Georgian Mathematical Journal
Volume 9 (2002), Number 1, 83–112

MACKEY CONTINUITY OF CHARACTERISTIC FUNCTIONALS

S. KWAPIEŃ AND V. TARIELADZE

Dedicated to N. Vakhania
on the occassion of his 70th birthday

Abstract. Problems of the Mackey-continuity of characteristic functionals and the localization of linear kernels of Radon probability measures in locally convex spaces are investigated. First the class of spaces is described, for which the continuity takes place. Then it is shown that in a non-complete sigma-compact inner product space, as well as in a non-complete sigma-compact metizable nuclear space, there may exist a Radon probability measure having a non-continuous characteristic functional in the Mackey topology and a linear kernel not contained in the initial space. Similar problems for moment forms and higher order kernels are also touched upon. Finally, a new proof of the result due to Chr. Borell is given, which asserts that any Gaussian Radon measure on an arbitrary Hausdorff locally convex space has the Mackey-continuous characteristic functional.

2000 Mathematics Subject Classification: Primary: 60B11. Secondary: 28C20, 60B15.
Key words and phrases: Radon probability measure, characteristic functional, kernel of a measure, weak topology, Mackey topology, presupport, Gaussian measure, covariance operator.

1. Introduction

Recall that for a Hausdorff locally convex space X the Mackey topology $\tau(X^*, X)$ is the topology in its topological dual X^* of uniform convergence on all weakly compact absolutely convex subsets of X. The aim of this paper is to answer the following three equivalent questions:

Question 1. Is the characteristic functional $\hat{\mu}$ of a Radon probability measure μ in a Hausdorff locally convex space X continuous in the Mackey topology $\tau(X^*, X)$?

Question 2. Is a Radon probability measure μ in a Hausdorff locally convex space X scalarly concentrated on the weakly compact absolutely convex subsets of X?

Question 3. Is for a Radon probability measure μ in a Hausdorff locally convex space X the initial space X a weak presupport of μ?

ISSN 1072-947X / \$8.00 /

A negative answer will provide a similar answer to the following related question (cf., Proposition 3.7 below):

Question 4. Is the (linear) kernel $\mathcal{H}_\mu$ of a Radon probability measure μ in a Hausdorff locally convex space contained in the initial space X?

It is standard to see that for a convex-tight probability measure μ the characteristic functional $\hat{\mu}$ is Mackey-continuous and $\mathcal{H}_\mu \subset X$. Therefore a negative answer to the above questions would imply a similar answer also to the next one:

Question 5. Is any Radon probability measure μ in a Hausdorff locally convex space X convex-tight?

The fact that the latter question has a negative answer is probably known. However it remains an open problem whether any Gaussian Radon measure in an arbitrary Hausdorff locally convex space is convex-tight (cf. [18, p. 457, Problem 3], [29] see also [1, p. 111, Corollary 3.4.3], where a positive solution of the problem is stated for a sequentially complete locally convex space). Nevertheless Chr. Borell [3, Theorem 2.1] showed that for any Gaussian Radon measure μ in any Hausdorff locally convex space we have that $\hat{\mu}$ is Mackey-continuous and $\mathcal{H}_\mu \subset X$ (for symmetric Gaussian Radon measures a similar assertion is stated also in [2, p. 359]). Borell reduced the problem to complete locally convex spaces (i.e., to the case in which all Radon probability measures are convex-tight) and then, using a precise description of the set of all admissible translates of a Gaussian Radon measure in the complete case, showed that the kernel is located in the initial space. The discussed result with Borell's original proof is already included in the recent books devoted to the Gaussian measures (see, e.g., [1, p. 101, Th. 3.2.3]). In Section 5, we give a "completion free" proof of Borell's theorem.

What happens in the case of non-Gaussian Radon probability measures? An answer to this question seems not to have appeared in the literature so far.

The paper is organized as follows:

In Section 2, the terminology is fixed, and the known, but not very easily available, facts from the theory of topological vector spaces are collected. It contains Theorem 2.10 which is a particular case of Valdivia's closed graph theorem and which improves essentially the related statement of [36].

In Section 3, all notions are defined concerning measures in topological vector spaces, which are used in this section without explanation. All known positive results about the continuity and localization of kernels are formulated. Proposition 3.7 clarifies the relationship between Question 1 and Question 4.

In Section 4, we prove that even in a sigma-compact inner product (or metrizable nuclear) space X a Radon probability measure μ may exist for which the kernel is not contained in the initial space and hence its characteristic functional is not Mackey-continuous (see Corollary 4.6 and Corollary 4.7). The proof is based on the result of S. Mazur [19]. In either of the cases the constructed measure μ has additional good properties: it is scalarly non-degenerate, X^* is locally convex with respect to μ-convergence and $\mathcal{H}_\mu$ contains X. This gives rise to the following natural problem which remains open: find an example of the Radon probability measure μ in a normed space X with similar good properties, whose kernel has a trivial intersection with X.

In Section 5, we deal with p-th order measures and kernels. It is shown that whenever $p > 1$, for a weak p-th order probability measure μ in a locally complete space the Mackey continuity of $\hat{\mu}$ implies the Mackey continuity of its p-th moment form. Previously, a similar statement was known for complete spaces [36]. The section is concluded by our proof of Borell's theorem.

2. Preliminaries

Let E be a vector space over the field $\mathbb{R}$ of real numbers. We denote by E^a the algebraic dual space of E. Recall that E^a consists of all linear functionals given on E and is a vector space over $\mathbb{R}$.

For nonempty $A, B \subset E$ and $t \in \mathbb{R}$ we write $A + B := \{a + b : a \in A,\ b \in B\}$ and $tA := \{tx : x \in A\}$.

A subset B of a vector space is called a *disk* if it is absolutely convex. Let B be a disk; then $E_B := \bigcup_{n=1}^{\infty} nB$ is a vector subspace and the Minkowski functional p_B of B is a seminorm on E_B (which is a norm if and only if the only vector subspace of E contained in B is $\{0\}$).

If B is a disk such that p_B is a norm on E_B and the obtained normed space (E_B, p_B) is complete, then B is called a *Banach disk*.

A Hausdorff topological vector space X is called:

- *complete* if any Cauchy net consisting of elements of X is convergent.
- *quasi-complete* if for each closed bounded $B \subset X$ any Cauchy net consisting of elements of B is convergent.
- *von Neumann complete* if for each closed precompact $B \subset X$ any Cauchy net consisting of elements of B is convergent.
- *sequentially complete* if any Cauchy sequence consisting of elements of X is convergent.
- *locally complete* if for each closed bounded disk $B \subset X$ is a Banach disk.

Remark 2.1. (1) We have that 'complete' $\Rightarrow$' quasi-complete' $\Rightarrow$' von Neumann-complete' $\Rightarrow$ 'sequentially complete' $\Rightarrow$ 'locally complete' and, in the case of a metrizable locally convex space, all of the notions are equivalent. It is known that in general none of these implications can be reversed.

(2) [10, Theorem 1] Let X be a locally complete space, G be a metrizable topological vector space, $G_0 \subset G$ be a dense vector subspace and $v_o : G_0 \to X$ be a continuous linear operator. Then there exists a continuous linear operator $v : G \to X$ such that $v|_{G_0} = v_o$. We shall use this property of locally complete spaces at the end of this section.

For a fixed topology $\mathcal{T}$ in a vector space E we denote by $(E, \mathcal{T})^*$ the set of all $\mathcal{T}$-continuous linear functionals given on E and by $(E, \mathcal{T})^s$ the set of all *sequentially* $\mathcal{T}$-continuous linear functionals given on E. Clearly, $(E, \mathcal{T})^* \subset (E, \mathcal{T})^s \subset E^a$.

For a topological vector space X we denote X^* the topological dual space of X. A generic element of X^* will be denoted by x^*. In X^* no topology will be fixed in advance and we shall always specify the topology under consideration.

A normed space $(X, \|\cdot\|_X)$ will be viewed as a topological vector space (with respect to its norm topology) and $B_X := \{x \in X : \|x\|_X \leq 1\}$. We denote by $\|\cdot\|_{X^*}$ the dual norm in X^*.

A topological vector space X is called *Mazur* if $X^* = X^s$ (see [38, Definition 8.6.3]).

We say that a topological vector space X is *a dually separated space* if X^* separates the points of X. Clearly, X is dually separated if and only if for any $x \in X$, $x \neq 0$, there exists $x^* \in X^*$ such that $x^*(x) \neq 0$.

Let us also say that a topological vector space X is *an essentially dually separated space* if $X/cl(\{0\})$ is a dually separated space (here $cl(\{0\})$ is the closure of the one-element set $\{0\}$ into X).

We have: X is essentially dually separated if and only if for any $x \in X \setminus cl(\{0\})$ there exists $x^* \in X^*$ such that $x^*(x) \neq 0$.

Thanks to the Hahn–Banach theorem, any Hausdorff locally convex space is dually separated and any locally convex space is essentially dually separated.

A pair (E, F), where E is a vector space and $F \subset E^a$, is called *duality*. A duality (E, F) is called *separated* if F separates the points of E. Plainly, if X is a topological vector space, then (X, X^*) is a duality which is separated if and only if X is a dually separated space.

For a duality (E, F) and non-empty $A \subset E$, $B \subset F$ we write

$$A^\circ := \{f \in F : |f(x)| \leq 1, \ \forall x \in A\}, \ \ B^\circ := \{x \in E : |f(x)| \leq 1, \ \forall f \in B\}.$$

For a duality (E, F) the notations $\sigma(E, F)$ and $\sigma(F, E)$ have their usual meaning, i.e., $\sigma(E, F)$ is the topology in E generated by F, and $\sigma(F, E)$ the topology of pointwise convergence in F.

We shall use the *bipolar theorem* in the following form: for a duality (E, F) and a non-empty $A \subset E$, the bipolar $(A^\circ)^\circ$ of A equals to the $\sigma(E, F)$-closure of the absolutely convex hull of A [14, Theorem 8.2.2 (p. 149)].

For a topological vector space X the topology $\sigma(X, X^*)$ is called the weak topology of X and $\sigma(X^*, X)$ is called the weak* topology of X^*. We note that if X is an infinite-dimensional locally bounded dually separated space, then $(X, \sigma(X, X^*))$ is not metrizable and hence $\sigma(X, X^*)$ is strictly coarser than the original topology of X.

Remark 2.2. Let X be a (not necessarily Hausdorff) topological vector space. The following facts concerning weak topologies are supposed to be known:

(1) If U is a neighborhood of zero in X, then U° is $\sigma(X^*, X)$-compact.

(2) If X is locally convex and A is a closed convex subset of X, then A is weakly closed.

(3) If X is locally convex and A is a weakly bounded subset of X, then A is bounded in X.

(4) If X is locally convex, then X is locally complete if and only if $(X, \sigma(X, X^*))$ is locally complete (this follows from (2) and (3)).

(5) If $X := (c_0, \sigma(c_0, l_1))$, then X is locally complete, but is not sequentially complete (the local completeness of X follows from (4) since c_0 is complete).

In what follows, for a separated duality (E, F) *we shall always identify the vector space* E *with its canonical image into* F^a *(i.e., any* $x \in E$ *will be identified with the linear functional* $f \to f(x)$*) (and in this way* (F, E) *will also be treated as a duality).*

For a duality (E, F) we have $(E, \sigma(E, F))^* = F$, and hence, for a separated duality (E, F), we have $(F, \sigma(F, E))^* = E$ (note that the equality $(E, \sigma(E, F))^* = F$ needs an easy proof, while for a topological vector space X the equality $(X, \sigma(X, X^*))^* = X^*$ is evident).

Let (E, F) be a duality and $\mathfrak{S}$ be a non-empty family of weakly (i.e., $\sigma(E, F)$-) bounded subsets of E such that $\varnothing \notin \mathfrak{S}$. In what follows $\mathfrak{S}(F, E)$ will denote the topology in F of uniform convergence on the members of $\mathfrak{S}$ and $F_{\mathfrak{S}} := (F, \mathfrak{S}(F, E))$. For a separated duality (E, F) and a non-empty family $\mathfrak{S}'$ of weakly (i.e., $\sigma(F, E)$-) bounded subsets of F such that $\varnothing \notin \mathfrak{S}'$ the notations $\mathfrak{S}'(E, F)$ and $E_{\mathfrak{S}'}$ will have a similar meaning.

Let us say that the family $\mathfrak{S}$ is *polar* [38, Definition 5.5.1] if it is directed upward with respect to the set-theoretic inclusion and has also the property: for each $K \in \mathfrak{S}$ there exists $A \in \mathfrak{S}$ such that $2K \subset A$.

It is easy to see that if $\mathfrak{S}$ is a polar family, then the family $\{K^\circ : K \in \mathfrak{S}\}$ is a fundamenthal system of neighborhoods of zero for the topology $\mathfrak{S}(F, E)$.

For a general separated duality (E, F) the following topologies have a special name and notation:

- the *Mackey topology*, $\tau(F, E) := \mathfrak{S}(F, E)$, where $\mathfrak{S}$ stands for the family of all $\sigma(E, F)$-compact absolutely convex subsets of E,
- the *strong topology* $\beta(F, E) := \mathfrak{S}(F, E)$, where $\mathfrak{S}$ stands for the family of all $\sigma(E, F)$-bounded subsets of E.

In what follows, for a Hausdorff topological vector space X in the topological dual space X^* we shall also deal with:

- the *compact-open topology* $k(X^*, X) := \mathfrak{S}(X^*, X)$, where $\mathfrak{S}$ stands for the family of all compact subsets of X and
- the topology $kc(X^*, X) := \mathfrak{S}(X^*, X)$, where $\mathfrak{S}$ stands for the family of all compact absolutely convex subsets of X.

We have the following relationship between these topologies:

$$\sigma(X^*, X) \subset kc(X^*, X) \subset \tau(X^*, X) \subset \beta(X^*, X)$$

and

$$\sigma(X^*, X) \subset kc(X^*, X) \subset k(X^*, X) \subset \beta(X^*, X).$$

We shall see in the next section that the questions from the Introduction make sense because the equality $kc(X^*, X) = k(X^*, X)$ does not hold always or the inclusion $k(X^*, X) \subset \tau(X^*, X)$ does not always take place. To clarify this point in a bit more detail, let us say following [22] (or [38, Definition 9-2-8] that X has the *convex compactness property* (or, briefly, the ccp) if the closed absolutely convex hull of each compact subset of X is compact.

Remark 2.3. Let X be a Hausdorff locally convex space. Then:

(a) The equality $kc(X^*, X) = k(X^*, X)$ holds if and only if X has the ccp (this is evident and true for a dually separated X too).

(b) The inclusion $k(X^*, X) \subset \tau(X^*, X)$ holds if and only if X has the ccp. (Assume that $A_1 \subset X$ is compact. Since $k(X^*, X) \subset \tau(X^*, X)$, there is a weakly compact absolutely convex $A_2 \subset X$ such that $A_1 \subset A_2$. Denote by B_1 the closed absolutely convex hull of A_1. Since X is *locally convex*, B_1 is precompact and weakly closed. Hence $B_1 \subset A_2$. Consequently, B_1 is precompact and weakly compact and by [6, Ch.4,§1, Proposition 3] is compact.)

(c) [6, Ch. 2, §4, Prop. 3, Corollary] If X is von Neumann complete, then X has the ccp (for Banach spaces this statement is due to S.Mazur; a sequentially complete Hausdorff locally convex space needs not to have the ccp [22, Example 2.1]).

(d) (Krein's theorem, [6, Ch.4,§5, Theorem 3], cf. [38, Prob. 14-2-301]) If X is quasi-complete, then $(X, \sigma(X, X^*))$ has the ccp.

(e) If $(X, \sigma(X, X^*))$ has the ccp, then X has the ccp (see [38, Th. 9-2-11] for a stronger statement).

(e$'$) (D. H. Fremlin, [38, Prob. 14-6-110]) X may have the ccp, but $(X, \sigma(X, X^*))$ may not have it (cf. [22, Question (1)]).

(f) [10, Theorem 1, $(b) \Leftrightarrow (f)$] (cf., [14, Theorem 10.3.4, (1) $\Leftrightarrow$ (3)]) The closed absolutely convex hull of any null-sequence of elements of X is compact if and only if X is locally complete.

(f$'$) If X has the ccp, then X is locally complete (this follows from (f); the ccp may not imply the sequential completeness: let $Y = (c_0, \sigma(c_0, l_1))$, then Y has the ccp (by (d)), but is not sequentially complete).

(g) If X is metrizable and has the ccp, then X is complete (this is not hard to prove, see, e.g., [12, Ch.2, Sec.13, Exercise 3 (c) (p. 72)] or [22, Th.2.3], but a separate formulation of this statement makes sense because it provides many examples of spaces failing to have the ccp).

Let (E, F) be a duality and $\mathfrak{T}$ be a (not necessarily vector or locally convex) topology in E. We shall say that $\mathfrak{T}$ is *compatible* with the duality (E, F) if $(E, \mathfrak{T})^* = F$. Similarly, we shall say that $\mathfrak{T}$ is *subcompatible* with the duality (E, F) if $(E, \mathfrak{T})^* \subset F$.

As we noted above, $(E, \sigma(E, F))^* = F$; hence the topology $\sigma(E, F)$ is compatible with the duality (E, F). Clearly, if X is a topological vector space, then its original topology is compatible with the duality (X, X^*)

By the *Mackey–Arens theorem* for a given duality (E, F) the Mackey topology $\tau(E, F)$ is the finest locally convex vector topology compatible with the duality (E, F).[1] It follows that a locally convex vector topology $\mathfrak{T}$ given in E

[1] The existence of the finest locally convex vector topology in E compatible with (E, F) is a direct consequence of the analytic form of the Hahn–Banach theorem, while its identification with $\tau(E, F)$ needs extra work based on the geometric form of Hahn Banach theorem (cf.

is compatible with the duality (E, F) if and only if $\sigma(E, F) \subset \mathfrak{T} \subset \tau(E, F)$.

Since we shall apply these notions and results mainly in the "dual" situation, let us give their "dual" formulations.

For a separated duality (E, F) a topology $\mathfrak{T}$ given in F is called *compatible* with the duality (E, F) if $(F, \mathfrak{T})^* = E$ and $\mathfrak{T}$ is called *subcompatible* with the duality (E, F) if $(F, \mathfrak{T})^* \subset E$.

The "dual" formulation of the *Mackey–Arens theorem* says that for a given *separated* duality (E, F) the Mackey topology $\tau(F, E)$ is the finest locally convex vector topology compatible with the duality (E, F). Consequently, a locally convex vector topology $\mathfrak{T}$ given in F is compatible with the duality (E, F) if and only if $\sigma(F, E) \subset \mathfrak{T} \subset \tau(F, E)$.

Lemma 2.4. *Let X be a dually separated topological vector space and $\mathfrak{T}$ be a topology in X^*. We have:*

(a) *If $\mathfrak{T} \subset \tau(X^*, X)$, then $\mathfrak{T}$ is subcompatible with the duality (X, X^*).*
(b) *If $\mathfrak{T}$ is a locally convex vector topology and is subcompatible with the duality (X, X^*), then $\mathfrak{T} \subset \tau(X^*, X)$.*

Proof. (a) We have: $(X^*, \mathfrak{T})^* \subset (X^*, \tau(X^*, X))^* = X$.

(b) Let $Y := (X^*, \mathfrak{T})$. Take any neighbourhood of zero B_1 in Y. We need to find a $\tau(X^*, X)$-neighborhood of zero A such that $A \subset B_1$. Since Y is a locally convex space, there is a closed and, hence, $\sigma(Y, Y^*)$-closed, absolutely convex neighborhood of zero B in Y such that $B \subset B_1$. Then B° is compact in $\sigma(Y^*, Y)$. As $\mathfrak{T}$ is subcompatible, we get that $B^\circ \subset Y^* \subset X$. Hence B° is $\sigma(X, X^*)$-compact. Then $A := (B^\circ)^\circ$ is a $\tau(X^*, X)$-neighbourhood of zero and is the $\sigma(X^*, X)$-closure of B. Since $\sigma(Y, Y^*) \subset \sigma(X^*, X)$ (as $Y^* \subset X$) and B is $\sigma(Y, Y^*)$-closed, we get that $A \subset B$. $\square$

Remark 2.5. For a dually separated space X we have:

(1) $kc(X^*, X)$ is compatible with the duality (X, X^*) (this follows directly from the Mackey–Arens theorem as $\sigma(X^*, X) \subset kc(X^*, X) \subset \tau(X^*, X)$).
(2) If X is locally convex, then the topology $k(X^*, X)$ is compatible with the duality (X, X^*) if and only if X has the ccp (the "if" part follows from (1), while the "only if" part follows from the Mackey–Arens theorem and Remark 2.3(b)).
(3) If X is locally convex and the topology $\beta(X^*, X)$ is compatible with the duality (X, X^*), then X called semireflexive. If X is locally convex, then it is semireflexive if and only if the bounded subsets of X are weakly relatively compact [6, Ch. IV, §2, Theorem 1]. Note that if X is a semireflexive space such that $X^* \neq X^a$, then $(X, \sigma(X, X^*))$ provides an example of a quasi-complete locally convex space which is not complete.

[38, Theorems 8.2.14, 9.2.3]). In general, there may not exist the finest vector topology in E compatible with the duality (E, F) (see [15], cf. [7, Prop. 2.1(b,c)], which asserts that if X is a metrizable infinite-dimensional topological vector space and $\mathfrak{M}$ is the least upper bound of the family of all compatible vector topologies for (X, X^*), then $(X, \mathfrak{M})^* = X^a$).

(4) If X is metrizable non-complete and locally convex, then the topology $k(X^*, X)$ is not compatible with the duality (X, X^*), i.e., $(X^*_k)^* \neq X$ (this follows from (1) and Remark 2.3 (e)).

For dually separated X and fixed $\alpha \in \{\sigma, \tau, k, \beta\}$ it would be convenient to put $X^*_\alpha := (X^*, \alpha(X^*, X))$.

Recall that the space X^*_β is called the strong dual space of X and the vector space $X^{**} := (X^*_\beta)^*$ is the second dual space of X. A (locally convex) space is called reflexive if $\beta(X^*, X)$ is compatible with the duality (X, X^*) and $\beta(X, X^*)$ coincides with the original topology of X.

Remark 2.6. Let X be a Hausdorff locally convex space. Then:

(1) We have $(X^*_k)^* \subset X^{**}$. In general, this inclusion may be strict (see the next point). This is important to note because, as we shall see in Section 3, the kernel of a Radon probability measure in X is always a subset of $(X^*_k)^*$ and so it is not 'too far' from X.

(2) Suppose X is a dense proper vector subspace of a Frechet space F. Then $(X^*_k)^* = F$ [12, Ch. 2,§13, Exercise 4(b)(p. 72)]. Therefore if F is non-reflexive, then in (1) we have the strict inclusion (as $X^{**} = F^{**}$).

(3) If X is metrizable (or is a Mazur space), then X^*_k and X^*_β are complete locally convex spaces.

(4) If X is metrizable and is not normable, then X^*_β is not metrizable. This will be important in Section 5.

The rest of the material of this section will not be used until Section 5. We shall need the following known statement:

Proposition 2.7. *Let X be a dually separated space, F a locally convex space and $u : X^*_\tau \to (F, \sigma(F, F^*))$ a continuous linear mapping. Then u is continuous as a mapping from X^*_τ into F.*

Proof. Let V be a closed absolutely convex neighborhood in F. It is sufficient to show that $u^{-1}(V)$ is a neighborhood of zero in $(X^*, \tau(X^*, X))$, i.e., we need to find weakly compact absolutely convex $K \subset X$ such that $K^\circ \subset u^{-1}(V)$. Since $(X^*_\tau)^* = X$ (by the Mackey–Arens theorem), the dual operator u^* maps F^* into $(X^*_\tau)^* = X$ and u^* is continuous as a mapping from $(F^*, \sigma(F^*, F))$ into $(X, \sigma(X, X^*))$. Since V° is $\sigma(F^*, F)$-compact, we get that $K := u^*(V^\circ)$ is a weakly compact absolutely convex subset of X. Using the bipolar theorem, we get $K^\circ = u^{-1}(V)$. $\square$

Let X be a dually separated space and $R : X^* \to X$ be a linear operator. With R we can associate the bilinear form $k_R : X^* \times X^* \to \mathbb{R}$ and the quadratic form $k_R : X^* \to \mathbb{R}$ as follows:

$$k_R(x_1^*, x_2^*) := x_2^*(Rx_1^*), \;\; q_R(x_1) := x_1^*(Rx_1^*), \;\; x_1^*, x_2^* \in X^*.$$

If k_R is a symmetric bilinear form, then R is called symmetric, and if $q_R \geq 0$, then R is called positive.

The operator R is said to be *symmetrically Hilbertian* if there is a Hilbert space H and a weakly continuous linear operator $T : H \to X$ such that $TT^* = R$.

Proposition 2.8. *Let X be a dually separated space and $R : X^* \to X$ be a linear operator.*

(a) *R is symmetrically Hilbertian if and only if R is a symmetric positive operator such that q_R is continuous in the Mackey topology $\tau(X^*, X)$.*

(b) *If $(X, \sigma(X, X^*))$ is locally complete and R is a symmetric positive operator, then R is symmetrically Hilbertian.*

Proof. (a) The "only if" part. Let H be a Hilbert space and $T : H \to X$ be a weakly continuous linear operator such that $TT^* = R$. Then, clearly, R is symmetric positive and $q_R(x^*) = \|T^*x^*\|_H^2$, $\forall x^* \in X^*$. Since T is weakly continuous, $T^* : X^*_\sigma \to (H, \sigma(H, H))$ is continuous. Then by Proposition 2.7 the mapping $u := T^* : X^*_\tau \to H$ is continuous. Hence, q_R is continuous on X^*_τ.

The "if" part. Let H_0 be the inner product space associated with the pre-Hilbert space (X^*, k_R) and H be the completion of H_0. Denote by u the natural mapping of X^* into H. Then $q_R(x^*) = \|ux^*\|_H^2$, $\forall x^* \in X^*$. Since, by supposition, q_R is continuous on X^*_τ, we get that $u : X^*_\tau \to H$ is continuous. Consequently, the operator $T := u^* : H \to (X^*_\tau)^* = X$ is weakly continuous and $TT^* = R$.

(b) To use (a), we need to show that q_R is continuous on X^*_τ. Let H_0, H and u be as in the proof of the "if" part of (b). Consider the linear operator $v_o : H_0 \to X$ defined as follows:

$$v_o u x^* = R x^*, \ \forall x^* \in X^*.$$

Let us show that $v_o(B_{H_0})$ is bounded in $(X, \sigma(X, X^*))$. We have

$$\begin{aligned} |y^*(v_o u x^*)| &= |y^*(Rx^*)| = |k_R(x^*, y^*)| = |(ux^*|uy^*)_H| \\ &\le \|ux^*\|_H \cdot \|uy^*\|_H, \quad \forall x^*, y^* \in X^*. \end{aligned}$$

Consequently, $\{|t| : t \in y^*(v_o B_{H_0})\} \subset [0, \|uy^*\|]$ for any $y^* \in X^*$. Therefore $v_o(B_{H_0})$ is bounded in $(X, \sigma(X, X^*))$ and hence, $v_o : H_0 \to (X, \sigma(X, X^*))$ is continuous. Since $(X, \sigma(X, X^*))$ is locally complete and H_0 is dense in H, according to Remark 2.1(2), v_o admits a continuous extension $v : H \to (X, \sigma(X, X^*))$. This and the equality $y^*(v_o u x^*) = (ux^*|uy^*)_H$, $\forall x^*$, $y^* \in X^*$ give the relation

$$(h|uy^*)_H = y^*(vh), \ \forall y^* \in X^*, \ \forall h \in H,$$

which implies that $u : X^*_\tau \to (H, \sigma(H, H))$ is continuous. Now by Proposition 2.7 we get the continuity of $u : X^*_\tau \to H$ and the required continuity of $q_R : X^*_\tau \to \mathbb{R}_+$. □

Remark 2.9. (1) The part (b) of this proposition has a little history: in [25, Prop. 10 (§ 5, pp. 154–157)] it was proved in two different ways for a quasi-complete complex locally convex space (it is used in monograph [26, p. 338] without the proof for which the reader is referred to [25]). Later, this statement was rediscovered for Banach spaces and was named as "the factorization lemma" in [32]; in [34, Lemma 4 (p.13)] it is stated for a sequentially complete locally convex space.

(2) If X is a Hausdorff locally convex space and $R : X^* \to X$ is a symmetrically Hilbertian operator, then it is easy to see that for any two representations of R in the form $TT^* = R = T_1T_1^*$ we have $T(H) = T_1(H_1)$. It follows that the vector subspace $T(H)$ depends only on R. Put $H(R) := T(H)$. If we equip $H(R)$ with the Hilbert space structure transformed from H through T, we get the 'Hilbert subspace of X associated with the "kernel" R' in the sense of [25]. In general, $R(X^*)$ is a proper subset of $H(R)$.

The next result is a "dual" version of Valdivia's Closed Graph Theorem (see the remark following the proof).

Theorem 2.10. *Let X be a Hausdorff locally convex space which is locally complete, F be a* ***reflexive*** *Banach space, or any Hausdorff locally convex space such that F^*_τ is* ***metrizable****, $u : X^* \to F$ be a linear operator.*

If in F there is a ***Hausdorff*** *topology $\mathcal{T}$ coarser than the original topology of F such that $u : X^*_\tau \to (F, \mathcal{T})$ is continuous, then u is continuous as a mapping from X^*_τ to F.*

Proof. Let $E =: X^*_\tau$. Then $E^* = X$ (by the Mackey–Arens theorem).

Claim: $u : E \to (F, \sigma(F, F^*))$ is continuous. This will be shown in the 4th step.

(1) $u : E \to F$ has the closed graph. Indeed, since $u : E \to (F, \mathcal{T})$ is continuous and $\mathcal{T}$ a Hausdorff topology, the graph of u is closed in $E \times (F, \mathcal{T})$. Since $\mathcal{T}$ is coarser than the topology of F, we get that the graph of u is closed in $E \times F$.

(2) Consider the set $G := \{y^* \in F^* : y^* \circ u \text{ is continuous on } E\}$. Then G is a vector subspace of F^* and $y^* \circ u \in X$ for each $y^* \in G$ (as $E^* = X$). Since the graph of u is closed in $E \times F$, the set G separates the points of F (see [38, Lemma 12.5.2]).

(3) Consider the linear mapping $v_o : G \to X$ defined by the equality $v_o y^* = y^* \circ u$, $y^* \in G$. It is clear that v_o is continuous with respect to $\sigma(G, F)$ and $\sigma(X, X^*)$. Take $\sigma(G, F)$-bounded $B \subset G$, then $v_o(B)$ is bounded in $(X, \sigma(X, X^*))$, and hence is bounded in X too. It follows, in particular, that v_o transforms the $\tau(F^*, F)$-bounded subsets of G into the bounded subsets of X. Since $(G, \tau(F^*, F)|_G)$ is a metrizable topological vector space, this implies that $v_o : (G, \tau(F^*, F)|_G) \to X$ is continuous. Since, by item (2), G separates the points of F, G is dense in F^*_τ. Since X is locally complete and F^*_τ is metrizable, according to Remark 2.1(2) v_o admits a continuous extension $v : F^*_\tau \to X$.

(4) The equality $x^*(v_o y^*) = y^*(ux^*)$ is true for each $y^* \in G$ and any fixed $x^* \in X^*$. Since G is dense in F^*_τ and v is continuous, we get

$$x^*(vy^*) = y^*(ux^*) \quad \forall y^* \in F^*, \ \forall x^* \in X^*.$$

This relation implies that $u : X^*_\tau \to (F, \sigma(F, F^*))$ is continuous; so our claim is achieved and by Proposition 2.7 we get a conclusion. □

Remark 2.11. (1) The remarkable Closed Graph theorem of Valdivia [37, Theorem 2] is the following: let E be a Hausdorff locally convex space such that E^*_σ is locally complete, F be a Λ_r-space. If u is a linear mapping of E into F with closed graph, then u is weakly continuous. Since F in Theorem 2.10 is a Λ_r-space,[2] we could finish the proof after step (1). The proof is included for the sake of self-containedness and because it does not involve directly the notion of a Λ_r-space. Note also that in [14, Theorem 10.3.4, (1) $\Leftrightarrow$ (5)] Valdivia's theorem is proved for $F = l_p$, $1 < p < \infty$.

(2) The local completeness of X is necessary in Theorem 2.10: if X is not locally complete, then for any given infinite-dimensional Banach space F there exists a non-continuous linear operator $u : X^*_\tau \to F$ which is continuous as a mapping from X^*_τ into $(F, \sigma(F, G))$ for some separating vector subspace $G \subset F^*$[37, Theorem 1]

(3) In the case of Banach spaces, the reflexivity of F is really needed in Theorem 2.10: take a non-reflexive Banach space X and $F = (X^*, \beta(X^*, X))$. Then for $\mathcal{T} = \tau(X^*, X) \subset \beta(X^*, X)$, the identity mapping $u : X^*_\tau \to (F, \mathcal{T})$ is trivially continuous, but $u : X^*_\tau \to F$ is not continuous, because $\tau(X^*, X) \neq \beta(X^*, X)$.

(4) This theorem improves essentially [36, Th.1] (cf., also [35, Th. 6.5.1 (p. 414)]), in which a similar statement was proved with supposition that X is a *complete* Hausdorff locally convex space, $F = L_p$, $p > 1$, and $\mathcal{T}$ is the topology of convergence in measure.

(5) The given proof shows that Theorem 2.10 remains valid for a dually separated X for which $(X, \sigma(X, X^*))$ is locally complete.

3. Continuity and Linear Kernels

Let X be a Hausdorff topological vector space. Fix a non-empty family $\mathfrak{S}$ of subsets of X, an algebra $\mathcal{B}$ of subsets of X and a set function $\mu : \mathcal{B} \to \mathbb{R}_+ := [0, \infty[$. Then μ is called:

- $\mathfrak{S}$-*regular* if
$$\mu(B) = \sup\{\mu(K) : K \in \mathfrak{S} \cap \mathcal{B},\ K \subset B\},\ \ \forall B \in \mathcal{B}.$$
- $\mathfrak{S}$-*tight* if $\forall \varepsilon > 0$ $\exists K \in \mathfrak{S}$ such that
$$B \in \mathcal{B},\ B \cap K = \varnothing \Rightarrow \mu(B) < \varepsilon.$$
- *scalarly concentrated on* $\mathfrak{S}$ if $\{x \in X : |x^*(x)| \leq 1\} \in \mathcal{B}$, $\forall x^* \in X^*$, and for any $\varepsilon > 0$ there exists $K \in \mathfrak{S}$ such that
$$\mu\{x \in X : |x^*(x)| \leq 1\} > \mu(X) - \varepsilon,\ \ \forall x^* \in K^\circ.$$

A vector subspace $E \subset X$ is called:

[2] The fact that any reflexive Banach space is a Λ_r-space is noted in [37, p. 651]; while in [9, Theorem 2] it is asserted that a space F for which F^*_τ is metrizable has a stronger property of being a Σ_r-space.

- a $\mathfrak{S}$-*presupport* of μ if μ is scalarly concentrated on the members of $\mathfrak{S}$ contained in E.
- a *presupport* of μ if E is $\mathfrak{S}$-*presupport* of μ, where $\mathfrak{S}$ is the family of all compact absolutely convex subsets of X.
- a *weak presupport* of μ if E is $\mathfrak{S}$-*presupport* of μ, where $\mathfrak{S}$ is the family of all weakly compact absolutely convex subsets of X.[3]

In what follows the expression "μ is a measure in X" will mean that μ is a measure defined on a σ-algebra $\mathcal{B}$ of subsets of X, with respect to which all continuous linear functionals are measurable.

For given $x \in X$, δ_x will stand for the Dirac measure.

For a measure μ in X we shall put

$$\mathcal{N}_\mu := \{x^* \in X^* : x^* = 0 \ \ \mu\text{-a.e.}\}.$$

A finite non-negative Borell measure μ given on the Borell σ-algebra of X is called:

- *Radon* if is $\mathfrak{S}$-regular, where $\mathfrak{S}$ is the family of all compact subsets of X,
- *tight* if is $\mathfrak{S}$-tight, where $\mathfrak{S}$ is the family of all compact subsets of X,
- and *convex-tight* if is $\mathfrak{S}$-tight, where $\mathfrak{S}$ is the family of all compact absolutely convex subsets of X.

A probability measure μ in X is called:

- *scalarly non-degenerate* (or *full*) if $\mathcal{N}_\mu = \{0\}$.
- *topologically non-degenerate* if μ is Borell and $\mu(U) > 0$ for each non-empty open $U \subset X$.
- *discrete* if μ is Borell and there exists an at most countable subset X_0 of X such that $\mu(X_0) = 1$.
- *algebraically trivial* if μ is Borell and there exists a vector subspace X_0 of X with at most countable algebraic dimension, such that $\mu(X_0) = 1$.

Fix a finite positive measure μ in X. We denote by $\mathcal{T}_\mu$ the topology in X^* of convergence in measure μ. Then $\mathcal{T}_\mu$ is a pseudometrizable vector topology in X^*. It is known, e.g., that the (non-homogeneous) pseudonorm $\|\cdot\|_{o,\mu}$ defined by the equality

$$\|x^*\|_{o,\mu} = \inf\{r \in]0,1[: \mu\{x \in X : |x^*(x)| > r\} < r\}, \quad x^* \in X^*,$$

generates a translation-invariant pseudometric on X^*, which induces $\mathcal{T}_\mu$. In general, the topology $\mathcal{T}_\mu$ may be neither Hausdorff nor locally convex. It is a Hausdorff topology if and only if μ is a scalarly non-degenerate measure

For a finite measure μ in X, its *characteristic functional (Fourier transform)* $\hat{\mu} : X^* \longrightarrow \mathbb{C}$ is defined by the equality

$$\hat{\mu}(x^*) = \int_X \exp\{ix^*(x)\} d\mu(x), \quad x^* \in X^*.$$

[3]The notions of scalar concentration and a (weak) presupport will not be really needed in the sequel, their definitions are recalled for the justification of the equivalence of the three questions from the Introduction; see [8] and [27] for more information.

Remark 3.1. (1) The fact that the correspondence between probability measures in a finite-dimensional space and characteristic functionals is injective (i.e., the uniqueness theorem for the Fourier thansform) was probably established for first time rigorously by P. Levy. This readily implies that the same is true for measures given on the cylindrical σ-algebra of an infinite-dimensional space. The uniqueness theorem remains valid for Radon measures in arbitrary Hausdorff locally convex spaces (Yu. V. Prokhorov, see [35]).

(2) The usefulness of characteristic functionals in a finite-dimemsional case is as a rule motivated by Levy's continuity theorem: a sequence (μ_n) of probability measures given in a finite-dimensional X converges weakly to a probability measure μ in X if and only if the sequence $(\hat{\mu}_n)$ converges pointwise to $\hat{\mu}$. Since the sufficiency part of this theorem fails for infinite-dimensional normed spaces, there is an opinion that characteristic functionals are less useful in the infinite-dimensional case. However, they remain important as the only available tool for the description of measures.

(3) The Bochner theorem asserts that for a finite-dimensional X a functional $\chi : X^* \to \mathbb{C}$ is the characteristic functional of a finite positive measure in X if and only χ is continuous and positive definite. The sufficiency part of this theorem also fails for an infinite-dimensional Banach space X if under the continuity is understood the continuity in the norm topology of X^*. However it is well-known that, e.g., in the case of a Hilbert space a characteristic functional of a probability measure is always continuous in a topology much coarser than the norm topology and this topology allows us to save the Bochner theorem for Hilbert spaces (Sazonov's theorem; we refer to [20], [26] and [35] for more detailed information and comments). This discussion shows that the study of the continuity properties of characteristic functionals is related in a certain sense with the Bochner theorem.

(4) The fact that the Bochner theorem in its usual formulation is not true for general infinite-dimensional spaces, gave rise to the theory of cylindrical mesures and Radonifying operators, for which characteristic functionals are not at a glance so important (but without them the exposition even of the theory of Gaussian measures seems rather complicated, see, e.g., the corresponding part in [26]).

Proposition 3.2. *Let X be a topological vector space and μ be a probability measure in X. Then:*

(a) *$\hat{\mu}$ is continuous on $(X^*, \mathcal{T}_\mu)$.*

(a′) *If $\mathfrak{T}$ is a vector topology in X^*, then $\hat{\mu}$ is (sequentially) continuous on $(X^*, \mathfrak{T})$ if and only if $\|\cdot\|_{o,\mu}$ is (sequentially) continuous on $(X^*, \mathfrak{T})$.*

(a″) *$\mathcal{T}_\mu$ is the coarsest topology among the vector topologies in X^* with respect to which $\hat{\mu}$ continuous.*

Proof. (a) follows from the easily established inequality

$$|\widehat{\mu}(x^*) - \widehat{\mu}(y^*)| \leq 3\|x^* - y^*\|_{o,\mu} \quad (x^*, y^* \in X^*).$$

(a′) The "if" part follows from (a), the "only if" part needs proving [11, Th. 2.1], see, also [26, Th. 1, p. 193] or [35, Prop. 4.3.4, p. 231].

(a″) follows from (a′). □

Proposition 3.3. *Let X be a topological vector space, $\mathfrak{S}$ be a family of weakly bounded subsets of X and μ be a probability measure in X. Then:*

(a) *If μ is a $\mathfrak{S}$-tight, then $\hat{\mu}$ and $\|\cdot\|_{o,\mu}$ are continuous on $X^*_{\mathfrak{S}}$.*

(b) *If $\mathfrak{S}$ is a polar family, then $\hat{\mu}$ or, $\|\cdot\|_{o,\mu}$, is continuous on $X^*_{\mathfrak{S}}$ if and only if μ is scalarly concentrated on $\mathfrak{S}$.*

Proof. The proof is standard. □

Proposition 3.4. *Let X be a Hausdorff topological vector space and μ be a probability measure in X. Then:*

(a) *$\hat{\mu}$ and $\|\cdot\|_{o,\mu}$ are* ***sequentially continuous*** *on X^*_σ.*

(b) *If μ is tight, then $\|\cdot\|_{o,\mu}$ and $\hat{\mu}$ are continuous on X^*_k.*

(b′) *If μ is convex-tight, then $\|\cdot\|_{o,\mu}$ and $\hat{\mu}$ are continuous on X^*_{kc} and hence are Mackey-continuous too.*

(c) *If X has the ccp and μ is a tight, in paticular, a Radon measure, then $\|\cdot\|_{o,\mu}$ and $\hat{\mu}$ are continuous on X^*_{kc} and hence are Mackey-continuous too.*

(c′) *If X is a von Neumann complete* (*in particular, complete or quasi-complete*) *locally convex space and μ is Radon, then $\|\cdot\|_{o,\mu}$ and $\hat{\mu}$ are continuous on X^*_{kc} and hence are Mackey-continuous too.*

(d) *$\hat{\mu}$, resp. $\|\cdot\|_{o,\mu}$, is continuous on X^*_τ if and only if μ is scalarly concentrated on the weakly compact absolutely compact subsets of X.*

(d′) *$\hat{\mu}$, resp. $\|\cdot\|_{o,\mu}$, is continuous on X^*_τ if and only if X is a weak presupport of μ.*

Proof. (a) is a consequence of the countable additivity of μ via the Lebesgue theorem about the convergence of integrals (which is true in particular for any pointwise convergent, unformly bounded *sequence* of measurable functions). (b) and (b′) follow from Proposition 3.3(a).

(c) follows from (b′).

(c′) follows from (c) since any Hausdorff von Neumann complete locally convex space has the ccp.

(d) follows from Proposition 3.3(b).

(d′) follows from (d) and the definition of the weak presupport. □

Remark 3.5. (1) In (a) the term "sequentially" cannot be omitted in general, as for a Radon probability measure μ in the dually separated X the *continuity* of $\hat{\mu}$ on X^*_σ implies that μ is algebraically trivial [11, Th. 3.1] (cf., also [35, Th. 6.3.3, p.400]).

(2) In general, (a) is not a consequence of (b) since the continuity of a functional X^*_k may not imply its sequential continuity on X^*_σ.

(3) In the case of an arbitrary X the Mackey continuity of $\hat{\mu}$ cannot be derived from (b) because of Remark 2.3(b, e).

(4) The statements (d) and (d′) imply that the first three questions stated in the Introduction are equivalent.

(5) Let us say for a moment that a dually separated space X has the *Measure Mackey Continuity Property* or, briefly, the MMCP, if the Fourier transform $\hat{\mu}$ of any Radon probability measure μ in X is continuous in the Mackey topology $\tau(X^*, X)$. From Proposition 3.4(c) we get that the ccp implies the MMCP. However, the MMCP may not imply the ccp: if X is any dually separated space with countable algebraic dimension, then X has the MMCP (by Proposition 3.4(b′))), but has no ccp in the metrizable case (see Remark 2.3). It follows that any space which has no MMCP must not have the ccp and must have an uncountable algebraic dimension.

(6) Let us say, again for a moment, that a dually separated space X has the *Measure Luzin Property* or, briefly, the MLP, if any Radon probability measure in X is convex-tight. Then from Proposition 3.4(b) we get that the MLP implies the MMCP. Whether the converse is true we do not know.

To discuss Question 4, let us recall first the corresponding notion.

The *kernel* of a measure μ in a dually separated space X is defined as the topological dual space $\mathcal{H}_\mu$ of $(X^*, \mathcal{T}_\mu)$, i.e., $\mathcal{H}_\mu := (X^*, \mathcal{T}_\mu)^*$.

Observe that since $\mathcal{T}_\mu$ is a pseudometrizable vector topology in X^*, we can say that a functional $f \in X^{*a}$ belongs to $\mathcal{H}_\mu$ if and only if for an arbitrary sequence (x_n^*) in X^* which converges to zero in measure μ, we have $\lim_{n\to\infty} f(x_n^*) = 0$. Also, it is easy to see that a functional $f \in X^{*a}$ belongs to $\mathcal{H}_\mu$ if and only if for an arbitrary sequence (x_n^*) in X^* which converges to zero μ-a.e., we have $\lim_{n\to\infty} f(x_n^*) = 0$.

Remark 3.6. Let μ be a probability measure in a dually separated space X.

(1) From the definition we have $\mathcal{H}_\mu \subset X^{*a}$ and the validity of the inclusion $\mathcal{H}_\mu \subset X$ means precisely that the topology $\mathcal{T}_\mu$ is subcompatible with the duality (X, X^*).

(2) In [33] the set $\mathcal{K}_\mu := \mathcal{H}_\mu \cap X$ is called the initial kernel of μ. It is clear that the set A_μ of all admissible translates of μ is a subset of $\mathcal{K}_\mu$, which in general may be proper.

(3) Denote by $[X^*]_\mu$ the Hausdorff topological vector space associated with $(X^*, \mathcal{T}_\mu)$. Then we can identify $\mathcal{H}_\mu$ with $[X^*]_\mu^*$.

(4) In general, even for a scalarly non-degenerate μ one may have that $\mathcal{H}_\mu = \{0\}$ (see, e.g., Lemma 4.2(d) below). If $(X^*, \mathcal{T}_\mu)$ is essentially dually separated (and $[X^*]_\mu$ is non-trivial), then $\mathcal{H}_\mu$ is sufficiently “rich”. However, it can be “very rich” only in exceptional cases, e.g., if X is a separable Frechet space and μ is scalarly non-degenerate, then $\mu(\mathcal{K}_\mu) = 1$ if and only if $(X^*, \mathcal{T}_\mu)$ is a nuclear locally convex space [16] (see also [33], where the possibility of the equality $\mathcal{K}_\mu = X$ is studied). We refer to [8, 13, 16, 23, 27, 28] for more information and results concerning kernels.

(5) For a (non-Radon) Gaussian μ the inclusion $\mathcal{H}_\mu \subset X$ may not hold. In fact, in [11] an example is given of a (countably additive) probability measure μ on the cylindrical σ-algebra of a Hausdorff locally convex space X with the

property: $\hat{\mu} = \exp(if)$ with $f \in X^{*a} \setminus X$. Clearly, for this μ the kernel $\mathcal{H}_\mu$ is the one-dimensional vector subspace of X^{*a} generated by f and hence $\mathcal{H}_\mu \cap X = \{0\}$. The fact that μ does not admit a Radon extension can be shown directly (cf., Proposition 5.8(b) below).

The problem of "localization" of $\mathcal{H}_\mu$ inside X and the problem of the Mackey-continuity of $\hat{\mu}$ are related as follows.

Proposition 3.7. *Let X be a dually separated space and μ be a probability measure on X. We have:*

(a) *If $\hat{\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$, then $\mathcal{H}_\mu \subset X$.*

(a′) *If $\mathcal{H}_\mu \subset X$ and $\mathcal{T}_\mu$* ***is a locally convex topology****, then $\hat{\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$.*

Proof. (a) By Proposition 3.2(a′) we have $\mathcal{T}_\mu \subset \tau(X^*, X)$. Hence, by Lemma 2.4(a), $\mathcal{T}_\mu$ is subcompatible with the duality (X, X^*) and so $\mathcal{H}_\mu \subset X$.

(a′) Since, by supposition, $\mathcal{T}_\mu$ is subcompatible with the duality (X, X^*) and is locally convex, by Lemma 2.4(b) we get that $\mathcal{T}_\mu \subset \tau(X^*, X)$. This and Proposition 3.2(a) imply that $\hat{\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$. □

The following statement is a "kernel" version of Proposition 3.4.

Proposition 3.8. *Let X be a dually separated topological vector space and μ be a probability measure in X. Then:*

(a) $\mathcal{H}_\mu \subset (X^*, \sigma(X^*, X))^s$.

(a′) *If X is a complete separable locally convex space, then $\mathcal{H}_\mu \subset X$.*

(a″) *If X is a complete separable locally convex space and $\mathcal{T}_\mu$ is locally convex, then $\hat{\mu}$ is Mackey-continuous.*

(b) *If μ is a tight, in particular a Radon, measure, then $\mathcal{H}_\mu \subset (X_k^*)^* \subset X^{**}$.*

(c) *If μ is convex-tight, then $\mathcal{H}_\mu \subset X$.*

(d) *If X has the ccp and μ is a tight, in particular a Radon, measure, then $\mathcal{H}_\mu \subset X$.*

(d′) *If X a von Neumann complete (in particular, complete or quasi-complete) locally convex space and μ is a Radon measure, then $\mathcal{H}_\mu \subset X$.*

Proof. (a) follows from the fact that since μ is countably additive, any pointwise convergent *sequence* of measurable functions converges also in measure μ.

(a′) follows from (a) and Grothendieck's theorem, which implies that under our suppositions, one has $(X^*, \sigma(X^*, X))^s = X$ [14, Corollary 9.2.3 (p.176)].

(a″) follows from (a′) via Proposition 3.7(a′).

(b) By Proposition 3.4(b) the tightness of μ implies $\mathcal{T}_\mu \subset k(X^*, X)$. Hence $\mathcal{H}_\mu = (X^*, \mathcal{T}_\mu)^* \subset (X^*, k(X^*, X))^*$.

(c) By Proposition 3.4(b) the convex-tightness of μ implies $\mathcal{T}_\mu \subset kc(X^*, X)$. Hence $\mathcal{H}_\mu = (X^*, \mathcal{T}_\mu)^* \subset (X^*, kc(X^*, X))^*$. Since $kc(X^*, X)$ is compatible with (X, X^*), we get $\mathcal{H}_\mu \subset X$.

(d) follows from (c). □

Remark 3.9. (1) The statement (a$''$) is not covered by Proposition 3.4.

(2) The statement (b) provides a "location" place for the kernel (see Remark 2.6$(1,2)$).

(3) In the case of non-complete metrizable locally convex X the validity of the inclusion $\mathcal{H}_\mu \subset X$ cannot be derived from (b) because of Remark 2.5(4).

We see that in many important cases the kernel is located in the initial space and the characteristic functional is continuous in the Mackey topology. As we have noted in the Introduction, the same is true for any Gaussian Radon measure in any Hausdorff locally convex space. In the next section we shall show that not for all Radon probability measures the kernel is contained in the initial space and not always the Mackey-continuity of characteristic functionals takes place.

4. Radon Measures with Badly Located Kernels

As usual, $\mathbb{R}^{\mathbb{N}}$ will denote the space of all real sequences $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ with the product topology and $\mathbb{R}_0^{\mathbb{N}}$ will stand for the vector space of all real eventually zero sequences.

For $1 \le p \le \infty$, l_p will denote the classical Banach space of real p-summable sequences with its usual norm and topology. The natural inner product of the Hilbert space l_2 will be denoted by $(\cdot|\cdot)$.

We need one slightly less classical sequence space. To introduce it, fix a strictly increasing sequence $\mathbf{m} := (m_j)_{j\in\mathbb{N}}$ of natural number, and put

$$\Lambda(\mathbf{m}) := \{\mathbf{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{j=1}^{\infty} |y_j| \cdot t^{m_j} < \infty, \ \ \forall t \in]0,1[.$$

Clearly, $\Lambda(\mathbf{m})$ is a vector subspace of $\mathbb{R}^{\mathbb{N}}$.[4] We shall equipp $\Lambda(\mathbf{m})$ with the locally convex topology by means of norms

$$\mathbf{y} = (y_j)_{j\in\mathbb{N}} \to \sum_{j=1}^{\infty} |y_j| \cdot t^{m_j}, \ t \in]0,1[.$$

The obtained locally convex space is called a *power-series space of (finite) type* 1. It is known that $\Lambda(\mathbf{m})$ is a nuclear Frechet space [14, Prop. 20.6.3].

Let us also put

$$\Lambda^{\times}(\mathbf{m}) := \{\mathbf{z} = (z_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sup_{j\in\mathbb{N}}(|z_j| \cdot t^{-m_j}) < \infty, \text{ for some } t \in]0,1[\}.$$

Clearly, $\Lambda^{\times}(\mathbf{m})$ is also a vector subspace of $\mathbb{R}^{\mathbb{N}}$.[5]

For any real sequences $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ and $\mathbf{y} = (y_j)_{j\in\mathbb{N}}$ such that the series $\sum_j x_j y_j$ is convergent, we write

$$< \mathbf{x}, \mathbf{y} > := \sum_{j=1}^{\infty} x_j y_j.$$

[4] By using the "root test" we get $\Lambda(\mathbf{m}) = \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : \limsup_{j\to\infty} |y_j|^{1/m_j} \le 1\}$.

[5] We have $\Lambda^{\times}(\mathbf{m}) = \{\mathbf{z} \in \mathbb{R}^{\mathbb{N}} : \limsup_{j\to\infty} |z_j|^{1/m_j} < 1\}$.

In view of the equalities (where, as usual, $p' := \frac{p}{p-1}$, $1 < p < \infty$, $1' := \infty$))

$$(\mathbb{R}^{\mathbb{N}})^* = \{< \cdot, \mathbf{y} >: \mathbf{y} \in \mathbb{R}_0^{\mathbb{N}}\},$$
$$(l_p)^* = \{< \cdot, \mathbf{y} >: \mathbf{y} \in l_{p'}\}, \quad 1 \le p < \infty$$

and

$$(\Lambda(\mathbf{m}))^* = \{< \cdot, \mathbf{y} >: \mathbf{y} \in \Lambda^\times(\mathbf{m})\}$$

the topological dual spaces of the considered spaces will be identified with the corresponding sequence spaces.

Let us associate with the sequence $\mathbf{m} := (m_j)_{j\in\mathbb{N}}$ the mapping $\mathbf{f_m} : [0, 1[\to \mathbb{R}^{\mathbb{N}}$ defined by the equality

$$\mathbf{f_m}(t) = (t^{m_1}, t^{m_2}, \dots), \quad t \in [0, 1[, \tag{4.1}$$

and denote by $E_{\mathbf{m}}$ the vector subspace of $\mathbb{R}^{\mathbb{N}}$ generated by $\mathbf{f_m}([0, 1[)$.

Clearly, we have $E_{\mathbf{m}} \subset \Lambda^\times(\mathbf{m})$.

The introduced notation will be fixed during the discussion in this section.

Lemma 4.1.
(a) *We have* $\mathbf{f_m}([0, 1[) \subset l_1$; *moreover,* $\mathbf{f_m} : [0, 1[\to l_1$ *is a continuous mapping.*
(b) $E_{\mathbf{m}}$ *is a dense sigma-compact vector subspace of* l_1.
(c) $E_{\mathbf{m}} \cap \mathbb{R}_0^{\mathbb{N}} = \{0\}$.

Proof. All statements are easy to check and we leave this to the reader as an excercise. □

In what follows λ will be the Lebesgue measure in $[0, 1[$.

Lemma 4.2. *Let F be any space $\mathbb{R}^{\mathbb{N}}$, l_p, $1 \le p < \infty$, with their usual topologies, and μ be the image of the Lebesgue measure λ under the mapping* $\mathbf{f_m} : [0, 1[\to F$. *Then:*
(a) *μ is a scalarly non-degenerate Radon probability measure in F.*
(b) $\mu(\mathbf{f_m}([0, 1[)) = 1$ *and* $\mu(E_{\mathbf{m}}) = 1$.
(c) *$\mathcal{T}_\mu$ in F^* is a Hausdorff topology.*
(d) [27] *If* $\sum_{j=1}^{\infty} \frac{1}{m_j} = \infty$, *then* $\mathcal{H}_\mu = \{0\}$.

Proof. (a) μ is Radon for general reasons (in this case this follows also from the fact that λ is a Radon probability measure in $[0, 1[$ and from the continuity of $\mathbf{f_m} : [0, 1[\to F$). Let now $\mathbf{y} = (y_j)_{j\in\mathbb{N}}$ be such that $< \cdot, \mathbf{y} >\in F^*$ and $< \cdot, \mathbf{y} >= 0$ μ-a.e. Then $< \mathbf{f_m}(\cdot), \mathbf{y} >= 0$ λ-a.e. From this since $< \mathbf{f_m}(\cdot), \mathbf{y} >$ is continuous, we get

$$\sum_{j=1}^{\infty} t^{m_j} y_j = 0 \quad \forall t \in [0, 1[.$$

This relation, as it is well-known, implies that $\mathbf{y} = 0$ and so μ is scalarly non-degenerate.

(b) The first equality is evident, hence we have the second equality too (note that, thanks to the continuity of $\mathbf{f_m}$, the sets $\mathbf{f_m}([0, 1[)$ and $E_{\mathbf{m}}$ are sigma-compact subsets of F and hence are in the domain of μ).

(c) $\mathcal{T}_\mu$ is Hausdorff by (a).

(d) Since $\sum_{j=1}^{\infty} \frac{1}{m_j} = \infty$, according to Muntz's theorem (see, e.g., [21, p. 42]), the vector subspace $S = \{< \mathbf{f_m}(\cdot), \mathbf{y} >: \mathbf{y} \in \mathbb{R}_0^{\mathbb{N}}\}$ is dense in $\mathbf{L}_2([0,1[,\lambda)$. This implies that it is dense in $\mathbf{L}_0([0,1[,\lambda)$ too. From this and from the known fact that $(\mathbf{L}_0([0,1[,\lambda))^* = \{0\}$, we get a conclusion. □

Now we shall show that a special choice of $\mathbf{m} := (m_j)_{j\in\mathbb{N}}$ will allow us to get measures with "too rich" kernels. First let us introduce the "functional realization" of $\Lambda(\mathbf{m})$. Denote for a fixed $\mathbf{y} = (y_j)_{j\in\mathbb{N}} \in \Lambda(\mathbf{m})$ by $f_\mathbf{y}$ the function defined by the equality:

$$f_\mathbf{y}(t) = \sum_{j=1}^{\infty} y_j \cdot t^{m_j}, \ t \in [0,1[.$$

The needed examples can be obtained by the next statement.

Theorem 4.3 ([19]). [6] *Let the sequence* $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ *satisfy the condition*

$$m_{j+1}/m_j > 5, \quad j = 2, 3, \dots, \tag{4.2}$$

and let $(\mathbf{y}_n)_{n\in\mathbb{N}}$ *be a sequence of elements of* $\mathbb{R}_0^{\mathbb{N}}$ *such that the corresponding sequence of functions* $(f_{\mathbf{y}_n})_{n\in\mathbb{N}}$ *converges* λ*-almost everywhere on the interval* $[0,1[$. *Then:*

(a) *For each* $j \in \mathbb{N}$ *the sequence* $(y_{n,j})_{n\in\mathbb{N}}$ *converges to a limit* $y_j \in \mathbb{R}$.

(b) $\mathbf{y} := (y_j)_{j\in\mathbb{N}} \in \Lambda(\mathbf{m})$.

(c) *For each* $0 < r < 1$ *the sequence* $(f_{\mathbf{y}_n})_{n\in\mathbb{N}}$ *converges uniformly to* $f_\mathbf{y}$ *on the interval* $[0,r]$.

Corollary 4.4. *Let* $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ *be a sequence which satisfies condition* (4.2) *and*

$$\mathbb{A}(\mathbf{m}) := \{f_\mathbf{y} : \mathbf{y} \in \Lambda(\mathbf{m})\}.$$

Then:

(a) *The set* $\mathbb{A}(\mathbf{m})$ *is a closed vector subspace of the space* $\mathbf{C}([0,1[,\mathbb{R})$ *of continuous functions with respect to the topology* τ_c *of uniform convergence on compact subsets of* $[0,1[$.

(b) *The mapping* $u : \Lambda(\mathbf{m}) \to \mathbb{A}(\mathbf{m})$ *defined by the equality:* $u\mathbf{y} = f_\mathbf{y}$, $\mathbf{y} \in \Lambda(\mathbf{m})$ *is a linear homeomorphism between* $\Lambda(\mathbf{m})$ *and* $(\mathbb{A}(\mathbf{m}), \tau_c|_{\mathbb{A}(\mathbf{m})})$.

(c) *In* $\mathbb{A}(\mathbf{m})$ *the topology of convergence in measure* λ *coincides with the compact open topology.*

Proof. (a) follows directly from Theorem 4.3.

(b) It is clear that u is bijective and continuous. By (a) we have that $(\mathbb{A}(\mathbf{m}), \tau_c|_{\mathbb{A}(\mathbf{m})})$ is a Frechet space. Then by the open mapping theorem, u is a homeomorphism.

(c) follows directly from Theorem 4.3. □

[6] This theorem remains true under the supposition $\sum_{j=1}^{\infty} \frac{1}{m_j} < \infty$; this follows from [4, Theorem 6.2.1].

Proposition 4.5. *Let F and μ be as in Lemma* 4.2 *and the sequence* $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ *satisfy condition* (4.2)*. Then:*

(c$'$) *$\mathcal{T}_\mu$ in F^* coincides with the topology induced from $\Lambda(\mathbf{m})$. In particular, $(F^*, \mathcal{T}_\mu)$ is a metrizable nuclear locally convex space.*

(c$''$) *$\mathcal{H}_\mu = \Lambda^\times(\mathbf{m})$.*

In particular, $E_\mathbf{m} + \mathbb{R}^\mathbb{N}_0 \subset \mathcal{H}_\mu$ and $E_\mathbf{m} \neq \mathcal{H}_\mu$.

Proof. (c$'$) Since F^* is one of the spaces $\mathbb{R}^\mathbb{N}_0$, $l_{p'}$, $1 \le p < \infty$, we have $F^* \subset \Lambda(\mathbf{m})$. So our assumption makes sense. Clearly, it is sufficient to show that for any sequence $(\mathbf{y}_n)$ of elements of F^* which converges to zero μ-a.e., we have that it converges to zero in $\Lambda(\mathbf{m})$. So, fix $(\mathbf{y}_n)$ that converges to zero μ-a.e. Put $C := \{\mathbf{x} \in F : \lim_n <\mathbf{x}, \mathbf{y}_n>= 0.\}$ Then $\mu(C) = 1$. From this and Lemma 4.2 (b) we get $\mu(C \cap \mathbf{f_m}([0, 1[)) = 1$. Let $C' := C \cap \mathbf{f_m}([0, 1[)$ and $A := \mathbf{f}^{-1}_\mathbf{m}(C')$. Clearly, $\lambda(A) = 1$. Then

$$f_{\mathbf{y}_n}(t) =<\mathbf{f_m}(t), \mathbf{y}_n> \to 0 \ \ \forall t \in A.$$

Consequently, the sequence $(f_{\mathbf{y}_n})$ tends to zero λ-a.e. From this, according to Corollary 4.4(c), we conclude that the sequence $(f_{\mathbf{y}_n})$ converges to zero in $(\mathbb{A}(\mathbf{m}), \tau_c|_{\mathbb{A}(\mathbf{m})})$. Consequently, by Corollary 4.4(b), we get that $(\mathbf{y}_n)$ converges to zero in $\Lambda(\mathbf{m})$ and (c$'$) is proved.

(c$''$) follows from (c$'$) since F^* is dense in $\Lambda(\mathbf{m})$ and $(\Lambda(\mathbf{m}))^* = \Lambda^\times(\mathbf{m})$. □

Now we can formulate the results announced in the Abstract.

Corollary 4.6. *There exist a non-complete sigma-compact metrizable nuclear locally convex space X and a Radon probability measure ν in X such that:*

(1) *X^* is a metrizable nuclear locally convex space with respect to the topology $\mathcal{T}_\nu$ of convergence in measure ν.*

(2) *X is contained in the kernel $\mathcal{H}_\nu$ of ν, i.e., $X \subset \mathcal{H}_\nu$.*

(3) *$\mathcal{H}_\nu \not\subset X$, i.e., the topology $\mathcal{T}_\nu$ is not subcompatible with the duality (X, X^*).*

(4) *$\hat{\nu}$ is not continuous in the Mackey topology $\tau(X^*, X)$.*

Proof. Let $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ be a sequence which satisfies the condition $m_{j+1}/m_j > 5$, $j = 2, 3, \ldots$, (plainly, such a sequence exists) and let X be the vector space $E_\mathbf{m}$ with the topology induced by $F = \mathbb{R}^\mathbb{N}$. Then X is a metrizable nuclear sigma-compact locally convex space. Since $\mu(E_\mathbf{m}) = 1$, the restriction ν of μ on the Borell σ-algebra of X is a Radon probability measure in X.

(1) follows from Proposition 4.5(c$'$) since X is dense in F and the restriction mapping $x^* \to x^*|_X$ is a linear homeomorphism between $(F^*, \mathcal{T}_\mu)$ and $(X^*, \mathcal{T}_\nu)$.

(2) and (3) follow from Proposition 4.5(c'').

(4) follows from (3) via Proposition 3.7(a). □

Corollary 4.7. *There exist a non-complete sigma-compact inner product space X and a Radon probability measure ν in X such that:*

(1) *X^* is a metrizable nuclear locally convex space with respect to the topology $\mathcal{T}_\nu$ of convergence in measure ν.*

(2) *X is contained in the kernel $\mathcal{H}_\nu$ of μ, i.e., $X \subset \mathcal{H}_\nu$.*

(3) *$\mathcal{H}_\nu \not\subset X$, i.e., the topology $\mathcal{T}_\nu$ is not subcompatible with the duality (X, X^*).*

(4) *$\hat{\nu}$ is not continuous in the Mackey topology $\tau(X^*, X)$.*

Proof. Let $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ be a sequence which satisfies the condition $m_{j+1}/m_j > 5$, $j = 2, 3, \dots$, and let X be the vector space $E_{\mathbf{m}}$ equipped with the inner product induced by $F = l_2$. Then X is a sigma-compact inner product space. Since $\mu(E_{\mathbf{m}}) = 1$, the restriction ν of μ on the Borell σ-algebra of X is a Radon probability measure in X. The needed properties of ν can be proved as in the case of Corollary 4.6. □

Remark 4.8. (1) The above corollaries provide the announced in the Introduction negative answer to the five questions posed there. They show that not any metrizable nuclear locally convex space, resp., not any inner product space, has the MMCP (in the sense of Remark 3.5).

(2) In view of the property (2) of the measure ν in Corollary 4.6 (as well as in Corollary 4.7) and Dudley's example mentioned in Remark 3.6(5) it would be interesting to produce an example of a *Radon* probability measure μ in a Hausdorff locally convex space X such that $\mathcal{T}_\mu$ is a Hausdorff locally convex topology, but $\mathcal{H}_\mu \cap X = \{0\}$.

5. Mackey-Continuity of Moment Forms and Kernels of Gaussian Measures

In this section X will be a dually separated topological vector space and the measures will be supposed to be given on a σ-algebra $\mathcal{B}$ of subsets of X with respect to which all continuous linear functionals are measurable and which is invariant under translations. Let $0 < p < \infty$ and μ be a probability measure in X which has the weak p-th order, i.e., $X^* \subset \mathcal{L}_p(X, \mu)$.[7]

The functional $\psi_{p,\mu} : X^* \to \mathbb{R}_+$ defined by the equality

$$\psi_{p,\mu}(x^*) = \int_X |x^*(x)|^p d\mu(x), \quad x^* \in X^*,$$

is called the p-th *moment form* of μ.

[7] Although the fundamentals of the theory of Lebesgue integration for vector valued functions (which from the probabilistic point of view is equivalent to the study of measures in vector spaces) were already developed in the 30s of the 20th century by the works of S. Bochner, N. Dunford, I. M. Gelfand, A. N. Kolmogorov, B. J. Pettis and others, the importance and handiness of weak p-th order measures for the infinite-dimensional probability theory were emphasized in the works of N. Vakhania a relatively short time ago. In particular, the notion of a covariance operator of a weak second order probability measure in a Banach space first appeared only in [31], while earlier the notion was in common use for strong second order probability measures in a Hilbert space. Note also that this notion in fact already was used in [30], where the first complete description of Gaussian measures was given for "non-Hilbertian" Banach spaces l_p, $1 \le p < \infty$, and the first norm-integrability results were obtained for them.

Evidently, if $0 < p < 1$, then $\psi_{p,\mu}$ is an absolutely p-homogeneous pseudonorm and if $1 \le p < \infty$, then $(\psi_{p,\mu})^{1/p}$ is an absolutely homogeneous pseudonorm (i.e., is a seminorm).

Denote by $\mathcal{T}_{p,\mu}$ the topology in X^* induced by $\mathcal{L}_p(X,\mu)$. Then $\mathcal{T}_{p,\mu}$ is a pseudometrizable topology, which is finer than $\mathcal{T}_\mu$. Clearly, the pseudonorm $\psi_{p,\mu}^{\min(1,1/p)}$ generates $\mathcal{T}_{p,\mu}$.

We define the *p-kernel* $\mathcal{H}_{p,\mu}$ of μ as the topological dual space of $(X^*, \mathcal{T}_{p,\mu})$.

The following observation is immediate.

Lemma 5.1. *Let X be a dually separated space and μ be a probability measure on X. We have:*

(a) *If $0 < p < \infty$ and μ is of weak p-th order, then $\mathcal{T}_\mu \subset \mathcal{T}_{p,\mu}$ and so, $\mathcal{H}_\mu \subset \mathcal{H}_{p,\mu}$.*

(b) *If $0 < p < \infty$, μ is of weak p-th order and $\mathcal{T}_\mu = \mathcal{T}_{p,\mu}$, then $\mathcal{H}_\mu = \mathcal{H}_{p,\mu}$.*

(c) *If μ is a Gaussian measure, then $\mathcal{H}_\mu = \mathcal{H}_{p,\mu}$, $\forall p \in]0,\infty[$.*

(d) *If $0 < p < \infty$, μ is of weak p-th order and*

$$\exists r \in]0,p[, \quad \exists c \in]0,\infty[: \ \psi_{p,\mu}^{1/p} \le c\psi_{r,\mu}^{1/r}, \tag{5.1}$$

then $\mathcal{T}_\mu = \mathcal{T}_{p,\mu}$ and $\mathcal{H}_\mu = \mathcal{H}_{p,\mu}$.

Proof. (a) and (b) are evident. (c) follows from (b). According to [17, Corollary 0.2.1](5.1) implies that $\mathcal{T}_\mu = \mathcal{T}_{p,\mu}$. □

We have the following analog of Proposition 3.7 with the same proof.

Proposition 5.2. *Let X be a dually separated space and μ be a weak p-th order probability measure on X. We have:*

(a) *If $\psi_{p,\mu}$ is continuous in the Mackey topology $\tau(X^*,X)$, then $\mathcal{H}_{p,\mu} \subset X$.*

(a′) *If $\mathcal{H}_{p,\mu} \subset X$ and $\mathcal{T}_{\mu,p}$* ***is a locally convex topology*** *(this condition is superflous when $p \ge 1$), then $\psi_{p,\mu}$ is continuous in the Mackey topology $\tau(X^*,X)$.*

Proof. (a) The Mackey-continuity of $\psi_{p,\mu}$ implies $\mathcal{T}_{p,\mu} \subset \tau(X^*,X)$. Hence by Lemma 2.4(a) $\mathcal{T}_{p,\mu}$ is subcompatible with the duality (X,X^*) and so $\mathcal{H}_{p,\mu} \subset X$.

(a′) Since, by supposition, $\mathcal{T}_{p,\mu}$ is subcompatible with the duality (X,X^*) and is locally convex, by Lemma 2.4(b) we get that $\mathcal{T}_{p,\mu} \subset \tau(X^*,X)$. Consequently, $\psi_{p,\mu}$ is continuous in the Mackey topology $\tau(X^*,X)$. □

Remark 5.3. (1) The “higher order” versions of Question 1 and Question 4 from Introduction have an easy negative answer. Indeed, e.g., unlike the characteristic functional, the functional $\psi_{1,\mu}$ may not be continuous in the Mackey topology $\tau(X^*,X)$ even for a measure μ in a separable Banach space X: let $X = c_0$ and μ be the discrete measure defined by the equality $\mu = \sum_{j=1}^\infty 2^{-j}\delta_{2^j\mathbf{e}_j}$. Then we have $\psi_{1,\mu}(\mathbf{y}) = \|\mathbf{y}\|_{l_1}$, $\forall \mathbf{y} \in l_1 = X^*$, consequently, $\mathcal{H}_{1,\mu} = l_\infty \not\subset c_0$, while $\mathcal{H}_\mu = \mathbb{R}_0^{\mathbb{N}} \subset c_0$.

(2) It is easy to observe that an analog of Proposition 3.4(a,b) cannot be true for the functional $\psi_{p,\mu}$ corresponding to the general *weak* p-th order probability measure μ.

Theorem 5.4. *Let X be a Hausdorff locally convex space, $1 < p < \infty$, and μ be the weak p-th order probability measure on X. Suppose further that at least one of the following conditions is satisfied:*

$(c1)$ X is locally complete and $\hat{\mu}$ is continuous on X^*_τ.

$(c2)$ X is locally complete and μ is convex-tight.

$(c3)$ X has the convex compactness property and μ is Radon.

$(c4)$ X is von Neumann complete (in particular, is complete or quasi-complete) and μ is Radon.

Then

$(c5)$ $\psi_{p,\mu}$ is continuous on X^*_τ and hence $\mathcal{H}_{p,\mu} \subset X$.

Proof. $(c1) \Rightarrow (c5)$ Denote by u the natural mapping from X^* to $\mathbf{L}_p(X, \mu)$. From the continuity of $\hat{\mu}$ on X^*_τ according to Proposition 3.2(a$'$) we get that $\|\cdot\|_{o,\mu}$ is continuous on X^*_τ. Consequently, $u : X^*_\tau \to (\mathbf{L}_p(X, \mu), \mathcal{T}_\mu)$ is a continuous mapping. Then by Theorem 2.10 $u : X^*_\tau \to \mathbf{L}_p(X, \mu)$ is continuous. Since $\|ux^*\|_{\mathbf{L}_p} = (\psi_{\mu,p}(x^*))^{1/p}$, $\forall x^* \in X^*$, we conclude that $\psi_{p,\mu}$ is continuous on X^*_τ.

$(c2) \Rightarrow (c5)$ follows from $(c1) \Rightarrow (c5)$ and Proposition 3.4(b').

$(c3) \Rightarrow (c5)$ follows from $(c1) \Rightarrow (c5)$ and Proposition 3.4(c), since the ccp implies local completeness.

$(c3) \Rightarrow (c5)$ follows from $(c1) \Rightarrow (c5)$ and Proposition 3.4(c'). □

Remark 5.5. (1) The implication $(c1) \Rightarrow (c5)$ was proved earlier in [36, Th. 4] under the supposition that X is *complete* in a very different method.

(2) Note that $(c1) \Rightarrow (c5)$ may not be true without some completeness-like supposition of X. In fact, equip $X := \mathbb{R}^{\mathbb{N}}_0$ with the inner product induced from l_2 and let μ be the discrete measure in X, defined by the equality $\mu = \sum_{j=1}^{\infty} 2^{-j}\delta_{2^{j/2}\mathbf{e}_j}$. Then $\hat{\mu}$ is even $\sigma(X^*, X)$-continuous, but $\psi_{2,\mu}(\mathbf{y}) = \|\mathbf{y}\|^2_{l_2}$, $\forall \mathbf{y} \in l_2 = X^*$, consequently, $\mathcal{H}_{2,\mu} = l_2$.

Let $p \geq 1$ and μ be the weak p-th order measure in X.

The linear functional $l_\mu : X^* \to \mathbb{R}$ defined by the equality

$$l_\mu(x^*) = \int_X x^*(x) d\mu(x), \ x^* \in X^*,$$

is called the Dunford–Gelfand mean of μ. Evidently, $l_\mu \in \mathcal{H}_{p,\mu}$. If $l_\mu = 0$, then μ is called *(scalarly) centered.* When there exists an element $b_\mu \in X$ such that $x^*(b_\mu) = l_\mu(x^*)$, $\forall x^* \in X^*$, it is called the Pettis mean or *baricenter* of μ. From Remark 5.3(1) we can conclude that not any weak first order probability measure in c_0 has the baricenter, while if μ has the weak order $p > 1$ and at least one of the conditions of Theorem 5.4 is satisfied, then μ has the baricenter.

Let $p \geq 2$ and μ be a weak p-th order measure in X. The functional $v_\mu : X^* \to \mathbb{R}$ defined by the equality

$$v_\mu(x^*) = \psi_{2,\mu}(x^*) - l^2_\mu(x^*), \ x^* \in X^*,$$

is called the *variance* of μ. Clearly, v_μ is the non-negative quadratic form on X^*, induced by the symmetric bilinear form

$$r_\mu(x_1^*, x_2^*) = \int\limits_X x_1^*(x)x_2^*(x)d\mu(x) - l_\mu(x_1^*)l_\mu(x_2^*),\ x_1^*, x_2^* \in X^*,$$

which is called the *covariance* of μ.

The covariance r_μ of a weak second order measure μ in X is a semi-inner product in the vector space X^*. We denote by $\mathcal{H}(r_\mu)$ the topological dual space of the pre-Hilbert space (X^*, r_μ) and call it the *reduced kernel* of μ. Consequently,

$$\mathcal{H}(r_\mu) := (X^*, r_\mu)^*.$$ [8]

Clearly, for any scalarly centered weak second order μ we have $\mathcal{H}(r_\mu) = \mathcal{H}_{2,\mu}$.

Lemma 5.6. *Let μ be a weak second order probability measure in a dually separated space X. Then:*

(a) $\mathcal{H}_{2,\mu} = \mathcal{H}(r_\mu) + \mathbb{R}l_\mu$ *(in the sense of X^{*a}).*

(b) *If $\mathfrak{T}$ is a vector topology in X^*, then $\psi_{2,\mu}$ is $\mathfrak{T}$-continuous if and only if v_μ and l_μ are so.*

(c) *If $\psi_{2,\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$, then the baricenter b_μ of μ exists (i.e., $l_\mu \in X$), $\mathcal{H}(r_\mu) \subset X$, and*

$$\mathcal{H}_{2,\mu} = \mathcal{H}(r_\mu) + \mathbb{R}b_\mu;$$

moreover, we have $\mathcal{H}(r_\mu) = \mathcal{H}_{2,\mu_0}$, where $\mu_0(\cdot) := \mu(\cdot - b_\mu)$.

(c′) *If at least one of the conditions of Theorem* 5.4 *is satisfied, then the conclusions of* (c) *hold.*

Proof. (a) Since $v_\mu \le \psi_{2,\mu}$ and $l_\mu \in \mathcal{H}_{2,\mu}$ we have $\mathcal{H}(r_\mu) + \mathbb{R}l_\mu \subset \mathcal{H}_{2,\mu}$. Let now $f \in \mathcal{H}_{2,\mu}$. This implies that for some $h \in \mathcal{L}_2(X, \mu)$ we have

$$f(x^*) = \int\limits_X x^*(x)h(x)d\mu(x),\ \forall x^* \in X^*.$$

Then for $f_h : X^* \to \mathbb{R}$ defined by the equality

$$f_h(x^*) = \int\limits_X (x^*(x) - l_\mu(x^*))h(x)d\mu(x),\ x^* \in X^*,$$

we get $f_h \in \mathcal{H}(r_\mu)$ and $f = f_h + tl_\mu$, where $t := \int_X h(x)d\mu(x)$. Hence $f \in \mathcal{H}(r_\mu) + \mathbb{R}l_\mu$.

(b) Suppose $\psi_{2,\mu}$ is $\mathfrak{T}$-continuous. Since $|l_\mu| \le \psi_{2,\mu}^{1/2}$, we get that the linear functional l_μ is $\mathfrak{T}$-continuous at zero and hence is $\mathfrak{T}$-continuous everywhere. Consequently, $v_\mu = \psi_{2,\mu} - l_\mu^2$ is $\mathfrak{T}$-continuous.

[8]The reduced kernel $\mathcal{H}(r_\mu)$, as a topological dual space of the pre-Hilbert space (X^*, r_μ), carries the natural Hilbert space structure. In the case of a Gaussian measure μ, the obtained Hilbert space is often called *the reproducing kernel Hilbert space (RKHS)* of μ.

(c) Since $\psi_{2,\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$, from (b) we get that l_μ and v_μ are continuous in the Mackey topology $\tau(X^*, X)$ too. Consequently, $l_\mu \in (X^*; \tau(X^*, X))^* = X$ and $\mathcal{H}(r_\mu) = (X^*, r_\mu)^* \subset (X^*; \tau(X^*, X))^* \subset X$. The rest is clear.

(c′) follows from (c) via Theorem 5.4. □

Let μ be a weak second order probability measure in a dually separated space X. Since its covatiance r_μ is a bilinear form on X^*, it induces a linear operator $R_\mu : X^* \to X^{*a}$ in a natural way:

$$R_\mu x^*(\cdot) := r_\mu(x^*, \cdot), \ \ \forall x^* \in X^*.$$

The operator R_μ is called the *covariance operator* of μ. Evidently, $R_\mu(X^*) \subset \mathcal{H}(r_\mu)$. Therefore the question of localization of the range of R_μ into X is related with the same question for $\mathcal{H}(r_\mu)$.

Proposition 5.7. *Let μ be a weak second order probability measure in a dually separated space X.*

(a) *If $\psi_{2,\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$, then the baricenter b_μ of μ exists, $R_\mu(X^*) \subset X$ and $R_\mu : X^* \to X$ is a symmetrically Hilbertian operator.*

Moreover, if $R_\mu = TT^*$, where T is a weakly continuous linear operator from a Hilbert space H to X, then $\mathcal{H}(r_\mu) = T(H)$ and $\mathcal{H}_\mu = T(H) + \mathbb{R}b_\mu$.

(a′) *If at least one of the conditions of Theorem* 5.4 *is satisfied, then the conclusions of* (a) *hold.*

(b) *If the baricenter b_μ of μ exists, $R_\mu(X^*) \subset X$ and $R_\mu : X^* \to X$ is a symmetrically Hilbertian operator, then $\psi_{2,\mu}$ is continuous in the Mackey topology $\tau(X^*, X)$.*

(b′) *If $R_\mu(X^*) \subset X$ and X is locally complete in the topology $\sigma(X, X^*)$, then $R_\mu : X^* \to X$ is a symmetrically Hilbertian operator.*

Proof. (a) Since $R_\mu(X^*) \subset \mathcal{H}(r_\mu)$, the needed inclusion follows from Lemma 5.6(*c*). Clearly, the operator $R_\mu : X^* \to X$ is symmetric and positive and the quadratic form $v_\mu = \psi_{2,\mu} - l_\mu^2$ corresponding to it is Mackey-continuous. Then by Proposition 2.8(a) it is symmetrically Hilbertian.

The proof of the "moreover" part is left to the reader.

(a′) follows from (a) and Lemma 5.6(c′).

(b) By Proposition 2.8(b) the quadratic form $q_{R_\mu} = v_\mu$ is Mackey-continuous. Since the baricenter of μ exists, l_μ is also Mackey-continuous. It follows that $\psi_{2,\mu} = v_\mu + l_\mu^2$ is continuous in the Mackey topology $\tau(X^*, X)$.

(b′) follows from Proposition 2.8(b). □

Now we turn to the Gaussian measures. Strictly speaking, all previous statements about weak p-th order measures and p-kernels are not needed for the proof of Theorem 5.9, but its formulation requires concepts which make their natural sense also for non-Gaussian measures; that is why we gave them in a general setting and have commented on some of the results. The proof of Theorem 5.9 will use the following assertion, the first three statements of which are simple.

Proposition 5.8. *Let X be a dually separated space and μ be a **Radon** probability measure in X.*

(a) *The set $\mathfrak{N}_\mu (= \{x^* \in X^* : \|x^*\|_{o,\mu} = 0\})$ is closed in the topology $\sigma(X^*, X)$.*

(b) *If $\hat{\mu} = \exp(if)$, where $f : X^* \to \mathbb{R}$ is a linear functional, then $f \in X$.*

(c) *If $\hat{\mu} = \exp(-f^2)$, where $f : X^* \to \mathbb{R}$ is a linear functional, then $f \in X$.*

(d) [35, Cor. 1, p. 399] *If $\hat{\mu} = \chi_1 \cdot \chi_2$, where $\chi_1 : X^* \to \mathbb{R}$ and $\chi_2 : X^* \to \mathbb{C}$ are positive definite functionals, then there exist Radon probability measures μ_1, μ_2 in X such that $\hat{\mu}_1 = \chi_1$ and $\hat{\mu}_2 = \chi_2$.*

(e) *If $\hat{\mu} = \exp(-v)$, where $v : X^* \to \mathbb{R}_+$ is a quadratic form and $f : X^* \to \mathbb{R}$ is a linear functional such that $f^2 \le v$, then $f \in X$.*

Proof. (a) Put $\mathcal{E}_\mu := \bigcap_{x^* \in \mathfrak{N}_\mu} ker(x^*)$. Since μ is a Radon probability measure and $\mu(ker(x^*)) = 1$ for each $x^* \in \mathfrak{N}_\mu$, we get $\mu(\mathcal{E}_\mu) = 1$. Let us show that

$$(\mathcal{E}_\mu)^\circ = \mathfrak{N}_\mu. \tag{5.2}$$

Let $x^* \in (\mathcal{E}_\mu)^\circ$, then (as $(\mathcal{E}_\mu)$ is a vector subspace), $x^*(x) = 0$, $\forall x \in \mathcal{E}_\mu$. Since $\mu(\mathcal{E}_\mu) = 1$, we have $x^* = 0$, μ-a.e. Hence $x^* \in \mathfrak{N}_\mu$.

Let now $x^* \in \mathfrak{N}_\mu$. Then $ker(x^*) \supset \mathcal{E}_\mu$. This implies $x^*(x) = 0$, $\forall x \in \mathcal{E}_\mu$. Hence, $x^* \in (\mathcal{E}_\mu)^\circ$.

From (5.2) it is plain that $\mathfrak{N}_\mu$ is $\sigma(X^*, X)$-closed.

(b) Let us show that

$$ker(f) = \mathfrak{N}_\mu. \tag{5.3}$$

Let $x^* \in ker(f)$. Fix arbitrary $t \in \mathbb{R}$. Then $tx^* \in ker(f)$. This implies $\hat{\mu}(tx^*) = 1$. Consequently, for the image μ_{x^*} of μ under x^* we have $\hat{\mu_{x^*}}(t) = 1$, $\forall t \in \mathbb{R}$. This (by the uniqueness theorem for the Fourier transform) implies $\mu_{x^*} = \delta_0$. Hence $x^* \in \mathfrak{N}_\mu$.

Let now $x^* \in \mathfrak{N}_\mu$. Fix arbitrary $t \in \mathbb{R}$. Then $tx^* \in \mathfrak{N}_\mu$. This implies $\hat{\mu}(tx^*) = 1$. Consequently, $\exp(itf(x^*)) = 1$, $\forall t \in \mathbb{R}$, i.e., $f(x^*) = 0$ and so $x^* \in ker(f)$.

Equality (5.3) and (a) give that $ker(f)$ is $\sigma(X^*, X)$-closed. Since f is linear, we get that f is $\sigma(X^*, X)$-continuous, hence $f \in X$.

(c) In this case equality (5.3) holds trivially, so the conclusion follows from (a), as in the previous case.

(e) Let $q := v - f^2$, then $q \ge 0$ and is a quadratic form. It is well-known that then $\exp(-q)$ is a positive definite functional on X^*. For the same reason, the functional $\exp(-f^2)$ is also positive definite. Therefore, $\hat{\mu} = \exp(-f^2) \cdot \exp(-q)$ and we can apply (d) and write $\exp(-f^2) = \hat{\mu}_1$ for some Radon probability measure μ_1 in X. Then by (c) we get $f \in X$. $\square$

Theorem 5.9. *Let X be a dually separated topological vector space and μ be a Gaussian Radon measure in X. Then:*

(a) *The functional $\psi_{2,\mu}$, and hence $\hat{\mu}$ too, is continuous in the Mackey topology $\tau(X^*, X)$.*

(b) *The baricenter b_μ of μ exists, the covariance operator R_μ maps X^* into X and R_μ is a symmetrically Hilbertian operator.*

Moreover, if $R_\mu = TT^*$, where T is a weakly continuous linear operator from a Hilbert space H to X, then

$$\mathcal{H}(r_\mu) = T(H) \subset X, \quad \mathcal{H}_\mu = \mathbb{R}b_\mu + T(H) \subset X.$$

Proof. (a) Since μ is a Gaussian measure in X, we have

$$\hat{\mu}(x^*) = \exp\left\{ il_\mu(x^*) - \frac{1}{2}v_\mu(x^*)\right\}, \ x^* \in X^*. \tag{5.4}$$

Equality (5.4) implies that $\hat{\mu}$ is the product of two positive definite functionals $\exp(il_\mu)$ and $\exp(-\frac{1}{2}v_\mu)$, the second of which is real. Since μ is a Radon measure, according to Proposition 5.8(d) there are Radon probability measures μ_1 and μ_2 in X such that[9]

$$\hat{\mu}_2 = \exp\{il_\mu\}, \quad \hat{\mu}_1 = \exp(-\frac{1}{2}v_\mu).$$

By Proposition 5.8(b) the first equality implies that $l_\mu \in X$, i.e., l_μ is induced by an element $b_\mu \in X$, which is the needed baricenter of μ.

Let $\gamma := \mu_1$. Clearly, $v_\mu = \psi_{2,\gamma}$. Consider now the inclusion mapping $u : X^* \longrightarrow \mathbf{L}_2(X,\gamma) := H$. Then

$$v_\gamma(x^*) = \| ux^* \|_H^2, \quad x^* \in X^*,$$

and so

$$\hat{\gamma}(x^*) = exp\{-\frac{1}{2} \| ux^* \|_H^2\}, \quad x^* \in X^*. \tag{5.5}$$

Let us show that u as an operator from $(X^*, \sigma(X^*, X))$ to $(H, \sigma(H, H^*))$ is continuous. Fix $h \in H$ with $\|h\| = 1$ and consider the linear functional f_h defined by the equality $f_h(x^*) = (ux^*|h), \ \ x^* \in X^*$. We have $f_h^2(x^*) \le \|ux^*\|_H^2$, $\forall x^* \in X^*$. From this inequality and from (5.5) by Proposition 5.8(e) we get $f_h \in X$. This means that there is an element $x_h \in X$ such that $f_h(x^*) = x^*(x_h)$ for all $x^* \in X^*$. So we obtain the equality $(ux^*|h) = x^*(x_h), \ \ x^* \in X^*, h \in H$, which implies that u is continuous with respect to the topologies $\sigma(X^*, X)$ and $\sigma(H, H^*)$. This, according to Proposition 2.7, implies that u is continuous as mapping from $(X^*, \tau(X^*, X))$ into $(H, \|\cdot\|_H)$. Consequently, the functional $x^* \to \|ux^*\|_H^2 = v_\gamma(x^*) = v_\mu(x^*)$ is continuous in topology $\tau(X^*, X)$. Since $\psi_{2,\mu}(x^*) = v_\mu(x^*) + |x^*(b_\mu)|^2$, $\forall x^* \in X^*$, we get (a).

[9] The existence of μ_1 can be seen directly; it is simply the image of $\mu \times \mu$ with respect to the mapping $(x_1, x_2) \to \frac{x_1 - x_2}{\sqrt{2}}$.

(b) follows from (a) by Proposition 5.7(a) (since by Lemma 5.1(c) $\mathcal{H}_\mu = \mathcal{H}_{2,\mu}$ in this case). □

Acknowledgements

The work on the final version of this paper was finished during the second author's sabbatical stay at the Department of Geometry and Topology of the Faculty of Mathematical Sciences of University of Complutense of Madrid (conv. SA58/00). He is grateful to the professor of the Department E. Martín-Peinador for her assistance and helpful "topological" hints.

The authors are grateful to the referees for their helpful comments and advice as to the trend of further studies.

References

1. V. I. Bogachev, Gaussian measures. *Surveys and Monograps* 62, *American Math. Soc., Providence, Rhode Island,* 1998.
2. A. Badrikian and S. Chevet, Mesures cylindriques, espaces de Wiener et fonctions aléatoires Gaussiennes. *Lecture Notes in Math.* **379** *Springer, Berlin etc.,* 1974.
3. Chr. Borell, Gaussian Radon measures on locally convex spaces. *Math. Scand.* **38**(1976), 265–284.
4. P. Borwein and T. Erdelyi, Polynomials and polynomial inequalities. *Graduate Texts in Mathematics* 161, *Springer, New York,* 1995.
5. N. Bourbaki, Topologie générale. *Actualités Scientifiques. Hermann, Paris,* 1960.
6. N. Bourbaki, Espaces vectoriels topologiques. *Masson, Paris etc.,* 1981.
7. M. J. Chasco, E. Martín-Peinador, and V. Tarieladze, On Mackey topology for groups. *Studia Math.* **132**(1999), 257–284.
8. S. Chevet, Kernel associated with a cylindrical measure. *Lecture Notes in Math.* **860**(1981), 51–84.
9. S. Diaz, Σ-completeness and closed graph theorems. *Collect. Math.* **42**(1991), 253–263.
10. P. Dierolf, Une caractérisation des espaces vectoriels topologiques complets au sens de Mackey. *C.R. Acad. Sci. Paris Sér. A-B* **283**(1976), No. 5, A245–A248.
11. R. M. Dudley, Random linear functionals. *Trans. Amer. Math. Soc.* **136**(1969), 1–24.
12. A. Grothendieck, Topological vector spaces. *Gordon and Breach Science Publishers,* 1973.
13. M. U. Hapizov, On the kernel of a Radon measure in a Banach space. (Russian) *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* (1993), No. 1, 3–8.
14. H. Jarchow, Locally convex spaces. *B.G. Teubner, Stuttgart,* 1981.
15. J. Kakol, Note on compatible vector topologies. *Proc. Amer. Math. Soc.* **99**(1987), 690–692.
16. S. Kwapień and W. Smoleński, On the nuclearity of a dual space with the convergence in probability topology. *Z. Wahrscheinlichkeitstheor. verw. Geb.* **59**(1982), 197–201.
17. S. Kwapień and W. A. Woyczyński, Random series and stochastic integrals: single and multiple. *Birkhäuser,* 1992.
18. M. Ledoux and M. Talagrand, Probability in Banach spaces. *Springer, Berlin etc.,* 1991.

19. S. Mazur, On the convergence of lacunary polinomials. *Studia Math.* **89**(1988), 75–78.

20. D. H. Mushtari, Probabilities and topologies on linear spaces. *Kazan Mathematics Foundation,* 1996.

21. I. P. Natanson, Constructive function theoery. II. *Frederic Ungar Publishing Co.,* 1965.

22. E. G. Ostling and A. Wilansky, Locally convex topologies and the convex compactness property. *Proc. Camb. Phil. Soc.* **75**(1974), 45–50.

23. K. Pietruska-Pałuba and W. Smoleński, Topology of the convergence in probability on a linear span of a sequence of independent random variables. *Probab. Math. Statist.* **10**(1989), 161–168.

24. H. H. Schaefer, Topological vector spaces. *Graduate Texts in Mathematics* 3, 2nd ed., *Springer, New York,* 1999.

25. L. Schwartz, Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associes. (Noyaux reproduisants). *J. Anal. Math.* **13**(1964), 115–256.

26. L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures. *Tata Institute of Fundamental Research Studies in Mathematics*, No. 6. *Published for the Tata Institute for Fundamental Research, Bombay by Oxford University Press, London,* 1973.

27. W. Smoleński, Pre-supports and kernels of probability measures in Frechet spaces. *Demonstratio Math.* **10**(1977), 751–762.

28. W. Smoleński, On the kernel of probability measures on a Banach space. *Demonstratio Math.* **21**(1988), 569–572.

29. M. Talagrand, Are all sets of positive measure essentially convex? *Operator Theory, Advances and Applications* 77, 294–301, *Birkhäuser, Basel,* 1995.

30. N. N. Vakhania, Sur les répartitions des probabilités dans les espaces de suites numériques. *C.R. Acad. Sci. Paris* **260**(1965), 1560–1562.

31. N. N. Vakhania, Covariance operator of a probability distribution in a Banach space. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **51**(1968), 35–40.

32. N. N. Vakhania, Probability distributions in linear spaces. (Russian) *Metsniereba, Tbilisi,* 1971; English translation: *North-Holland, Amsterdam, etc.,* 1981.

33. Nguyen Duy Tien and V. Tarieladze, Probability measures with big kernels. *Georgian Math. J.* **8**(2001), No. 2, 333–346.

34. N. N. Vakhania and V. I. Tarieladze, Covariance operators of probability distributions in locally convex spaces. *Theory Probab. Appl.* **23**(1978), 1–21.

35. N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, Probability distributions on Banach spaces. *D. Reidel Publishing Co., Dordrecht*, 1987.

36. N. N. Vakhania, V. I. Tarieladze, and A. Tortrat, On the continuity of random linear operators. *Theory Probab. Appl.* **26**(1981), 167–173.

37. M. Valdivia, Mackey convergence and the closed graph theorem. *Arch. Math.* **25**(1974), 649–656.

38. A. Wilansky, Modern methods in topological vector spaces. *McGraw-Hill, Inc.,* 1978.

(Received 21.05.2001)

Authors' addresses:

Stanislaw Kwapień
Institute of Mathematics, Warsaw University
ul. Banacha, 2. 02-087, Warsaw, Poland

E-mail: kwapstan@mimuw.edu.pl

V. Tarieladze
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia
E-mail: tar@gw.acnet.ge
visit01@dma.uvigo.es