1}(z)) = \{p \in L^n&[a, a+2\tau] :\\ &\exists f \in \Phi(P_2(y), Q_2^{\dot{x}_1}(z)), f = \dot{x}_1 \text{ on } [a, a+\tau], p = f \text{ on } [a, a+2\tau]\}.\end{aligned} \tag{17}$$

As the images of the mapping $\Phi(\cdot,\cdot)$ are decomposable, it follows from the definition of the mapping $\Phi_2(\cdot,\cdot)$ that the equality

$$\Phi_2(P_2(y), Q_2^{\dot{x}_1}(z))|_{[a+\tau, a+2\tau]} = \Phi(P_2(y), Q_2^{\dot{x}_1}(z))|_{[a+\tau, a+2\tau]}$$

is true for all $(y,z) \in \Omega_{x_1} \times L^n[a+\tau, a+2\tau]$. Therefore the mapping $\Phi_2(P_2(\cdot), Q_2^{\dot{x}}(\cdot))$ defined by equality (17) is continuous.

Let us note that as the mapping $\Phi(\cdot,\cdot)$ is τ-Volterra in the second argument, the relation

$$\Phi_2(P_2(y), Q_2^{\dot{x}_1}(z)) = \Phi_2(P_2(y), Q_1(\dot{x}_1)).$$

is fulfilled for all $(y,z) \in \Omega_{x_1} \times L^n[a+\tau, a+2\tau]$. As the mapping $\Phi_2(P_2(\cdot), Q_2^{\dot{x}_1}(\cdot))$ is continuous, the mapping $\Phi_2(W^d(P_2(\cdot)), Q_1(\dot{x}_1)) : \Omega_{x_1} \to \Pi[L^n[a, a+2\tau]]$ is also continuous. Therefore, according to [15], there exists a continuous mapping $g_2 : \Omega_{x_1} \to L^n[a, a+2\tau]$, possessing the property: the inclusion

$$g_2(y) \in \Phi_2(W^d(P_2(y)), Q_1(\dot{x}_1)) \tag{18}$$

is fulfilled for every $y \in \Omega_{x_1}$ and the estimate

$$\|\dot{q} - g_2(y)\|_{L(\mathcal{U})} \leqslant \rho_{L(\mathcal{U})}[\dot{q}, \Phi_2(W^d(P_2(y)), Q_1(\dot{x}_1))] + \varepsilon\mu(\mathcal{U}) \tag{19}$$

holds for every measurable $\mathcal{U} \subset [a, a+2\tau]$ and every $y \in \Omega_{x_1}$. Note that according to the definition of the mapping $\Phi_2(P_2(\cdot), Q_2^{\dot{x}_1}(\cdot))$ for every $y \in \Omega_{x_1}$ the restriction of the function $g_2(y)$ on $[a, a+\tau]$ is $\dot{x}_1$.

Let $V_2 : L^n[a, a + 2\tau] \to C^n[a, a + 2\tau]$ be an operator of integration. On the set $\Omega_{x_1} \subset C^n[a, a + 2\tau]$ defined by equality (16) we consider an equation

$$x = x_0 + V_2(g_2(x)). \tag{20}$$

As the operator defined by the right hand side of equation(20) is continuous and maps a convex closed set Ω_{x_1} into it, and the image of $\Phi_2(W^d(\Omega_{x_1}), Q_1(\dot{x}_1))$ is a weakly compact set in the space $L^n[a, a + 2\tau]$, according to the Schauder theorem equation (20) has a solution.

Let x_2 be a solution of equation (20). Note that from the previous arguments it follows that the restriction of the function x_2 to the segment $[a, a + \tau]$ is the solution x_1. Analogously to the proof of the inequality $\|x_1\|_{C[a,a+\tau]} \leqslant \|u\|_{C[a,b]} + 1$, it is possible to prove that $\|x_2\|_{C[a,a+2\tau]} \leqslant \|u\|_{C[a,b]} + 1$, where u is an upper solution of problem (5). Therefore, from the τ-Volterra property of the operator $\Phi(\cdot, \cdot)$ and the definition of the operator $W^d(\cdot)$ we deduce that the inclusion

$$\dot{x}_2 \in \Phi_2(P_2(x_2), Q_2(\dot{x}_2))$$

is valid. Moreover, it follows from inequalities (14), (19) that the inequality

$$\|\dot{q} - \dot{x}_2\|_{L(\mathcal{U})} < \rho_{L(\mathcal{U})}[\dot{q}, \Phi(P_2(x_2), Q_2(\dot{x}_2))] + \varepsilon\mu(\mathcal{U})$$

is fulfilled for every measurable set $\mathcal{U} \subset [a, a + 2\tau]$. Proceeding in a similar way, we find that there exists a function $x = x_m \in D^n[a, b]$ which satisfies the theorem. $\square$

Let for the function $q \in D^n[a, b]$ there exists a function $\varkappa \in L^1_+[a, b]$ such that the inequality

$$\rho_{L(\mathcal{U})}[\dot{q}; \Phi(q, \dot{q})] \leqslant \int_{\mathcal{U}} \varkappa(t)\, dt \tag{21}$$

is fulfilled for every measurable set $\mathcal{U} \subset [a, b]$.

We say that a mapping $\Phi : C^n[a, b] \times L^n[a, b] \to \Pi[L^n[a, b]]$ *possesses property* $\mathcal{D}_0$ if it possesses property $\mathcal{A}$ and there exists a continuous isotonic mapping $T^1 : C^1_+[a, b] \times L^1_+[a, b] \to L^1_+[a, b]$ possessing property $\mathcal{A}$ for which the following conditions are fulfilled: $T^1(0, 0) = 0$; for all $x_1, x_2 \in C^n[a, b]$, $y_1, y_2 \in L^n[a, b]$ and every measurable set $\mathcal{U} \subset [a, b]$ the inequality

$$h_{L(\mathcal{U})}[\Phi(x_1, y_1); \Phi(x_2, y_2)] \leqslant \|T^1(Z(x_1 - x_2), Z(y_1 - y_2))\|_{L(\mathcal{U})} \tag{22}$$

is valid; for all $\varepsilon \geqslant 0$ and $\theta \in L^1_+[a, b]$ there exists a continuous isotonic Volterra mapping $\Xi(\varepsilon, \theta) : C^1_+[a, b] \to L^1_+[a, b]$ such that for all $x \in C^1_+[a, b]$ the estimate $\Xi(\varepsilon, \theta)(x) \geqslant A(\varepsilon, \theta)(x)$ is correct, where the operator $A(\varepsilon, \theta) : C^1_+[a, b] \to L^1_+[a, b]$, at fixed $\varepsilon \geqslant 0$ and function $\theta \in L^1_+[a, b]$, is defined by the equalities

$$A(\varepsilon, \theta)(x) = \Lambda^m(\varepsilon, \theta)(x, 0), \quad \Lambda(\varepsilon, \theta)(x, y) = \varepsilon + \theta + T^1(x, y). \tag{23}$$

We say that a mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$, possessing property $\mathcal{D}_0$, possesses property $\mathcal{D}$ if for $\varepsilon > 0$ and $\theta = \varkappa$ satisfying estimate (21) the problem

$$\dot{x} = \Xi(\varepsilon,\theta)(x), \quad x(a) = \nu \tag{24}$$

has an upper solution, where the mapping $\Xi(\varepsilon,\theta) : C^1_+[a,b] \to L^1_+[a,b]$ is defined by the property $\mathcal{D}_0$ for every $x \in C^1_+[a,b]$. If the relation

$$\lim_{\substack{\varepsilon\to 0+0 \\ \theta\to 0+0}} \Xi(\varepsilon,\theta)(x) = \Xi(0,0)(x)$$

exists for a continuous operator $\Xi(0,0) : C^1_+[a,b] \to L^1[a,b]$ and for all $x \in C^1_+[a,b]$ and problem (24) has only a zero solution at $\nu = 0$, $\varepsilon = 0$, $\theta = 0$, then we say that a mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$, possessing property $\mathcal{D}_0$, *possesses property* $\mathcal{D}^*$.

Remark 1. In properties $\mathcal{D}$ and $\mathcal{D}^*$ for every $\varepsilon \geqslant 0$ and every $\theta \in L^1_+[a,b]$ it is natural to call the mapping $\Xi(\varepsilon,\theta) : C^1_+[a,b] \to L^1_+[a,b]$ a majorizing mapping and problem (24) a majorant problem. The introduction of the majorant problem is caused by that in many cases (see the examples below) it is easier to find such a majorant operator $\Xi(\varepsilon,\theta) : C^1_+[a,b] \to L^1_+[a,b]$, that it facilitates the finding or the estimation of a solution of problem (24) instead of solving it at $\Xi(\varepsilon,\theta)(\cdot) \equiv A(\varepsilon,\theta)(\cdot)$, where the mapping $A(\varepsilon,\theta)(\cdot)$ is defined by equalities (23).

Theorem 2. *Let the function $q \in D^n[a,b]$ and $\varepsilon > 0$. Further, let the mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possess properties $\mathcal{C}$ and $\mathcal{D}$ and let the function $\xi(\varepsilon,\varkappa) \in D^1[a,b]$ be an upper solution of the problem* (24) *at $\nu = |y(a) - x_0|$ and $\theta = \varkappa$, where the function $\varkappa \in L^1_+[a,b]$ satisfies estimate* (21)*. Then for every solution $x \in D^n[a,b]$ of the problem* (1)*, which satisfies equality* (10) *for every measurable set $\mathcal{U} \subset [a,b]$, the inequalities*

$$Z(x-q) \leqslant \xi(\varepsilon,\varkappa), \quad Z(\dot{x}-\dot{q}) \leqslant \Xi(\varepsilon,\varkappa)(\xi(\varepsilon,\varkappa)) \tag{25}$$

are fulfilled.

Proof. Let x, q satisfy the conditions of the theorem and $\varepsilon > 0$. Then from equalities (10), (22) for every measurable $\mathcal{U} \subset [a,b]$ we obtain the estimates

$$\begin{aligned} \rho_{L(\mathcal{U})}[\dot{q},\Phi(x,\dot{x})] &\leqslant \rho_{L(\mathcal{U})}[\dot{q},\Phi(q,\dot{q})] + h_{L(\mathcal{U})}[\Phi(x,\dot{x}),\Phi(q,\dot{q})] \\ &\leqslant \rho_{L(\mathcal{U})}[\dot{q},\Phi(q,\dot{q})] + \|T^1(Z(x-q),Z(\dot{x}-\dot{q}))\|_{L_1(\mathcal{U})}. \end{aligned} \tag{26}$$

The inequality

$$\begin{aligned} Z(\dot{q}-\dot{x}) &\leqslant \varepsilon + \varkappa + T^1(Z(x-q),Z(\dot{x}-\dot{q})) \\ &= \Lambda(\varepsilon,\varkappa)(Z(x-q),Z(\dot{x}-\dot{q})) \end{aligned} \tag{27}$$

results from estimates (21), (10) and (26). Let the absolutely continuous function $z : [a, b] \to \mathbb{R}$ be defined by the equality

$$z(t) = |q(a) - x_0| + \int_a^t |\dot{q}(s) - \dot{x}(s)| ds.$$

As $\dot{z} = Z(\dot{q} - \dot{x})$ and $z \geqslant Z(q - x)$, inequality (27) implies the relation

$$\dot{z} \leqslant \Lambda(\varepsilon, \varkappa)(z, \dot{z}). \tag{28}$$

Estimate (28) and condition $\mathcal{D}$ imply that $\dot{z} \leqslant \Xi(\varepsilon, \varkappa)(z)$. Hence, according to the lemma we obtain relations (25). $\square$

We say that a mapping $F : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{comp}[\mathbb{R}^n]$ possesses property $\mathcal{F}$ if the following conditions are fulfilled: for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ the mapping $F(\cdot, x, y)$ is measurable; there exist functions $\alpha \in L^1[a, b]$ and $\beta \in L^1_\infty[a, b]$ such that for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and almost every $t \in [a, b]$ the estimate

$$h[F(t, x_1, y_1), F(t, x_2, y_2)] \leqslant \alpha(t)|x_1 - x_2| + \beta(t)|y_1 - y_2| \tag{29}$$

is correct; there exists a function $\gamma \in L^1[a, b]$ such that the estimate

$$\|F(t, 0, 0)\| \leqslant \gamma(t) \tag{30}$$

is fulfilled for almost every $t \in [a, b]$.

Let for almost every $t \in [a, b]$ the measurable function $f : [a, b] \to \mathbb{R}^1$ satisfy the inequality $f(t) \leqslant t$, and the measurable function $r : [a, b] \to \mathbb{R}^1$ possess the property: there exists a number $\tau \in (0, b - a)$ such that the inequality $r(t) \leqslant t - \tau$ is fulfilled for all $t \in [a, b]$; and the relation $\sup\limits_{e \subset [a,b], \mu(e) \neq 0} \frac{\mu[r^{-1}(e)]}{\mu(e)} < \infty$ is valid ($r^{-1}(e)$ is a preimage of a measurable set e).

Let us define the continuous operators $S_f : C^n[a, b] \to L^n_\infty[a, b]$, $S_r : L^n[a, b] \to L^n[a, b]$ (see [18, p. 707]) by the equalities

$$\begin{gathered} (S_f x)(t) = \begin{cases} x[f(t)], & \text{if} \quad f(t) \in [a, b], \\ 0, & \text{if} \quad f(t) \notin [a, b], \end{cases} \\ (S_r x)(t) = \begin{cases} x[r(t)], & \text{if} \quad r(t) \in [a, b], \\ 0, & \text{if} \quad r(t) \notin [a, b]. \end{cases} \end{gathered} \tag{31}$$

We define the continuous operators $\widetilde{S}_f : C^1_+[a, b] \to L^1_+[a, b]$ and $\widetilde{S}_r : L^1_+[a, b] \to L^1_+[a, b]$ by analogous equalities.

Consider the following problem as an application of Theorem 2:

$$\dot{x}(t) \in F(t, (S_f x)(t), (S_r \dot{x})(t)), \quad t \in [a, b], \quad x(a) = x_0. \tag{32}$$

We say that a function $x \in D^n[a, b]$ is a solution of problem (32) if inclusion (32) for all $t \in [a, b]$ and the equality $x(a) = x_0$ are satisfied.

Define the continuous multi-valued Nemytsky operator $N : L^n_\infty[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ generated by the mapping $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{comp}[\mathbb{R}^n]$ by the equality

$$N(x,y) = \{z \in L^n[a,b] : z(t) \in F(t, x(t), y(t)) \text{ for almost all } t \in [a,b]\}.$$

Then problem (32) can be stated in the following way:

$$\dot{x} \in N(S_f(x), S_r(\dot{x})), \quad x(a) = x_0. \tag{33}$$

and therefore problem (32) can be reduced to problem (1).

Here the mapping $\Phi_S : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possessing property $\mathcal{A}$ is defined by the equality

$$\Phi_S(x,y) = N(S_f(x), S_r(y)). \tag{34}$$

As the mapping $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{comp}[\mathbb{R}^n]$ possesses property $\mathcal{F}$, the mapping $\Phi_S(\cdot,\cdot)$ defined by equality (34) possesses property $\mathcal{C}$. Indeed, in this case the operator $T_S : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ possessing property $\mathcal{B}$ and satisfying inequality (6) with the mapping $\Phi(\cdot,\cdot) \equiv \Phi_S(\cdot,\cdot)$, is defined by the equality

$$T_S(x,y) = \alpha \widetilde{S}_f(x) + \beta \widetilde{S}_r(y) + \gamma;$$

here and further the functions $\alpha, \gamma \in L^1[a,b]$ and $\beta \in L^1_\infty[a,b]$ satisfy inequalities (29), (30). Moreover, the mapping $\Phi_S(\cdot,\cdot)$ satisfies inequality (22), where the continuous isotonic mapping $T^1_S : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$, possessing property $\mathcal{A}$ is defined by the equality

$$T^1_S(x,y) = \alpha \widetilde{S}_f(x) + \beta \widetilde{S}_r(y).$$

Therefore in this case for all $\varepsilon \geqslant 0$ and $\theta \in L^1_+[a,b]$ the continuous isotonic operator $A_S(\varepsilon,\theta) : C^1_+[a,b] \to L^1_+[a,b]$, defining properties $\mathcal{D}$ and $\mathcal{D}^*$, is defined by the equality

$$A_S(\varepsilon,\theta)(x) = \Lambda^m_S(\varepsilon,\theta)(x,0),$$

where the mapping $\Lambda_S(\varepsilon,\theta) : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$ is defined by the equality

$$\Lambda_S(\varepsilon,\theta)(x,y) = \varepsilon + \theta + \alpha \widetilde{S}_f x + \beta \widetilde{S}_r y.$$

Hence, the equality

$$A_S(\varepsilon,\theta)(x) = \sum_{i=0}^{m-1} (\beta \widetilde{S}_r)^i (\varepsilon + \theta) + \sum_{i=0}^{m-1} (\beta \widetilde{S}_r)^i (\alpha \widetilde{S}_f(x)) \tag{35}$$

holds for all $x \in C^1_+[a,b]$, here and below $(\beta \widetilde{S}_r)^0(\cdot)$ is an identity operator.

Let the operator $M : C^1_+[a,b] \to C^1_+[a,b]$ be defined by the equality

$$(Mx)(t) = \max_{s \in [a,t]} x(s). \tag{36}$$

We define a majorant mapping $\Xi_S(\varepsilon,\theta) : C^1_+[a,b] \to L^1_+[a,b]$ by the relation

$$\Xi_S(\varepsilon,\theta)(x)(t) = \varphi_S(\varepsilon,\theta)(t) + \psi_S(t) M(x)(t), \tag{37}$$

where

$$\varphi_S(\varepsilon,\theta)(t)=\sum_{i=0}^{m-1}(\beta\widetilde{S}_r)^i(\varepsilon+\theta)(t),\quad \psi_S(t)=\sum_{i=0}^{m-1}(\beta\widetilde{S}_r)^i(\alpha)(t). \tag{38}$$

Note that for all $x\in C^1_+[a,b]$ the inequality $A_S(\varepsilon,\theta)(x)\leqslant \Xi_S(\varepsilon,\theta)(x)$ is deduced from the definitions of the mappings $A_S(\varepsilon,\theta)(\cdot)$ (see (35)) and $\Xi_S(\varepsilon,\theta)(\cdot)$ (see (37)).

Further, let us find a solution of problem (24) at the operator $\Xi(\varepsilon,\theta)(\cdot)=\Xi_S(\varepsilon,\theta)(\cdot)$, defined by equality (37). Let $\xi_S(\varepsilon,\theta)$ be a solution of this problem. As $\dot{\xi}_S(\varepsilon,\theta)\geqslant 0$, from the definition of the mapping $M(\cdot)$ (see (36)) for all $t\in[a,b]$ we deduce the equality $M(\xi_S(\varepsilon,\theta))(t)=\xi_S(\varepsilon,\theta)(t)$. Therefore for almost all $t\in[a,b]$ we obtain the relations

$$\dot{\xi}_S(\varepsilon,\theta)(t)=\Xi_S(\varepsilon,\theta)(\xi_S(\varepsilon,\theta))(t)=\varphi_S(\varepsilon,\theta)(t)+\psi_S(t)\xi_S(\varepsilon,\theta)(t).$$

Hence for every $t\in[a,b]$ we have the equality

$$\xi_S(\varepsilon,\theta)(t)=\nu e^{\int_a^t\psi_S(s)\,ds}+\int_a^t e^{\int_\tau^t\psi_S(s)\,ds}\varphi_S(\varepsilon,\theta)(\tau)\,d\tau, \tag{39}$$

where the functions $\varphi_S(\varepsilon,\theta),\psi_S\in L^1_+[a,b]$ are specified by equalities (38).

Let for $q\in D^n[a,b]$ there exist a function $\varkappa\in L^1_+[a,b]$ such that the relation

$$\rho[\dot q(t),F(t,(S_fq)(t),(S_r\dot q)(t))]\leqslant\varkappa(t) \tag{40}$$

is valid for all $t\in[a,b]$.

Theorem 2 gives rise to

Corollary 2. *Let the mapping $F:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathrm{comp}[\mathbb{R}^n]$ possess property $\mathcal{F}$ and the measurable functions $f:[a,b]\to\mathbb{R}$, $r:[a,b]\to\mathbb{R}$ satisfy the conditions formulated above. Then for every function $q\in D^n[a,b]$ and an arbitrary $\varepsilon>0$ there exists a solution $x\in D^n[a,b]$ of problem* (32) *such that the estimates*

$$Z(x-q)\leqslant\xi_S(\varepsilon,\varkappa),\quad Z(\dot x-\dot q)\leqslant\varphi_S(\varepsilon,\varkappa)+\psi_S\xi_S(\varepsilon,\varkappa) \tag{41}$$

are valid. Here the function $\varkappa\in L^1_+[a,b]$ satisfies inequality (40)*; the functions $\varphi_S(\varepsilon,\varkappa)$, $\xi(\varepsilon,\varkappa)$ are defined by equalities* (38) *and* (39) *when $\theta=\varkappa$ and $\nu=|q(a)-x_0|$.*

Remark 2. If in problem (40) $f(t)=t$, and in inequality (29) $\beta(t)\equiv 0$, then from estimates (25) we obtain the Filippov's estimates up to $\varepsilon>0$ (see [6, 7]).

Consider the problem

$$\dot x\in\overline{\mathrm{co}}\Phi(x,\dot x),\quad x(a)=x_0. \tag{42}$$

Let H, H_{co} be sets of solutions of problems (1) and (42), respectively.

We say that a mapping $\Phi:C^n[a,b]\times L^n[a,b]\to\Pi[L^n[a,b]]$ *possesses property $\mathcal{E}$,* if for arbitrary sequences $x_i\in C^n[a,b]$ and $y_i\in L^n[a,b]$, $i=1,2,\dots$, such that $x_i \to x$ in $C^n[a,b]$ and $y_i \to y$ weakly in $L^n[a,b]$ as $i \to \infty$, the relation $h_{L[a,b]}[\Phi(x_i, y_i); \Phi(x,y)] \to 0$ is fulfilled as $i \to \infty$.

Theorem 3. *Let the mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possesses properties $\mathcal{C}$ and $\mathcal{E}$. Then the set $H_{\rm co}$ is closed in the space $C^n[a,b]$.*

Proof. Let $x_i \in H_{\rm co}$ $i = 1, 2, \dots$ and $x_i \to x$ in $C^n[a,b]$ as $i \to \infty$. First, we prove that the function x is absolutely continuous. Indeed, as the mapping $\Phi(\cdot,\cdot)$ possesses property $\mathcal{C}$, there exists a Lebesgue integrable function $u \in L^1[a,b]$ such that the inequality $|\dot{x}_i(t)| \leqslant u(t)$ is fulfilled for every $i = 1, 2, \dots$ and almost every $t \in [a,b]$. Hence the function $x \in D^n[a,b]$ and $\dot{x}_i \to \dot{x}$ weakly in the space $L^n[a,b]$ as $i \to \infty$.

Now we shall prove that the function z satisfies inclusion (42). Let for every $i = 1, 2, \dots$ for function $y_i \in \overline{\rm co}\Phi(x, \dot{x})$ the equality

$$\|\dot{x}_i - y_i\|_{L[a,b]} = \rho_{L[a,b]}[\dot{x}_i, \overline{\rm co}\Phi(x, \dot{x})]$$

be valid. As the mapping $\Phi(\cdot,\cdot)$ possesses property $\mathcal{E}$, the mapping $\operatorname{co}\Phi(\cdot,\cdot)$ also possesses this property (see [19, p. 25]). Therefore from the inequalities

$$\|\dot{x}_i - y_i\|_{L[a,b]} \leqslant h_{L_1[a,b]}[\overline{\rm co}\Phi(x_i, \dot{x}_i); \overline{\rm co}\Phi(x, \dot{x})], \quad i = 1, 2, \dots,$$

we deduce that $\lim\limits_{i\to\infty} \|\dot{x}_i - y_i\|_{L[a,b]} = 0$. Hence $y_i \to \dot{x}$ weakly in $L^n[a,b]$ as $i \to \infty$. Therefore, according to [16, p. 177], x satisfies inclusion (42). □

We say that a mapping $K : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ *possesses property* $\mathcal{K}$ if the following conditions are fulfilled: the mapping $K(\cdot,\cdot)$ has elements measurable on $[a,b] \times [a,b]$; the function $k : [a,b] \to [0,\infty)$ defined by the equality

$$k(t) = \operatorname{vraisup}_{s\in[a,b]} |K(t,s)|$$

is Lebesgue integrable.

Let $\tau \in (0, b-a)$. Let us define the continuous τ-Volterra operators $K_\tau : L^n[a,b] \to L^n[a,b]$ and $\widetilde{K}_\tau : L^1_+[a,b] \to L^1_+[a,b]$ by the equalities

$$(K_\tau(y))(t) = \begin{cases} \int\limits_a^{t-\tau} K(t,s)y(s)\,ds, & \text{if } \ t \in [a+\tau, b], \\ 0, & \text{if } \ t \in [a, a+\tau), \end{cases} \tag{43}$$

$$(\widetilde{K}_\tau(y))(t) = \begin{cases} \int\limits_a^{t-\tau} |K(t,s)|y(s)\,ds, & \text{if } \ t \in [a+\tau, b], \\ 0, & \text{if } \ t \in [a, a+\tau), \end{cases} \tag{44}$$

where the mapping $K : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ possesses property $\mathcal{K}$.

Lemma 2. *Let the kernel $K : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ of the operator $K_\tau : L^n[a,b] \to L^n[a,b]$ ($\tau \in (0, b-a)$), defined by equality* (43), *possess property $\mathcal{K}$. Then a continuous τ-Volterra operator $K_\tau : L^n[a,b] \to L^n[a,b]$ maps every weakly converging sequence to a strongly converging sequence of the space $L^n[a,b]$.*

Indeed, as for almost every $t \in [a,b]$ the mapping $(K_\tau(\cdot))(t) : L^n[a,b] \to \mathbb{R}^n$, defined by equality (43), is a bounded linear vector-functional, it follows from the property $\mathcal{K}$ and the Lebesgue theorem that the operator $K_\tau : L^n[a,b] \to L^n[a,b]$ maps every weakly converging sequence to a strongly converging sequence of the space $L^n[a,b]$.

Let $\tau \in (0, b-a)$. Let us consider the problem

$$\dot{x} \in N(S_f(x), K_\tau(\dot{x})), \quad x(a) = x_0, \tag{45}$$

where the mapping $N : L^n_\infty[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ is the multi-valued Nemytsky operator generated by the mapping $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{comp}[\mathbb{R}^n]$, possessing property $\mathcal{F}$; the operators $S_f : C^n[a,b] \to L^n_\infty[a,b]$ and $K_\tau : L^n[a,b] \to L^n[a,b]$ are defined by equalities (31) and (43), respectively. Similarly to the previous example, problem (45) is a particular case of problem (1), where the mapping $\Phi_K : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$, possessing property $\mathcal{A}$, is specified by the equality

$$\Phi_K(x,y) = N(S_f(x), K_\tau(y)). \tag{46}$$

Moreover, the mapping $\Phi_K(\cdot,\cdot)$, defined by equality (46) possesses property $\mathcal{C}$. As the operator $T_K : C^1_+[a,b] \times L^1_+[a,b] \to L^1_+[a,b]$, possessing property $\mathcal{B}$ and satisfying inequality (6) with the mapping $\Phi(\cdot,\cdot) \equiv \Phi_K(\cdot,\cdot)$, can be defined by the mapping

$$T_K(x,y) = \alpha \widetilde{S}_f(x) + \beta \widetilde{K_\tau}(y) + \gamma,$$

where the operators $\widetilde{S}_f : C^1_+[a,b] \to L^1_\infty[a,b]$ and $\widetilde{K_\tau} : L^1_+[a,b] \to L^1_+[a,b]$ are defined by equalities (31) and (44), respectively, the functions $\alpha, \gamma \in L^1[a,b]$, $\beta \in L^1_\infty[a,b]$ satisfy equalities (29), (30).

Now let us prove that the mapping $\Phi_K : C^n[a,b] \times L^n[a,b] \to \Pi L^n[a,b]$ possesses property $\mathcal{E}$. Indeed, let $x_1, x_2 \in L^n_\infty[a,b]$, $y_1, y_2 \in L^n[a,b]$ and

$$h_{L(\mathcal{U})}[N(x_1,y_1); N(x_2,y_2)] = h^+_{L(\mathcal{U})}[N(x_1,y_1); N(x_2,y_2)],$$

where $\mathcal{U}$ is a measurable set. Then we obtain the equality

$$\begin{aligned} &h_{L(\mathcal{U})}[N(x_1,y_1); N(x_2,y_2)] \\ &\qquad = \int_{\mathcal{U}} h^+[F(s, x_1(s), y_1(s)); F(s, x_2(s), y_2(s))]\, ds \end{aligned} \tag{47}$$

(see [20]). From equality (47) and estimate (29) we derive the relation

$$\begin{aligned} &h_{L(\mathcal{U})}[N(x_1,y_1); N(x_2,y_2)] \\ &\qquad \leqslant \int_{\mathcal{U}} (\alpha(s)|x_1(s) - x_2(s)| + \beta(s)|y_1(s) - y_2(s)|)\, ds. \end{aligned} \tag{48}$$

From inequality (48) and the definition of the mapping $\Phi_K : C^n[a,b]\times L^n[a,b] \to \Pi[L^n[a,b]]$ for all $x_1, x_2 \in C^n[a,b]$ and all $y_1, y_2 \in L^n[a,b]$ we derive the estimate

$$h_{L(\mathcal{U})}[\Phi_K(x_1,y_1),\Phi_K(x_2,y_2)] \leqslant \int\limits_{\mathcal{U}} \alpha(s)|S_f(x_1-x_2)(s)|\,ds + \int\limits_{\mathcal{U}} \beta(s)|K_\tau(y_1-y_2)(s)|\,ds. \tag{49}$$

Therefore, from inequality (49) and Lemma 2 it follows that $\Phi_K(\cdot,\cdot)$ possesses property $\mathcal{E}$.

Consider the problem

$$\dot{x} \in \overline{\mathrm{co}}N(S_f(x); K_\tau(\dot{x})), \quad x(a) = x_0. \tag{50}$$

Let $\mathcal{H}$, $\mathcal{H}_{\mathrm{co}}$ be sets of solutions of problems (45) and (50), respectively.

From Theorem 3 we have

Corollary 3. *Let $\tau \in (0, b-a)$, and the function $f : [a,b] \to \mathbb{R}^1$ satisfy the conditions formulated above. Let the mapping $F : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathrm{comp}[\mathbb{R}^n]$ possess property $\mathcal{F}$, and the kernel $K : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ generating the operator $K_\tau : L^n[a,b] \to L^n[a,b]$ possess property $\mathcal{K}$. Then the set $\mathcal{H}_{\mathrm{co}}$ is closed in the space $C^n[a,b]$.*

Further we obtain an estimate of solutions for problem (45), analogous to Filippov's estimate. For this, we prove that the mapping $\Phi_K(\cdot,\cdot)$ defined by equality (46) possesses property $\mathcal{D}$. Indeed, it follows from inequality (49) that this mapping satisfies inequality (22), where a continuous isotonic operator $T_K^1 : C_+^1[a,b]\times L_+^1[a,b] \to L_+^1[a,b]$ is specified by the equality

$$T_K^1(x,y) = \alpha\widetilde{S}_f(x) + \beta\widetilde{K}_\tau(y).$$

Therefore for every $\varepsilon \geqslant 0$ and $\theta \in L_+^1[a,b]$ a continuous isotonic operator $A_K(\varepsilon,\theta) : C_+^1[a,b] \to L_+^1[a,b]$, defining properties $\mathcal{D}$ and $\mathcal{D}^*$, can be defined by the equality

$$A_K(\varepsilon,\theta)(x) = \Lambda_K^m(\varepsilon,\theta)(x,0),$$

where $m = \frac{b-a}{\tau}$, and the mapping $\Lambda_K : (\varepsilon,\theta) : C_+^1[a,b]\times L_+^1[a,b] \to L_+^1[a,b]$ is defined by the relation

$$\Lambda_K(\varepsilon,\theta)(x,y) = \varepsilon + \theta + \alpha\widetilde{S}_f(x) + \beta\widetilde{K}_\tau(y).$$

Hence, for every $x \in C_+^1[a,b]$

$$A_K(\varepsilon,\theta)(x) = \sum_{i=0}^{m-1}(\beta\widetilde{K}_\tau)^i(\varepsilon+\theta) + \sum_{i=0}^{m-1}(\beta\widetilde{K}_\tau)^i(\alpha\widetilde{S}_f(x)), \tag{51}$$

here and further the mapping $(\beta\widetilde{K}_\tau)^0$ is an identity operator.

We define the majorant mapping $\Xi_K(\varepsilon,\theta) : C_+^1[a,b] \to L_+^1[a,b]$ analogously to the previous problem by the equality

$$\Xi_K(\varepsilon,\theta)(x) = \varphi_K(\varepsilon,\theta) + \psi_K M(x),$$

where the operator $M : C^1_+[a,b] \to C^1_+[a,b]$ is specified by the relation (36), and the functions $\varphi_K(\varepsilon,\theta)$, $\psi_K \in L^1_+[a,b]$ have the form

$$\varphi_K(\varepsilon,\theta)(t) = \sum_{i=0}^{m-1} (\beta\widetilde{K_\tau})^i(\varepsilon+\theta)(t), \psi_K(t) = \sum_{i=0}^{m-1} (\beta\widetilde{K_\tau})^i(\alpha)(t). \tag{52}$$

Thus, from Theorem 2 we deduce

Corollary 4. *Let $\tau \in (0, b-a)$, and a function $f : [a,b] \to \mathbb{R}^1$ satisfy the conditions formulated above. Let the mapping $F : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \operatorname{comp}[\mathbb{R}^n]$ possess property $\mathcal{F}$, and the kernel $K : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ generating the operator $K_\tau : L^n[a,b] \to L^n[a,b]$ possess property $\mathcal{K}$. Then for every function $q \in D^n[a,b]$ and an arbitrary $\varepsilon > 0$ there exists a solution $x \in D^n[a,b]$ of the problem* (45)*, so that the estimates*

$$Z(x-q) \leqslant \xi_K(\varepsilon,\varkappa), \quad Z(\dot{x}-\dot{q}) \leqslant \varphi_K(\varepsilon,\varkappa) + \psi_K\xi_K(\varepsilon,\varkappa) \tag{53}$$

are valid. Here the function $\varkappa \in L^1_+[a,b]$ satisfies the inequality (40) *where $\widetilde{S}_f\dot{q} \equiv \widetilde{K_\tau}\dot{q}$; the function $\xi_K(\varepsilon,\varkappa)$ is specified by the equality* (39) *at $\theta = \varkappa$ and $\nu = |q(a)-x_0|$, where $\xi_S(\varepsilon,\varkappa) \equiv \xi_K(\varepsilon,\varkappa)$, $\psi_S \equiv \psi_K$, $\varphi_S(\varepsilon,\varkappa) \equiv \varphi_K(\varepsilon,\varkappa)$ for which the functions ψ_K, $\varphi_K(\varepsilon,\varkappa)$ are defined by the relation* (52)*.*

Let the mapping $\Gamma : [a,b]\times C^n[a,b]\times L^n[a,b] \to \operatorname{comp}[\mathbb{R}^n]$ possess the property: for all $x \in C^n[a,b]$, $y \in L^n[a,b]$ the mapping $\Gamma(\cdot,x,y)$ is measurable and satisfies the equality

$$\Phi(x,y) = \{z \in L^n[a,b] : z(t) \in \Gamma(t,x,y) \text{ for almost all } t \in [a,b]\}.$$

Let us note that such mapping does exist (see [13, 20]).

Theorem 4. *Let the mapping $\Phi : C^n[a,b]\times L^n[a,b] \to \Pi[L^n[a,b]]$ possess properties $\mathcal{C}$, $\mathcal{D}^*$ and $\mathcal{E}$. Then the equation is correct $\overline{H} = H_{\rm co}$, where $\overline{H}$ is a closure of the set H in the space $C^n[a,b]$.*

Proof. It is evident that $H \subset H_{\rm co}$. Since according to the theorem 3 the set $H_{\rm co}$ is closed then $\overline{H} \subset H_{\rm co}$. Further, we shall prove that $H_{\rm co} \subset \overline{H}$. Let $z \in H_{\rm co}$. Let us consider the Cauchy problem for a functional-differential inclusion in the space $C^n[a,b]$

$$\dot{x} \in \Phi(x,\dot{z}), \quad x(a) = x_0.$$

According to [17] for z, there exists such a sequence of absolutely continuous functions $y_i : [a,b] \to \mathbb{R}^n$, $i = 1,2,\dots$, that $y_i \to z$ in $C^n[a,b]$ at $i \to \infty$, $\dot{y}_i \to z$ weakly in $L^n[a,b]$, for every $i = 1,2,\dots$ the inclusion $\dot{y}_i \in \Phi(z,\dot{z})$ and the equality $y_i(a) = x_0$ are fulfilled. As the mapping $\Phi(\cdot,\cdot)$ possesses property $\mathcal{C}$, then according to the theorem 1 for every $\dot{y}_i \in \Phi(z,\dot{z})$, $i = 1,2,\dots$ there exists such $x_i \in H$, $i = 1,2,\dots$, that for every measurable set $\mathcal{U} \subset [a,b]$ the relation

$$\|\dot{y}_i - \dot{x}_i\|_{L(\mathcal{U})} \leqslant \frac{1}{i}\mu(\mathcal{U}) + \rho_{L(\mathcal{U})}[\dot{y}_i; \Phi(x_i,\dot{x}_i)], \quad i = 1,2,\dots, \tag{54}$$

is valid. From estimate (54) for every $i = 1, 2, \dots$ and every measurable set $\mathcal{U} \subset [a, b]$ we derive the inequality

$$\|\dot{y}_i - \dot{x}_i\|_{L(\mathcal{U})} \leqslant \frac{1}{i}\mu(\mathcal{U}) + \rho_{L(\mathcal{U})}[\dot{y}_i; \Phi(y_i, \dot{y}_i)] + h_{L(\mathcal{U})}[\Phi(x_i, \dot{x}_i), \Phi(y_i, \dot{y}_i)].$$

Therefore, according to property $\mathcal{D}^*$ the inequality

$$\begin{aligned}\|\dot{y}_i - \dot{x}_i\|_{L(\mathcal{U})} &\leqslant \frac{1}{i}\mu(\mathcal{U}) + \rho_{L(\mathcal{U})}[\dot{y}_i; \Phi(y_i, \dot{y}_i)] \\ &\quad + \|T^1(Z(x_i - y_i), Z(\dot{x}_i - \dot{y}_i))\|_{L(\mathcal{U})}\end{aligned} \tag{55}$$

holds for every $i = 1, 2, \dots$ and every measurable set $\mathcal{U} \subset [a, b]$. Let the integrable function $\omega_i : [a, b] \to [0, \infty)$ be defined by the equality

$$\omega_i(t) = \frac{1}{i} + \rho[\dot{y}_i(t), \Gamma(t, y_i, \dot{y}_i)] \tag{56}$$

for every $i = 1, 2, \dots$ As inequality (55) is valid for every measurable set $\mathcal{U} \subset [a, b]$, the relation

$$Z(\dot{x}_i - \dot{y}_i) \leqslant \omega_i + T^1(Z(x_i - y_i), Z(\dot{x}_i - \dot{y}_i)) \tag{57}$$

holds for every $i = 1, 2, \dots$.

Let us note that for every $i = 1, 2, \dots$ the function ω_i, defined by the equality (56), satisfies the estimate

$$\|\omega_i\|_{L[a,b]} \leqslant \frac{b-a}{i} + h^+_{L[a,b]}[\Phi(z, \dot{z}), \Phi(y_i, \dot{y}_i)].$$

Therefore, from the condition $\mathcal{E}$ we derive that $\omega_i \to 0$ in the space $L^1[a, b]$ as $i \to \infty$.

For every $i = 1, 2, \dots$ consider an equation

$$z = V(\omega_i) + V(Q_1(T^1(P_2(z), 0))), \tag{58}$$

in the cone $C^1_+[a, a + 2\tau]$, where $V : L^1_+[a, b] \to C^1_+[a, b]$ is an operator of integration and the continuous mappings $P_2 : C^1_+[a, a + 2\tau] \to C^1_+[a, b]$ and $Q_1 : L^1_+[a, b] \to L^1_+[a, b]$ are defined by equalities (2) at $\tau \equiv a+2\tau$ and $\tau \equiv a+\tau$, respectively. As the superposition $Q_1(T^1(P_2(\cdot), 0))$ is a Volterra weakly compact operator and $\omega_i \to 0$ in $L[a, b]$ as $i \to \infty$, according to [21, 22] for every $i = 1, 2, \dots$ for equation (58) there exists an interval $[a, \nu_i) \subset [a, a + 2\tau]$ such that all solutions of equation (58) are defined on it (i.e. every local solution of problem (58), defined on the segment $[a, \tau] \subset [a, \nu_i)$ can be extended to the segment $[a, \nu_i)$). Since, according to the condition $\mathcal{D}^*$, equation (58) at $\omega_i \equiv 0$ has only a zero solution on the segment $[a, a + 2\tau]$, by [21, 22] we have $\nu_i \to a + 2\tau$ as $i \to \infty$. Therefore, starting with a certain i_0 the relation $\nu_i > a + \tau$ is fulfilled for every$i \geqslant i_0$. Further, without loss of generality we assume that for every $i = 1, 2, \dots$ there exists an estimate $\nu_i > a + \tau$. Therefore, as the superposition $V(Q_1(T^1(P_2(\cdot), 0)))$ is isotonic and according to [17] for every $i = 1, 2, \dots$ equation (58) has an upper solution ξ_i defined on the segment $[a, a + \tau]$. From the latter estimate we deduce that $\xi_i \to 0$ in the space $C^1[a, a+\tau]$ as $i \to \infty$. As according to Lemma 1 for every $i = 1, 2, \ldots$ the inequalities

$$Z(x_i - y_i) \leqslant \xi_i, \quad Z(\dot{x}_i - \dot{y}_i) \leqslant \dot{\xi}_i, \tag{59}$$

are fulfilled on $[a, a+\tau]$, it follows from the first estimate (59) that

$$\lim_{i\to\infty} \|y_i - x_i\|_{C[a,a+\tau]} = 0.$$

Further, for every $i = 1, 2, \ldots$ we consider inequalities (57) on $[a, a+2\tau]$. As the operator $T^1(\cdot,\cdot)$ is isotonic and τ-Volterra, inequalities (59) take the form of

$$Z(\dot{x}_i - \dot{y}_i) \leqslant \omega_i + T^1(Z(x_i - y_i), Q_1(\dot{\xi}_i))$$

on the segment $[a, a+2\tau]$. Consider an equation

$$y = V\omega_i + V(Q_2(T^1(P_3(y), Q_1(\dot{\xi})))), \tag{60}$$

for every $i = 1, 2, \ldots$ in the cone $C^1_+[a, a+3\tau]$, where the continuous mappings $P_3 : C^1_+[a, a+3\tau] \to C^1_+[a, b]$ and $Q_2 : L^1_+[a, b] \to L^1_+[a, b]$ are defined by the equalities (2) at $\tau = a+3\tau$ and $\tau = a+2\tau$, respectively. Let the mapping $\Sigma_i : C^1_+[a, a+3\tau] \to C^1_+[a, a+3\tau]$ be defined by equality (60) for every $i = 1, 2, \ldots$. Further, we prove that for every bounded set $U \subset C^1_+[a, a+3\tau]$ the set $\bigcup\limits_{i=1}^{\infty} \Sigma_i(U)$ is relatively compact in $C^1[a, a+3\tau]$. Indeed, let $l \geqslant 0$ so that for every $p \in U$ the inequality $\|p\|_{C[a,a+3\tau]} \leqslant l$ is fulfilled. Without loss of generality we assume that $i = 1, 2, \ldots$ $\|y_i\|_{C[a,b]} \leqslant l$ and $\|\xi_i\|_{C[a,a+\tau]} \leqslant l$. Further, let the function $\eta \in L^1_+[a, b]$ be such that for all $y \in \Phi(z, \dot{z})$ the estimate $Z(y) \leqslant \eta$ is fulfilled in the cone $L^1_+[a, b]$. Then for every $i = 1, 2, \ldots$ the relation

$$\omega_i \leqslant 1 + \eta + T(l, \eta) \tag{61}$$

follows from property $\mathcal{C}$ and the definition of the function ω_i (see (56)). Therefore for every $i = 1, 2, \ldots$ on $[a, a+\tau]$ we obtain the relation

$$\dot{\xi}_i \leqslant 1 + \eta + T(l, \eta) + T^1(l, (1 + \eta + T(l, \eta))). \tag{62}$$

From inequalities (61) and (62) and the definition of the mapping $\Sigma_i(\cdot)$, $i = 1, 2, \ldots$, it follows that the set $\bigcup\limits_{i=1}^{\infty} \Sigma_i(U)$ is relatively compact in the space $C^1[a, a+3\tau]$. Thus, according to [21, 22] an upper solution ξ_i, $i = 1, 2, \ldots$, defined on the segment $[a, a+\tau]$ can be extended to the segment $[a, a+2\tau]$. To be brief, we denote this extended upper solution by ξ_i, $i = 1, 2, \ldots$, as well. For these solutions inequalities (62) are fulfilled already on $[a, a+2\tau]$. Proceeding in a similar way, it is possible to extend these upper solutions to the whole segment $[a, b]$. We also denote these solutions by ξ_i, $i = 1, 2, \ldots$. Inequalities (62) are fulfilled for these upper solutions on the whole segment $[a, b]$. As $\xi_i \to 0$ in $C^1[a, b]$ at $i \to \infty$, we have $\lim\limits_{i\to\infty} \|x_i - y_i\|_{C[a,b]} = 0$. Hence, $x_i \to z$ in the space $C^n[a, b]$ as $i \to \infty$. $\square$

Remark 3. If the mapping $\Phi : C^n[a,b] \times L^n[a,b] \to \Pi[L^n[a,b]]$ possesses property $\mathcal{D}^*$ and at $\nu = 0$ problem (24) has an upper solution in a vicinity of the point $(0,0) \in [0,\infty) \times L^1_+[a,b]$, then Theorem 4 follows directly from Theorem 2.

Remark 4. If the equality $\overline{H} = H_{\text{co}}$ is fulfilled for a differential inclusion then it is sometimes said (see [10]) that for a differential inclusion the density principle is fulfilled. The density principle is the fundamental property in the theory of differential inclusions as this property is an necessary and sufficient condition for the stability of solutions in relation to internal and external disturbances (see [10, 23, 24, 25]). So Theorem 4 determines sufficient conditions at which the density principle is fulfilled for problem (1).

From Theorem 4 for problem (45) we obtain

Corollary 5. *Let $\tau \in (0, b-a)$ and a measurable function $f : [a,b] \to \mathbb{R}^1$ satisfy the condition formulated above. Let the mapping $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \text{comp}[\mathbb{R}^n]$ possess property $\mathcal{F}$, and the kernel $K : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$, generating the operator $K_\tau : L^n[a,b] \to L^n[a,b]$ possess property $\mathcal{K}$. Then the equality $\overline{\mathcal{H}} = \mathcal{H}_{\text{co}}$ is valid, where $\overline{\mathcal{H}}$ is a closure in the space $C^n[a,b]$ of the set $\mathcal{H}$.*

Remark 5. Let us note that Corollary 5 contains Filippov's theorem (see [6, 7]).

Remark 6. Theorems 1–4 and their corollaries are extensions of the results obtained in [26, 27] for differential inclusions of neutral type. Moreover, they specify and complement the results of [28].

Acknowledgements

The authors are grateful to E. S. Zhukovskii for discussions of the results of the paper and helpful hints that contributed to the improvement.

The researches are supported by the Russian Fund of Basic Research, Grant No. 01-01-00140.

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(Received 3.08.2001)

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