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HYPERCONVEX SPACES AND FIXED POINTS

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Abstract. New fixed point results for a very general class of maps on hyperconvex spaces are given. Our theory relies on the fact that hyperconvex compact spaces are absolute retracts. Also maps on k-CAR sets are discussed.

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1. Introduction

In Section 2 new fixed point results are presented for multivalued maps in hyperconvex spaces. This notion was introduced by Aronszajn and Panitchpakdi [4] and recently many papers on fixed point theory in these spaces have appeared in the literature (see [5–8, 10] and the references therein). Using a recent fixed point result of Agarwal, O'Regan and Park [3] we are able to establish a new and very general fixed point theorem in hyperconvex spaces. In particular we show if X is a hyperconvex compact space and $F \in \mathcal{U}_c^k(X,X)$ (defined below), then F has a fixed point. The class $\mathcal{U}_c^k$ is very general and contains for example Aronszajn, acyclic, O'Neill, admissible and permissible maps. The proof of the above fixed point is elementary and relies on the fact [4] that a hyperconvex compact space is an absolute retract. Also in Section 2 we replace the compactness of the space with the compactness (or condensingness) of the map. In Section 3 we present new fixed point theory for maps defined on k-CAR sets, and our result extend those in [1].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Let X and Y be Hausdorff topological vector spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ Fix\, F \neq \varnothing \text{ for all } \ F \in \mathcal{X}(Z,Z)\}.$$

A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

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(iii) for any polytope P, $F \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.1. $F \in \mathcal{U}_c^k(X, Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall a space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable W and $A \subseteq W$ closed each $f : A \to Z$ extends over W to a map $f : W \to Z$. The following fixed point result of Agarwal, O'Regan and Park [3] will be needed in Section 2.

Theorem 1.1. *Let $X \in AR$ be a topological vector space and $F \in \mathcal{U}_c^k(X, X)$ a compact map. Then F has a fixed point.*

Let (E, d) be a pseudometric space. For $S \subseteq E$, let $B(S, \epsilon) = \{x \in E : d(x, S) \leq \epsilon\}$, $\epsilon > 0$, where $d(x, S) = \inf_{y \in Y} d(x, y)$. The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\epsilon > 0 : \ M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F : S \to 2^E$; here $S \subseteq E$. The map F is said to be a countably P-concentrative mapping if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \leq \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e. X is not precompact in the pseudonormed space (E, p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (E, p).

The following fixed point result of Agarwal and O'Regan [2] will be needed in Section 3.

Theorem 1.2. *Let Ω be a nonempty, closed, convex subset of a Fréchet space E (P is a defining system of seminorms). Suppose $F \in \mathcal{U}_c^k(\Omega, \Omega)$ is a countably P-concentrative mapping. Then F has a fixed point in Ω.*

Finally, for completeness, we also give the definition of countably condensing maps. Let X be a metric space and $P_B(X)$ the bounded subsets of X. The Kuratowskii measure of noncompactness is the map $\alpha : P_B(X) \to [0, \infty)$ defined by

$$\alpha(A) = \inf\{\epsilon > 0 : \ A \subseteq \cup_{i=1}^n X_i \text{ and } \operatorname{diam}(X_i) \leq \epsilon\};$$

here $A \in P_B(X)$. Let S be a nonempty subset of X and let $H : S \to 2^X$. H is called (i). condensing if $\alpha(H(\Omega)) \leq \alpha(\Omega)$ for all bounded sets Ω of S and $\alpha(H(\Omega)) < \alpha(\Omega)$ for all bounded sets Ω of S with $\alpha(\Omega) \neq 0$, (ii). countably condensing if $\alpha(H(\Omega)) \leq \alpha(\Omega)$ for all countably bounded sets Ω of S and $\alpha(H(\Omega)) < \alpha(\Omega)$ for all countably bounded sets Ω of S with $\alpha(\Omega) \neq 0$, (iii). k-set contractive ($k \geq 0$) if $\alpha(H(\Omega)) \leq k\,\alpha(\Omega)$ for all bounded sets Ω of S, and (iv). countably k-set contractive ($k \geq 0$) if $\alpha(H(\Omega)) \leq k\,\alpha(\Omega)$ for all countably bounded sets Ω of S.

2. Hyperconvex Spaces

A metric space (X, d) is hyperconvex if $\cap_\alpha B(x_\alpha, r_\alpha) \neq \varnothing$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in X for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. Fixed point theorems in hyperconvex spaces have been presented in [5–8, 10]. We begin by presenting a fixed point result which enables us to improve considerably most results in the literature. The class of maps we consider is very general and contains for example acyclic, approachable and permissible maps [9].

Theorem 2.1. *Let X be a hyperconvex compact topological vector space and $F \in \mathcal{U}_c^k(X, X)$. Then F has a fixed point.*

Proof. Since X is hyperconvex and compact then $X \in AR$ (see [4 pp. 422]). Now Theorem 1.1 guarantees that F has a fixed point. $\square$

Remark 2.1. It is possible to remove the assumption that X is a topological vector space in Theorem 2.1 if $F : X \to X$ is a continuous single valued map. The idea in this case is to apply the Generalized Schauder Theorem (i.e., if $Y \in AR$ and $f : Y \to Y$ is a continuous compact map then f has a fixed point) instead of Theorem 1.1.

It is also possible to remove the assumption that X is a topological vector space in Theorem 2.1 for certain classes of multivalued maps if we use Theorem 2.2 below instead of Theorem 1.1. Let Z and W be Hausdorff topological spaces. A class of maps $\mathcal{R}(Z, W)$ is said to be *admissible* if

(i) $\mathcal{R}$ contains the class $\mathcal{C}$;

and

(ii) each $F \in \mathcal{R}_c$ is upper semicontinuous and closed valued.

The same reasoning as in [3] establishes the following result.

Theorem 2.2. *Let $\mathcal{R}$ be an admissible class of maps, $X \in AR$ and $F \in \mathcal{R}_c(X, X)$ a compact map. If $B^n = \{x \in R^n : \|x\| \leq 1\}$ is in $\mathcal{F}(\mathcal{R}_c)$ for all $n \geq 1$, then F has a fixed point.*

This immediately yields the following result for hyperconvex spaces.

Theorem 2.3. *Let $\mathcal{R}$ be an admissible class of maps, X a hyperconvex compact space and $F \in \mathcal{R}_c(X, X)$ with $B^n \in \mathcal{F}(\mathcal{R}_c)$ for all $n \geq 1$. Then F has a fixed point.*

We next replace the compactness of the space with the compactness (or condensingness) of the map. Indeed the argument to establish this is now standard (see [6, 7]) but for completeness we include it here. We first however need the following concepts. A mapping of metric spaces $e : X \to E$ is called a hyperconvex hull of X if E is hyperconvex, e is an isometric embedding, and no hyperconvex proper subspace of E contains $e(X)$. A function $f \in C(X)$ (continuous functions from X to R) is an extremal function over X if for all $x,\ y \in X$ we have $f(x) + f(y) \geq d(x, y)$ and f is pointwise minimal (i.e. if

g is another function with the same property such that $g(x) \leq f(x)$ for all $x \in X$ then $g = f$). We let

$$\epsilon\, X = \{f \in C(X) : \ f \text{ is extremal}\};$$

we refer the reader to [6] for a discussion of the above ideas.

Theorem 2.4. *Let X be a hyperconvex, bounded metric topological vector space and let $F \in \mathcal{U}_c^k(X, X)$ be condensing. Then F has a fixed point.*

Proof. Fix $x_0 \in X$, and let

$$\Omega = \{A : \ x_0 \in A, \ A \subseteq X, \ A \text{ hyperconvex and } \ F(A) \subseteq A\}.$$

Note $X \in \Omega$ so $\Omega \neq \varnothing$. We may now apply Zorn's Lemma since it is immediate from [5, Theorem 7] (or [6, Theorem 3]) that every chain in Ω has a lower bound. As a result there exists a minimal element Y of Ω. Now [6, Lemma 4] guarantees that there exists a subset B of X isometric to $\epsilon\,(F(Y) \cup \{x_0\})$ with B hyperconvex and $F(Y) \cup \{x_0\} \subseteq B \subseteq Y$. This immediately implies $F(B) \subseteq F(Y) \subseteq B$, and so $x_0 \in B$ with B hyperconvex, $F(B) \subseteq B$ and $B \subseteq Y$. As a result $B = Y$. Next notice

$$\alpha(Y) = \alpha(B) = \alpha(\epsilon\,(F(Y) \cup \{x_0\})). \tag{2.1}$$

Also [6, Corollary, p. 135] yields

$$\alpha(\epsilon\,(F(Y) \cup \{x_0\})) = \alpha(F(Y) \cup \{x_0\})$$

and this together with (2.1) gives

$$\alpha(Y) = \alpha(F(Y)).$$

Now since F is condensing we have that $\overline{Y}$ is compact. In fact since hyperconvex spaces are closed [4] we have Y compact. Thus Y is a compact hyperconvex space with $F(Y) \subseteq Y$. In addition since $\mathcal{U}_c$ is closed under compositions we have $F|_Y \in \mathcal{U}_c^k(Y, Y)$. Now Theorem 2.1 establishes the result. $\square$

Remark 2.2. X a topological vector space can be removed in Theorem 2.4 if we are in the situations described after Theorem 2.1.

3. k-CAR Sets

Throughout this section E will be a Fréchet space.

Definition 3.1. A closed subset A of E is said to be k-CAR ($k \geq 0$) if there exists a continuous k-set contractive retraction R from $\overline{co}\,(A)$ to A; here $\overline{co}\,(A)$ denotes the closed convex hull of A.

Remark 3.1. It is well known [4, 5, 8] that a hyperconvex subset A of a Banach space is a 1-CAR set (in fact the retraction R is nonexpansive).

Definition 3.2. A closed subset A of E is said to be countably k-CAR ($k \geq 0$) if there exists a continuous countable k-set contractive retraction R from $\overline{co}\,(A)$ to A.

Our first results extends [1, Theorem 2.2] to a wider class of maps which include acyclic, O'Neill and permissible maps (see [9]).

Theorem 3.1. *Let E be a Fréchet space and C a nonempty closed countable k-CAR ($0 \leq k \leq 1$) subset of E. Suppose $F \in \mathcal{U}_c^k(C,C)$ is a countably condensing map with $F(C)$ bounded. Then F has a fixed point in C.*

Proof. Let $R : \overline{co}\,(C) \to C$ be the continuous countable k-set contractive retraction which is guaranteed since C is a countable k-CAR set. Notice since $\mathcal{U}_c^k$ is closed under compositions that $G = F\,R \in \mathcal{U}_c^k(\overline{co}\,(C), \overline{co}\,(C))$; note in fact that $G(\overline{co}\,(C)) \subseteq C$. It is easy to check that G is countably condensing and $G(\overline{co}\,(C))$ is bounded. Now Theorem 1.2 guarantees that there exists $x \in \overline{co}\,(C)$ with $x \in G\,(x)$. In fact since $G(\overline{co}\,(C)) \subseteq C$ we must have $x \in C$ and as a result $x \in F\,R(x) = F(x)$. $\square$

Remark 3.2. Of course it is possible to replace C countably k-CAR and F countably condensing in Theorem 3.1 with any conditions which guarantee that the map $G = F\,R$ is countably P-concentrative.

Essentially the same reasoning as in Theorem 3.1 yields the following result.

Theorem 3.2. *Let E be a Fréchet space and C a nonempty closed countable k-CAR ($k \geq 0$) subset of E. Suppose $F \in \mathcal{U}_c^k(C,C)$ is a countably β-set contractive ($\beta \geq 0$) map with $F(C)$ bounded and $k\,\beta < 1$. Then F has a fixed point in C.*

Next we establish two nonlinear alternatives of Leray–Schauder type for countably k-CAR sets.

Theorem 3.3. *Let E be a Fréchet space, C a closed countable k-CAR ($0 \leq k \leq 1$) subset of E, U an open subset of E with $0 \in U$ and $U \subseteq C$ convex. Suppose $F \in \mathcal{U}_c^k(\overline{U}, C)$ is a countably condensing map with $F(\overline{U})$ bounded and*

$$x \notin \lambda\,F\,x \quad \textit{for all } x \in \partial U \textit{ and } \lambda \in (0,1); \tag{3.1}$$

here ∂U denotes the boundary of U in C. Then F has a fixed point in $\overline{U}$.

Proof. Let $r : E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}},$$

where μ is the Minkowski functional on $\overline{U}$ i.e. $\mu(x) = \inf\{\alpha > 0 :\ x \in \alpha\,\overline{U}\}$. We know that $r : E \to \overline{U}$ is a 1-set contractive map. Now let $H = F\,r$. Clearly $H \in \mathcal{U}_c^k(C,C)$ and it is easy to see that H is a countably condensing map with $H(C)$ bounded. Now Theorem 3.1 guarantees that there exists $x \in C$ with $x \in F\,r(x)$. With $z = r(x)$ we have $z \in r(F(z))$ and so $z = r(w)$ for some $w \in F(z)$. There are two cases to consider, namely $w \in \overline{U}$ or $w \in C \setminus \overline{U}$. If

$w \in C \setminus \overline{U}$ then since $z = r(w) = \frac{w}{\mu(w)}$ we have $\mu(z) = 1$, so $z \in \partial_E U = \partial U$ since $int_C\, U = U$, and thus

$$z \in \lambda\, F(z) \text{ with } \lambda = \frac{1}{\mu(w)} \in (0,1) \text{ and } z \in \partial U.$$

This contradicts (3.1). Thus $w \in \overline{U}$ so $z = w \in F(z)$ and $z \in \overline{U}$. □

Remark 3.3. A remark similar to Remark 3.2 holds for Theorem 3.3.

If we use Theorem 3.2 instead of Theorem 3.1 we have the following result.

Theorem 3.4. *Let E be a Fréchet space, C a closed countable k-CAR ($k \geq 0$) subset of E, U an open subset of E with $0 \in U$ and $U \subseteq C$ convex. Suppose $F \in \mathcal{U}_c^k(\overline{U}, C)$ is countably β-set contractive ($\beta \geq 0$) with $F(\overline{U})$ bounded, $k\,\beta < 1$ and* (3.1) *holding. Then F has a fixed point in $\overline{U}$.*

We now introduce the notion of an essential map in this situation. For our next two theorems let E be a Fréchet space, C a closed countable k-CAR ($k \geq 0$) subset of E, U an open subset of E with $0 \in U$ and $U \subseteq C$ convex. We will assume $0 \leq k \leq 1$.

Definition 3.3. $D(\overline{U}, C)$ denotes the set of countably condensing maps $F \in \mathcal{U}_c^k(\overline{U}, C)$ with $F(\overline{U})$ bounded.

Remark 3.4. It is also possible to discuss countably β-set contractive maps $F \in \mathcal{U}_c^k(\overline{U}, C)$ with $F(\overline{U})$ bounded when $k \geq 0$ and $k\,\beta < 1$. We leave the details to the reader.

Definition 3.4. We let $F \in D_{\partial U}(\overline{U}, C)$ if $F \in D(\overline{U}, C)$ with $x \notin F\,(x)$ for $x \in \partial U$ (the boundary of U in C).

Definition 3.5. A map $F \in D_{\partial U}(\overline{U}, C)$ is essential in $D_{\partial U}(\overline{U}, C)$ if for every $G \in D_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G\,(x)$.

Theorem 3.5 (Normalization). *Let E, C and U be as above. Then the zero map is essential in $D_{\partial U}(\overline{U}, C)$.*

Proof. Let $\theta \in D_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. We must show there exists $x \in U$ with $x \in \theta(x)$. Let r and μ be as in Theorem 3.3 and let $G = \theta\, r$. Clearly $G \in \mathcal{U}_c^k(C, C)$ is a countably condensing map so Theorem 3.1 guarantees that there exists $x \in C$ with $x \in \theta\, r(x)$. With $z = r(x) \in \overline{U}$ we have $z \in r\,\theta(z)$ and so $z = r(w)$ for some $w \in \theta\,(z)$. If $z \in \partial U$ then $\mu(z) = 1$ and so

$$1 = \mu(z) = \frac{\mu(w)}{\max\{1, \mu(w)\}} \quad \text{since} \quad r(w) = \frac{w}{\max\{1, \mu(w)\}}.$$

Consequently $\mu(w) \geq 1$ and so $z = \frac{w}{\mu(w)} \in \lambda\,\theta(z) = \{0\}$ (here $\lambda = \frac{1}{\mu(w)}$) since $\theta|_{\partial U} = \{0\}$, a contradiction since $0 \in U$. Thus $z \in U$ so $\mu(z) < 1$ and so

$$1 > \mu(z) = \frac{\mu(w)}{\max\{1, \mu(w)\}}.$$

Thus $\mu(w) < 1$ and as a result $r(w) = w$, so $z \in U$ with $z = r(w) = w \in \theta(z)$. (Alternatively one could show, as in Theorem 3.3, that $w \in \overline{U}$ so $z \in F(z)$ with $z \in \overline{U}$, of course as above it is easy to see that $z \notin \partial U$.) $\square$

Next we obtain a generalization of Theorem 3.3.

Theorem 3.6 (Homotopy). *Let E, C and U be as above and suppose $F \in D(\overline{U}, C)$ with*

$$x \notin \lambda F x \ \ \text{for all} \ \ x \in \partial U \ \ \text{and} \ \ \lambda \in (0,1] \tag{3.2}$$

holding. Then F is essential in $D_{\partial U}(\overline{U}, C)$.

Proof. Let $\theta \in D_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = F|_{\partial U}$. We must show there exists $x \in U$ with $x \in \theta(x)$. Notice (3.2) together with $\theta|_{\partial U} = F|_{\partial U}$ guarantees that

$$x \notin \lambda \theta x \ \ \text{for all} \ \ x \in \partial U \ \ \text{and} \ \ \lambda \in (0,1]. \tag{3.3}$$

Now Theorem 3.2 (applied to θ) guarantees that θ has a fixed point in $\overline{U}$ (in fact in U from (3.3) with $\lambda = 1$). $\square$

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