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ON AN OPTIMAL DECOMPOSITION IN ZYGMUND SPACES

ALBERTO FIORENZA AND MIROSLAV KRBEC

Abstract. An optimal decomposition formula for the norm in the Orlicz space $L(\log L)^\alpha$ is given. New proofs of some results involving $L(\log L)^\alpha$ spaces are given and the decomposition is applied to apriori estimates for elliptic partial differential equations with the right-hand side in Zygmund classes.

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1. Introduction

A lot of attention has always been paid to various expressions for the norm in Orlicz spaces (Luxemburg norm, "dual" norm, Amemyia norm, see, e.g., [24], [28], [25]); indeed, the definition of a norm is one of very peculiar features of these spaces, which sometimes makes things rather difficult and – by the rule of thumb – frequently prevents one from straightforward usage of the L^p-space techniques and approach.

In general, the idea of decomposition has turned out to be highly fruitful in many areas of analysis. It is often helpful to consider functions decomposed into suitable elementary pieces which are easier to handle. In connection with extrapolation procedures there is even a chance to consider not only these small pieces in a given space, but also to treat them as elements of spaces "in proximity" of the considered space. We shall see that it is natural to arrive at $L(\log L)^\alpha$ via the scale of L^p-spaces with $p > 1$, breaking functions into suitable little pieces belonging to "better" L^p-spaces.

This has been done at the abstract extrapolation theory level in Jawerth and Milman [22], see also Milman [26]. The identification of the result of the Σ-method applied to L^p-spaces is done via including the K-functional into the considerations and then employing the fact that Zygmund spaces can also be interpreted as special Lorentz–Zygmund spaces (see [1, Chapter 4]).

A functional analysis argument leads to the same expression in Edmunds and Triebel [7] for the range $1 < p < \infty$. Here, the key estimate for extrapolation characterization of $L^p(\log L)^{-\alpha}$, $1 < p < \infty$ and $\alpha > 0$, is the estimate $c_1\|f\|_{p_1} \leq \|f\|_{p_1,p} \leq c_2\|f\|_{p_2}$ with suitable $p_1 < p_2 < p$, relating the Lorentz and Lebesgue norms. Then a duality argument is used for spaces $L^p(\log L)^\alpha$, $\alpha > 0$. The case

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$p = 1$ in [8] requires an extra effort since $L_{\exp t^{1/\alpha}}$ is not the predual of $L(\log L)^\alpha$ (cf. also Remark 3.7). In this connection let us also recall Fiorenza [9], where it has been shown in another situation that in some cases the norm in a dual space can be expressed in terms of a suitable decomposition.

The plan of the paper is the following: In Section 2 we introduce the notation and recall some relevant result framing our considerations. Section 3 contains the main results and their proofs. In the concluding Section 4 we give some examples showing how the decomposition norm can be employed to get straightforward proofs of some results involving $L \log L$-spaces and some new apriori estimates for elliptic partial differential equations.

2. Preliminaries

In the sequel we assume that Ω is an open bounded subset of $\mathbb{R}^N$ of Lebesgue measure $|\Omega| = 1$. If not otherwise specified, all the norms and all the spaces are tacitly assumed to concern Lebesgue measurable functions defined a.e. in Ω. All positive constants whose exact value is not important for our purposes are denoted by c, occasionally with additional subscripts within the same formula or the same proof.

Let either $\alpha \in \mathbb{R}$ and $1 < r < \infty$ or $\alpha \geq 0$ and $r = 1$. The *Orlicz space* $L^r(\log L)^\alpha$ is defined as the Banach space of all measurable functions f on Ω such that the following (Luxemburg) norm

$$\|f\|_{L^r(\log L)^\alpha} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f|}{\lambda} \right)^r \log^\alpha \left(e + \left(\frac{|f|}{\lambda} \right)^r \right) dx < 1 \right\}$$

is finite. If $\alpha = 0$ then the space $L^r(\log L)^\alpha$ reduces to the Lebesgue space L^r whose norm is denoted simply by $\|\cdot\|_r$. If $r = 1$ and $\alpha > 0$, we do speak about the *Zygmund classes*. We shall not discuss the case $r = \infty$, $\alpha < 0$; this leads to exponential Orlicz spaces $L^{\exp t^{1/\alpha}}$ (and gives justification to alternative notation $L^\infty(\log L)^\alpha$ for them). Sometimes another notation is employed: for $\beta > 0$, the space EXP_β (or simply EXP if $\beta = 1$) is defined as the Banach space of all measurable functions f on Ω such that

$$\|f\|_{\mathrm{EXP}_\beta} = \inf \left\{ \lambda > 0 : \int_\Omega e^{(|f|/\lambda)^\beta} \, dx < 2 \right\}$$

is finite. Later we shall make use of the well-known extrapolation characterization of EXP_β as of the space consisting of all f such that

$$\sup_{k \in \mathbb{N}} \frac{\|f\|_k}{k^{1/\beta}} < \infty \tag{2.1}$$

(see, e.g., [14, Exercise 17, p. 279]). Observe that the sup on the left hand side is an equivalent norm in EXP_β.

We use the notation $\lambda_j = (2^j)' = (1 - 2^{-j})^{-1}$, $j \in \mathbb{N}$. For later use let us recall the formulas from [7] and [8] for the norm in $L^r(\log L)^\alpha$: if $1 \leq r < \infty$,

$J \in \mathbb{N}$, then the functional

$$\|g\|_{r,\alpha} = \inf_{|g|=\sum_{j=J}^{\infty} g_j} \left(\sum_{j=J}^{\infty} 2^{j\alpha} \|g_j\|_{r\lambda_j}^r \right)^{1/r} \tag{2.2}$$

defines a norm in $L^r(\log L)^\alpha$ equivalent to the Luxemburg norm. Clearly, any sequence (g_j) such that for some j the function g_j is not in $L^{r\lambda_j}$ makes no contribution to the computation of the infimum, therefore without loss of generality we may assume that $g_j \in L^{r\lambda_j}$ for all $j \geq J$. Plainly, we can assume that $J = 1$. Also, we shall restrict ourself to representations of $|g|$ as sums of non-negative functions g_j and we shall consider only non-negative functions in more general representations of type $|g|^{1/q} = \sum |g_j|^{1/q}$ $(q \geq 1)$ later; it is easy to see that the infimum remains to be the same (provided the powers have sense). We call such decompositions *admissible* and in the sequel all the infima over decompositions of $|g|$ will be taken over admissible decompositions without an explicit recalling this assumption. Note that some more delicate considerations about similar decompositions into sign-changing functions can be found in [9].

The *non-increasing rearrangement* of a non-negative measurable function f on Ω is defined as $f^*(t) = \inf\{\lambda > 0;\, m(f,\lambda) \leq t\}$, $t > 0$, where $m(f,\lambda) = |\{x \in \Omega;\, f(x) > \lambda\}|$, or, alternatively, in terms of the function f, as $f^*(t) = \sup_{|E|=t} \operatorname*{ess\,inf}_{x\in E} f(x)$, $t \in (0,1]$ (see [12]). These definitions yield functions, which are equal up to a countable subset of $(0,1)$, thus the difference between them can be neglected as far as their integration is concerned. It can be easily proved that f^* is non-negative and non-increasing on $(0,1)$ and vanishing for $t \geq 1$. Moreover, Orlicz spaces are *rearrangement-invariant*, that is, the norm of a function in an Orlicz space is preserved when considering its non-increasing rearrangement. For more details about non-increasing rearrangements we refer to [1], [19], [33], [34].

We refer to [23], [28], [16], and [24] for the theory of Orlicz spaces and integral operators acting on them.

Remark 2.1. Let us recall, for completeness, various known formulas for norms of a function $f \in L(\log L)$, all of them equivalent to the Luxemburg norm of f; for simplicity, we restrict ourselves here to the case $\Omega = [0,1]$. The list follows:

$$\inf\left\{\lambda > 0 : \int_0^1 (|f(t)|/\lambda) \log(A + (|f(t)|/\lambda))\, dt < B\right\} \quad \text{for any } A \geq 1, B > 0;$$

$$\sup\left\{ \int_0^1 f(t)g(t)\, dt : \|g\|_{\mathrm{EXP}} \leq 1\right\} \quad \text{(the Orlicz norm, or the dual norm)};$$

$$\int_0^1 \left(\frac{1}{t}\int_0^t f^*\right) dt;$$

$$\int_0^1 f^*(t)\log(1/t)\,dt;$$

$$\int_0^1 f(t)\log\Big(e+\frac{f(t)}{\int_0^1 f}\Big)\,dt;$$

$$\inf\Big\{\frac{1}{\lambda}+\int_0^1 \lambda|f|\log(e+\lambda|f|)\,dx:\ \lambda>0\Big\}\qquad\text{(the Amemyia norm).}$$

The first expression is clear from the well-known imbedding theorems in Orlicz spaces and the second can be found, e.g., in [24]. The norms in the third and fourth line in terms of f^* appear for instance in [1, Chapter 4], and their equivalence follows from Herz's theorem (see, e.g., [1, Chapter 3, Theorem 3.8]). The last two expressions for the norm can be found, for instance, in [3] and [25].

Remark 2.2. The norms $\int_0^1 f^*(t)\log(1/t)\,dt$ above, and/or, more generally, $\int_0^1 f^*(t)[\log(1/t)]^\alpha\,dt$ in $L(\log L)^\alpha$ give information about the right compensation for the growth rate of the L_p-norm of a function in $L(\log L)^\alpha$ as $p\to 1$. Namely, we have

$$\int_0^1 f^*(t)\big[\log(1/t)\big]^\alpha\,dt \le \|f\|_p\bigg(\int_0^1\Big(\log(1/t)\Big)^{\alpha p'}\,dt\bigg)^{1/p'}$$

$$=\|f\|_p\bigg(\int_0^\infty \xi^{\alpha p'}e^{-\xi}\,d\xi\bigg)^{1/p'}=\|f\|_p[\Gamma(\alpha p'+1)]^{1/p'}=\|f\|_p(\alpha p')^{1/p'}[\Gamma(\alpha p')]^{1/p'},$$

where Γ is the gamma-function. According to Stirling's formula we have

$$[\Gamma(\alpha p')]^{1/p}\sim\Big[\sqrt{2\pi}(\alpha p')^{\alpha p'-1/2}e^{-\alpha p'}\Big]^{1/p'}\sim(p')^{\alpha-1/(2p')}\sim(p')^\alpha$$

as $p'\to\infty$. Altogether

$$\|f\|_{L(\log L)^\alpha}\le c(p')^\alpha\|f\|_p\qquad\text{as}\quad p\to 1 \tag{2.3}$$

with some c independent of p and f.

The following lemma is a simple observation about the relation between Luxemburg norms in different Orlicz spaces. The function Φ appearing in the statement can be any Young function (i.e., any real function on $[0,\infty)$, continuous, increasing, convex, and such that $\lim_{t\to 0}\Phi(t)/t=\lim_{t\to\infty}t/\Phi(t)=0$).

Lemma 2.3. *For any $p\ge 1$ and for any function $f\in L^{\Phi(t^p)}$ we have*

$$\|f\|_{\Phi(t^p)}=\|f^p\|_{\Phi(t)}^{1/p}.$$

The proof of Lemma 2.3 follows immediately by using the definition of the Luxemburg norm in a general Orlicz space L^Φ.

3. A decomposition Norm

Let $\mathcal{M}$ denote the set of all Lebesgue measurable functions on Ω.

Theorem 3.1. *For any $q \geq 1$ and $\alpha \geq 0$, the functional*

$$g \mapsto \|g\|_{1,\alpha,q} = \inf_{|g|=\left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|g_j\|_{\lambda_j}$$

has the following properties:

(i) $\|g\|_{1,\alpha,q} \geq 0, \quad g \in \mathcal{M}$;
(ii) $\|\lambda g\|_{1,\alpha,q} = |\lambda| \|g\|_{1,\alpha,q}, \quad g \in \mathcal{M}$;
(iii) $g \in \mathcal{M}$, $\|g\|_{1,\alpha,q} = 0 \Leftrightarrow g = 0$ *a.e. in* Ω;
(iv) $\|g+h\|_{1,\alpha,q} \leq \|g\|_{1,\alpha,q} + \|h\|_{1,\alpha,q}, \quad g, h \in \mathcal{M}$.

Proof. The proof of (i) is immediate, and (ii) is easy. As to (iii), we prove that if $g \in \mathcal{M}$, $\|g\|_{1,\alpha,q} = 0$, then $g = 0$ a.e. in Ω.

Let us consider first the easy case $q = 1$. Given $\varepsilon > 0$, there exists (g_j) such that

$$\|g\|_1 = \Big\| \sum_{j=1}^{\infty} g_j \Big\|_1 \leq \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j} < \varepsilon$$

and therefore we get the assertion. If $q > 1$, we make use of the following argument (in which, of course, we may assume $g \geq 0$) which we develop now only for $\alpha = 1$, $q = 2$; the remaining cases can be treated similarly.

Fix $0 < H < 1$ and a decomposition $g = \left(\sum_{j=1}^{\infty} g_j^{1/2}\right)^2$. We claim that $\Big|\{g > 20H\}\Big| \leq \Big|\bigcup_{j=1}^{\infty}\{g_j > (2/3)^j H\}\Big|$. Indeed, if this is not true, then there exists a set $E \subseteq \Omega$ of positive measure such that $g(x) > 20H$ for a.e. $x \in E$, $g_j(x) \leq (2/3)^j H$ for a.e. $x \in E$ and for every $j \in \mathbb{N}$. Therefore

$$g = \left(\sum_{j=1}^{\infty} g_j^{1/2}\right)^2 \leq \left[\sum_{j=1}^{\infty} \left((2/3)^j H\right)^{1/2}\right]^2 < 20H \qquad \text{for a.e. } x \in E,$$

which is absurd. Our claim is proved and therefore we have also

$$\Big|\{g > 20H\}\Big| \leq \sum_{j=1}^{\infty} \Big|\{g_j > (2/3)^j H\}\Big|.$$

Now we estimate $|\{g_j > (2/3)^j H\}|$ for a particular choice of (g_j). Let us fix $0 < \varepsilon < 1$, and denote again by (g_j) a sequence such that

$$g = \left(\sum_{j=1}^{\infty} g_j^{1/2}\right)^2 \quad \text{and} \quad \sum_{j=1}^{\infty} 2^j \|g_j\|_{\lambda_j} < \varepsilon.$$

We have $2^j \|g_j\|_{\lambda_j} < \varepsilon$, $j \in \mathbb{N}$, and therefore

$$2^{j\lambda_j} \Big|\{g_j > (2/3)^j H\}\Big| \left((2/3)^j H\right)^{\lambda_j} \leq 2^{j\lambda_j} \int_{\{g_j > (2/3)^j H\}} g_j^{\lambda_j}\, dx < \varepsilon.$$

Our estimate follows because

$$\left|\{g_j > (2/3)^j H\}\right| \le \varepsilon 2^{-j\lambda_j}\left(\frac{3^j}{2^j H}\right)^{\lambda_j} \le \frac{\varepsilon(3/4)^j}{H^2}, \qquad j \in \mathbb{N}.$$

Hence

$$\left|\{g > 20H\}\right| \le \sum_{j=1}^{\infty}\left|\{g_j > (2/3)^j H\}\right| \le \frac{\varepsilon}{H^2}\sum_{j=1}^{\infty}\left(\frac{3}{4}\right)^j \le \frac{3\varepsilon}{H^2}.$$

This inequality implies $|\{g > 20H\}| = 0$ for all $0 < H < 1$. Therefore $g = 0$ for a.e. $x \in \Omega$ and (iii) is proved.

Finally, we prove the triangle inequality (iv). First, we have

$$\inf_{|g+h| \le \left(\sum_{j=1}^{\infty} z_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|z_j\|_{\lambda_j} \le \inf_{\substack{|g| = \left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q \\ |h| = \left(\sum_{j=1}^{\infty} h_j^{1/q}\right)^q}} \sum_{j=1}^{\infty} 2^{j\alpha}\|g_j + h_j\|_{\lambda_j}.$$

This is easy to see: if (g_j), (h_j) are such that

$$|g| = \left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q \quad \text{and} \quad |h| = \left(\sum_{j=1}^{\infty} h_j^{1/q}\right)^q,$$

then for $z_j = g_j + h_j$, $j \in \mathbb{N}$, we have

$$|g+h| \le |g| + |h| \le \left(\sum_{j=1}^{\infty} z_j^{1/q}\right)^q,$$

where we used Minkowski's inequality with an exponent between 0 and 1 (see, e.g., [19, n. 25, (2.11.5), p. 31]). Now we have

$$\begin{aligned}
\|g+h\|_{1,\alpha,q} &= \inf_{|g+h| = \left(\sum_{j=1}^{\infty} z_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|z_j\|_{\lambda_j} = \inf_{|g+h| \le \left(\sum_{j=1}^{\infty} z_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|z_j\|_{\lambda_j} \\
&\le \inf_{\substack{|g| = \left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q \\ |h| = \left(\sum_{j=1}^{\infty} h_j^{1/q}\right)^q}} \sum_{j=1}^{\infty} 2^{j\alpha}\|g_j + h_j\|_{\lambda_j} \\
&\le \inf_{\substack{|g| = \left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q \\ |h| = \left(\sum_{j=1}^{\infty} h_j^{1/q}\right)^q}} \left(\sum_{j=1}^{\infty} 2^{j\alpha}\|g_j\|_{\lambda_j} + \sum_{j=1}^{\infty} 2^{j\alpha}\|h_j\|_{\lambda_j}\right) \\
&= \inf_{|g| = \left(\sum_{j=1}^{\infty} g_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|g_j\|_{\lambda_j} + \inf_{|h| = \left(\sum_{j=1}^{\infty} h_j^{1/q}\right)^q} \sum_{j=1}^{\infty} 2^{j\alpha}\|h_j\|_{\lambda_j} \\
&= \|g\|_{1,\alpha,q} + \|h\|_{1,\alpha,q}
\end{aligned}$$

and therefore (iv) is also proved. $\square$

Next step is to prove that the norm defined in Theorem 3.1 is equivalent to the Luxemburg norm in $L(\log L)^\alpha$. We shall make use of

Lemma 3.2. *Let Φ be any Young function. If $\mathcal{N}(f)$ is a norm in $X = \{f \in L^1 : \mathcal{N}(f) < \infty\}$ and if*

$$f \in L^1 \mapsto \Big[\mathcal{N}(|f|^p)\Big]^{1/p}$$

is a norm in $L^{\Phi(t^p)}$ equivalent to the Luxemburg norm, then $X = L^{\Phi(t)}$ and

$$f \in L^1 \mapsto \mathcal{N}(f)$$

is a norm in $L^{\Phi(t)}$ equivalent to the Luxemburg norm.

Proof. We know that there exist $c_1, c_2 > 0$ such that

$$c_1\|f\|_{\Phi(t^p)} \le [\mathcal{N}(|f|^p)]^{1/p} \le c_2\|f\|_{\Phi(t^p)}, \qquad f \in L^{\Phi(t^p)}.$$

Therefore $c_1\||f|^{1/p}\|_{\Phi(t^p)} \le [\mathcal{N}(f)]^{1/p} \le c_2\||f|^{1/p}\|_{\Phi(t^p)}$, $f \in L^{\Phi(t)}$. In view of Lemma 2.3, $c_1\|f\|^{1/p}_{\Phi(t)} \le [\mathcal{N}(f)]^{1/p} \le c_2\|f\|^{1/p}_{\Phi(t)}$, $f \in L^{\Phi(t)}$. Hence $c_1^p\|f\|_{\Phi(t)} \le \mathcal{N}(f) \le c_2^p\|f\|_{\Phi(t)}$, $f \in L^{\Phi(t)}$. $\square$

Proposition 3.3. *Let $q > 1$. Then $\|\cdot\|_{1,\alpha,q}$ is equivalent to the Luxemburg norm of $L(\log L)^\alpha$.*

Proof. We have

$$\|g^q\|^{1/q}_{1,\alpha,q} = \inf_{|g|^q = \left(\sum_{j=1}^\infty g_j^{1/q}\right)^q} \Big(\sum_{j=1}^\infty 2^{j\alpha}\|g_j\|_{\lambda_j}\Big)^{1/q}$$

$$= \inf_{|g| = \sum_{j=1}^\infty \gamma_j} \Big(\sum_{j=1}^\infty 2^{j\alpha}\|\gamma_j^q\|_{\lambda_j}\Big)^{1/q} = \inf_{|g| = \sum_{j=1}^\infty \gamma_j} \Big(\sum_{j=1}^\infty 2^{j\alpha}\|\gamma_j\|^q_{q\lambda_j}\Big)^{1/q}.$$

By (2.2) the last quantity is equivalent to the Luxemburg norm in $L^q(\log L)^\alpha$. Therefore, by virtue of Lemma 3.2, we get the assertion. $\square$

Our next step will be to prove that Proposition 3.3 remains true for $q = 1$. At the same time we find an *optimal* decomposition of g into a sum; this is the core of the paper. We observe that one inequality of the equivalence of $\|g\|_{1,\alpha,1}$ to the Luxemburg norm can be proved as a corollary of Proposition 3.3. We shall use, however, another argument, independent of [7], and we prefer to prove both inequalities in the desired equivalence in a direct way, giving us additional information about how to decompose a function in order to get an expression equivalent to the norm in $L(\log L)^\alpha$. Let us recall the well-known fact that $f \in \mathrm{EXP}_\beta$ if and only if $\sup_{k\in\mathbb{N}} k^{-1/\beta}\|f\|_k < \infty$ and that the latter expression is an equivalent norm in EXP_β. (See, e.g., [14, Chapter VI, Exercise 17].) Moreover, this statement can be improved (see [6]), namely, $f \in \mathrm{EXP}_\beta$ if and only if $\sup_{k\in\mathbb{N}} k^{-1/\beta}\|f\chi_{I_k}\|_k < \infty$, where χ_{I_k} is the characteristic function of the interval (e^k, e^{k-1}), $k = 1, 2, \ldots$ (and one gets an equivalent norm in this way, too).

Theorem 3.4. *Let $\alpha > 0$. Then*

$$\|g\|_{1,\alpha,1} = \inf_{|g|=\sum_{j=1}^{\infty} g_j} \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j} \tag{3.1}$$

and the Luxemburg norm are equivalent in $L(\log L)^\alpha$.

Proof. We may assume that $g \geq 0$ and without loss of generality, we consider the case $\Omega = (0,1) \subset \mathbb{R}^N 1$. Moreover, in order to show the first inequality we may consider the "dual norm" in $L(\log L)^\alpha$ instead of the Luxemburg norm:

$$\|g\|_{L(\log L)^\alpha} = \sup \left\{ \int_0^1 f(t) g(t)\, dt : \|f\|_{\mathrm{EXP}_{1/\alpha}} \leq 1 \right\}.$$

Fix any decomposition $g = \sum_{j=1}^{\infty} g_j$. If $g \in L(\log L)^\alpha$, then for any $\varepsilon > 0$ let $f_\varepsilon \in \mathrm{EXP}_{1/\alpha}$, $\|f_\varepsilon\|_{\mathrm{EXP}_{1/\alpha}} \leq 1$, $f_\varepsilon \geq 0$, be such that

$$\|g\|_{L(\log L)^\alpha} - \varepsilon \leq \int_0^1 f_\varepsilon g\, dx.$$

If $g \notin L(\log L)^\alpha$, then one can put any positive number instead of $\|g\|_{L(\log L)^\alpha} - \varepsilon$.

Then

$$\begin{aligned} \|g\|_{L(\log L)^\alpha} - \varepsilon &= \sum_{j=1}^{\infty} \int_0^1 f_\varepsilon g_j\, dx = \sum_{j=1}^{\infty} \int_0^1 \frac{f_\varepsilon}{2^{j\alpha}} 2^{j\alpha} g_j\, dx \\ &\leq \sum_{j=1}^{\infty} \left\| \frac{f_\varepsilon}{2^{j\alpha}} \right\|_{2^j} 2^{j\alpha} \|g_j\|_{\lambda_j} \leq \|f_\varepsilon\|_{\mathrm{EXP}_{1/\alpha}} \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j} \leq \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j}, \end{aligned}$$

where the last but one inequality follows from the standard extrapolation characterization of $\mathrm{EXP}_{1/\alpha}$ (see (2.1)). This yields

$$\|g\|_{L(\log L)^\alpha} \leq \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j}.$$

Since the decomposition of g as $g = \sum_{j=1}^{\infty} g_j$ can be arbitrary, we also have

$$\|g\|_{L(\log L)^\alpha} \leq \inf_{|g|=\sum_{j=1}^{\infty} g_j} \sum_{j=1}^{\infty} 2^{j\alpha} \|g_j\|_{\lambda_j}.$$

On the other hand, if $g \in L \log L$, then we have the following estimate, in which k' denotes the Hölder conjugate exponent of k. To avoid unnecessary technicalities and clumsy formulas, for $k = 2$ read $(k-1)'$ in the following estimates as any number greater than $4' = 4/3$. We have

$$\|g\|_{L(\log L)^\alpha} \geq c \int_0^1 g^*(t) [\log(1/t)]^\alpha\, dt$$

$$
\begin{aligned}
&\geq c\sum_{k=2}^{\infty}\int_{e^{-k}}^{e^{-k+1}} g^*(t)[\log(1/t)]^{\alpha}\,dt \geq c\sum_{k=2}^{\infty} k^{\alpha} g^*(e^{-k+1})e^{-k}\\
&\geq c\sum_{k=2}^{\infty} k^{\alpha}\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}} e^{-k}\Big[e^{-k+1}(e-1)\Big]^{-\frac{1}{(k-1)'}}\\
&\geq c\sum_{k=2}^{\infty} k^{\alpha}\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}}\\
&= c\sum_{j=1}^{\infty}\sum_{k=2^j}^{2^{j+1}-1} k^{\alpha}\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}}\\
&\geq c\sum_{j=1}^{\infty}\sum_{k=2^j}^{2^{j+1}-1} 2^{j\alpha}\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}}\\
&\geq c\sum_{j=1}^{\infty} 2^{j\alpha}\sum_{k=2^j}^{2^{j+1}-1}\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}}.
\end{aligned}
$$

Now, by Hölder inequality,

$$
\begin{aligned}
&\Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(k-1)'}\,dt\Big)^{\frac{1}{(k-1)'}}\\
&\qquad\geq \Big(\int_{e^{-k+1}}^{e^{-k+2}} [g^*(t)]^{(2^{j+1})'}\,dt\Big)^{\frac{1}{(2^{j+1})'}}\Big[e^{-k+2}-e^{-k+1}\Big]^{[(2^{j+1})'/(k-1)']-1}.
\end{aligned}
$$

Elementary estimates show that the second term on the right-hand side is equivalent to a positive constant as $k\to\infty$. Hence the norm in $L^{(k-1)'}$ can be estimated by the norm in $L^{(2^{j+1})'}$ from below for $j = 2^j,\dots,2^{j+1}-1$ and the constant in the estimate remains the same for all $k\in\mathbb{N}$. Now it suffices to apply the triangle inequality to conclude that

$$
\begin{aligned}
\|g\|_{L(\log L)^{\alpha}} &\geq c\sum_{j=1}^{\infty} 2^{j\alpha}\Big(\int_{e^{-2^{j+1}+2}}^{e^{-2^j+2}} [g^*(t)]^{(2^{j+1})'}\,dt\Big)^{\frac{1}{(2^{j+1})'}}\\
&\geq c\inf_{|g|=\sum_{j=1}^{\infty} g_j}\sum_{j=1}^{\infty} 2^{j\alpha}\|g_j\|_{\lambda_j}.
\end{aligned}
$$

The theorem is proved. $\square$

Corollary 3.5. *Let* $\alpha > 0$. *Then*

$$\|f\|_{L(\log L)} \approx \sum_{j=1}^{\infty} 2^{j\alpha} \|f^*\|_{(2^j)',J_j}, \tag{3.2}$$

where $(2^j)' = 2^j/(2^j-1)$, $J_j = (e^{-2^{j+1}+2}, e^{-2^j+2})$, $j = 1, \dots$.

Remark 3.6. After proving Theorem 3.4 it is clear that every member of the last chain of inequalities of the proof is equivalent to the Luxemburg norm in $L \log L$. The decomposition in (3.2) is one of them and since it can be considered as a particular argument of the infimum in (3.1), we have actually found one of the "best", or "*optimal*" decompositions realizing the infimum in (3.1) up to an equivalence. Note that we do not know whether there exist decompositions which realize exactly the infimum.

Remark 3.7. Expression (3.2) permits, in particular, an interpretation of the norm in $L \log L$ as a norm of a special sequence in ℓ_1.

Let us recall the following notation for the spaces of sequences: c_0 is the space of all numerical sequences converging to zero, equipped with the sup norm, ℓ_1 is the space of all numerical absolutely convergent sequences (and the sum of the series is the norm), ℓ_∞ is the space of all numerical bounded sequences with the sup norm. The links between spaces of functions and spaces of sequences could be illustrated by the following scheme, also summarizing the other known results. We shall denote by *exp* the closure of L^∞ in EXP. Then we have

$$\begin{array}{ccc} g \in exp & f \in L\log L & h \in \mathrm{EXP} \\ \Updownarrow & \Updownarrow & \Updownarrow \\ \dfrac{\|g\|_{k,I_k}}{k} \in c_0 & 2^j \|f^*\|_{(2^j)',I_j} \in \ell_1 & \dfrac{\|h\|_{k,I_k}}{k} \in \ell_\infty \end{array}$$

These three equivalencies express some characterizations of the spaces in question. For the first one in "global" terms of $k^{-1}\|g\|_k$, see [4], [15], [13], [29]. Let us note that this follows directly in terms of $k^{-1}\|g\|_k$ and $k^{-1}\|g\|_{k,I_k}$ from the fact that $\sup k^{-1}\|f\|_k$ and $\sup k^{-1}\|f\|_{k,I_k}$ respectively is an equivalent norm in EXP. Indeed, if $g \in exp$, then for any $\varepsilon > 0$ there exists $h \in L^\infty$ such that $\sup k^{-1}\|g-h\|_{k,I_k} \le \varepsilon$. Since $k^{-1}\|h\|_{k,I_k} \to 0$ as $k \to \infty$, we get $k^{-1}\|g\|_{k,I_k} \le k^{-1}\|g-h\|_{k,I_k} + k^{-1}\|h\|_{k,I_k} < \varepsilon$ for large k. The second equivalence is contained in our Theorem 3.4, and the third one is due to Edmunds and Krbec ([6]).

Observe that the spaces in the first and the second row are the preduals of the spaces occurring in the second and the third row, respectively.

4. Applications

Let us begin by recalling the following result (see, e.g., [33, Theorem 3.3, p. 124]).

Theorem 4.1. *If A is a sublinear operator bounded in L^p, $1 < p \le p_0 < \infty$ such that*

$$\|Af\|_p \le c_1 p^{1/\beta} \|f\|_p, \qquad f \in L^p,\ p > 1,$$

for some $\beta > 0$ and c_1 independent of p and f, then $Af \in \mathrm{EXP}_\beta$ for all $f \in L^\infty$.

After our Theorem 3.4 we can prove a "dual" result, namely, we derive estimates of operators in logarithmic spaces starting from analogous estimates in Lebesgue spaces. A central role is played by the order of infinity of the constant in the L^p estimates.

Theorem 4.2. *If $\mathcal{H}$ is a subadditive operator acting on L^1 and such that*

$$\|\mathcal{H}g\|_{r(p)} \le \frac{c_1}{(p-1)^\beta} \|g\|_p, \qquad g \in L^p, \quad 1 < p \le p_0 < \infty,$$

for some constants $c_1 > 0$, $\beta \ge 0$, $1 < p_0 < \infty$, independent of p, g and for some $r : [1, p_0] \to [1, \infty)$ such that $p \le r(1)p \le r(p)$, for every $p \in [1, p_0]$, independent of g, then there is $c_2 > 0$ such that for any $\alpha \ge \beta$,

$$\|\mathcal{H}f\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} \le c_2 \|f\|_{L(\log L)^\alpha}, \qquad f \in L^\infty,\ f \ge 0,$$

the inequality is true with $c_2 = c_1$ if the norms in the logarithmic spaces considered are given by (2.2) *and* (3.1)*.*

Proof. Given $f \in L^\infty$, $f \ge 0$, fix a decomposition $f = \sum_{j=1}^\infty f_j$, $f_j \ge 0$, $j \in \mathbb{N}$. For any $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that we have, by using the inequality $r(1) \le r(1)p_0 \le r(p_0)$ and the assumption $f_j \ge 0$, $j \in \mathbb{N}$,

$$\begin{aligned}
\|\mathcal{H}f\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} &\le \Big\|\mathcal{H}\Big(\sum_{j=1}^{\nu} f_j\Big)\Big\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} \\
&+ c\Big\|\mathcal{H}\Big(\sum_{j=\nu+1}^{\infty} f_j\Big)\Big\|_{r(p_0)} \le \Big\|\mathcal{H}\Big(\sum_{j=1}^{\nu} f_j\Big)\Big\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} \\
&+ \frac{cc_1}{(p_0-1)^\beta}\Big\|\sum_{j=\nu+1}^{\infty} f_j\Big\|_{p_0} \le \Big\|\mathcal{H}\Big(\sum_{j=1}^{\nu} f_j\Big)\Big\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} + \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\mathcal{H}f\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} &\le \sup_{m\in\mathbb{N}} \Big\|\mathcal{H}\Big(\sum_{j=1}^{m} f_j\Big)\Big\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} \\
&\le \sup_{m\in\mathbb{N}} \Big\|\sum_{j=1}^{m} \mathcal{H}f_j\Big\|_{L^{r(1)}(\log L)^{r(1)(\alpha-\beta)}} \\
&\le \Big(\sum_{j=1}^{\infty} 2^{r(1)(\alpha-\beta)j} \Big\|\mathcal{H}f_j\Big\|_{r(1)\lambda_j}^{r(1)}\Big)^{1/r(1)} \\
&\le \sum_{j=1}^{\infty} 2^{(\alpha-\beta)j} \|\mathcal{H}f_j\|_{r(\lambda_j)} \le c_1 \sum_{j=1}^{\infty} \frac{2^{(\alpha-\beta)j}}{(2^{-j}\lambda_j)^\beta} \|f_j\|_{\lambda_j}
\end{aligned}$$

$$\leq c_1 \sum_{j=1}^{\infty} 2^{\alpha j} \|f_j\|_{\lambda_j},$$

where the passage from the third line to the fourth one is justified by (2.2). Observe, however, that one can drop the fourth line completely – the corresponding argument for passing from the third line to the fifth one is estimate (2.3). Taking the infimum over all admissible decompositions of f, we get the assertion. $\square$

Let us recall the (local) maximal operator (see, e.g., [10]). For any measurable function f on Ω, let

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad x \in \Omega,$$

where the supremum is taken over all cubes Q in Ω containing x and with the sides parallel to the coordinate axes. The following *Hardy–Littlewood–Wiener maximal theorem* is well known:

$$\|Mf\|_p \leq c(p)\|f\|_p, \qquad f \in L^p,\ 1 < p \leq \infty,$$

where $c(p) = \mathcal{O}((p-1)^{-1})$ as $p \to 1$ (See, e.g., [32, Chapter 1], for the quantitative behaviour of the norms resulting from the interpolation of weak type operators.)

Applying Theorem 4.2 to the maximal operator in the case $r(p) = p$, $\beta = 1$, we get the following well-known result (see [31], [32, p. 23]), as a consequence of the Hardy–Littlewood–Wiener maximal theorem and of the a.e. monotone convergence properties of the maximal operator:

Corollary 4.3. *Let $\alpha \geq 0$. Then there is $c > 0$ such that*

$$\|Mg\|_{L(\log L)^\alpha} \leq c\|g\|_{L(\log L)^{\alpha+1}} \ \text{ for all } \ g \ \text{ in } \ L(\log L)^{\alpha+1}.$$

We observe that it is known that Corollary 4.3 follows from an extrapolation procedure (see, e.g., [33]), but we point out here that the proof of Theorem 4.2 follows immediately from the definition of the norm in $L \log L$ in terms of decomposition.

A result equivalent to the previous one (see, e.g., [1]), that can be proved independent of the concept of a non-increasing rearrangement (see also [19]), is the limiting case of the Hardy inequality as $p \to 1$. We shall get it as a consequence of Theorem 4.1 when applied to the Hardy operator

$$f \mapsto Tf = \frac{1}{x} \int_0^x f(t)\, dt, \qquad x \in (0,1) \subset \mathbb{R}^1,$$

whose boundedness in L^p-spaces is given by the classical Hardy inequality

$$\|Tf\|_p \leq \frac{p}{p-1} \|f\|_p.$$

Corollary 4.4. *For any f in $L(\log L)^{\alpha+1}(0,1)$, $\alpha \geq 0$, we have*

$$\|Tf\|_{L(\log L)^{\alpha}(0,1)} \leq c\|f\|_{L(\log L)^{\alpha+1}(0,1)}$$

with $c > 0$ independent of f.

Let us now give another application of our extrapolation procedure in the theory of partial differential equations, and let us consider as a model the Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^N n$, $n \geq 3$, with $\mathcal{C}^{1,1}$-boundary

$$\begin{cases} \operatorname{div} A(x)\nabla u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where $f \in L^p(\Omega)$, $1 < p < n$, and $A = (a_{ij})_{i,j=1\dots n}$ is such that $a_{ij} \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^N n)$, $a_{ij} = a_{ji}$ and there exist $0 < \lambda_0 \leq \Lambda_0 < \infty$ such that

$$\lambda_0|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \Lambda_0|\xi|^2$$

for all $\xi \in \mathbb{R}^N n$ and for a.e. $x \in \Omega$. For the sake of simplicity, we shall also assume that $|\Omega| = 1$.

It is well known (see [2]) that there exists one and only one (weak) solution u to the b.v.p. (4.1) in the Sobolev space $W_0^{1,np/(n-p)}(\Omega)$. Moreover, this is not true when $p = 1$ (see [11], [17]).

If $f \in L(\log L)$, the existence of a solution u in $W_0^{1,n/(n-1)}$ can be proved (see [30]), and actually a better estimate of the norm of Du has been proved ([27]) by using duality techniques:

$$\||Du|\|_{L^{n/(n-1)}(\log L)^{-1+[n\alpha/(n-1)]}} \leq K\|f\|_{L(\log L)^\alpha} \tag{4.2}$$

for all $0 < \alpha \leq 1$. As an application of our Theorem 4.2, we prove estimate (4.2) for any $\alpha \geq (n-1)/n$, extending thus the validity of (4.2) for all $\alpha > 0$. We shall use the following result, due to Di Fazio ([5], see also [20], [21]):

Theorem 4.5. *Under the above assumptions on A, if $|F| \in L^p$, $1 < p < \infty$, then the Dirichlet problem*

$$\begin{cases} \operatorname{div} A(x)\nabla u = \operatorname{div} F & \textit{in} \quad \Omega, \\ u = 0 & \textit{on} \quad \partial\Omega, \end{cases}$$

has a unique solution and, moreover, there exists a constant c such that

$$\||\nabla u|\|_p \leq c\||F|\|_p. \tag{4.3}$$

We note that the constant c in (4.3) can be chosen independent of p if p stays away from 1 and ∞.

We can prove now

Lemma 4.6. *Let $1 < p < n$, $f \in L^p$, and let u be a solution of* (4.1). *Then*

$$\||\nabla u|\|_{np/(n-p)} \leq \frac{c}{(p-1)^{(n-1)/n}}\|f\|_p.$$

Proof. Let us consider a solution F to the equation $\operatorname{div} F = f$, which can be expressed explicitly in terms of the vector Riesz potential

$$F(x) = \frac{1}{n\omega_n} \int\limits_{\Omega} \frac{x-y}{|x-y|^n} f(y)\, dy,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^N n$ (see [18]). By using the well-known estimates which come out from the Hardy–Littlewood–Wiener maximal theorem and the Sobolev inequality for Riesz potentials (see [34]) we get

$$\||F|\|_{np/(n-p)} \le \frac{c_1}{(p-1)^{(n-p)/n}} \|f\|_p \le \frac{c_2}{(p-1)^{(n-1)/n}} \|f\|_p,$$

which, by (4.3), yields

$$\||\nabla u|\|_{np/(n-p)} \le c_3 \||F|\|_{np/(n-p)} \le \frac{c_4}{(p-1)^{(n-1)/n}} \|f\|_p,$$

i.e., we get the assertion. □

We are going to prove

Corollary 4.7. *If $\alpha > 0$, $f \in L(\log L)^\alpha$, and let u be a solution of* (4.1). *Then the apriori estimate* (4.2) *holds for any $\alpha \ge (n-1)/n$.*

Proof. It is clear that the mapping $\mathcal{H}$ which maps a given $f \in L^p$ into $|Du| \in L^{np/(n-p)}$, where u is a solution of (4.1), is subadditive, satisfies the assumption of Theorem 4.2 with $r(p) = np/(n-p)$, $p_0 = n$ and $\beta = (n-1)/n$, by virtue of Lemma 4.6. Hence Theorem 4.2 applies and, after extrapolation, using a standard density argument, we get the inequality for all $f \in L(\log L)^\alpha$, $f \ge 0$.

Finally, writing $f = f^+ + f^-$, and calling u_+, u_- a solutions of (4.1) with f replaced by f^+, f^- respectively, we have

$$\begin{aligned}
\||Du|\|_{L^{n/(n-1)}(\log L)^{-1+[n\alpha/(n-1)]}} &= \||Du_+ + Du_-|\|_{L^{n/(n-1)}(\log L)^{-1+[n\alpha/(n-1)]}} \\
&\le \||Du_+|\|_{L^{n/(n-1)}(\log L)^{-1+[n\alpha/(n-1)]}} + \||Du_-|\|_{L^{n/(n-1)}(\log L)^{-1+[n\alpha/(n-1)]}} \\
&\le K\|f^+\|_{L(\log L)^\alpha} + K\|f^-\|_{L(\log L)^\alpha} \le 2K\|f\|_{L(\log L)^\alpha}. \qquad \square
\end{aligned}$$

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Authors' addresses:

A. Fiorenza
Dipartimento di Costruzioni e Metodi Matematici
in Architettura, Università degli Studi di Napoli "Federico II"
via Monteoliveto 3, 80134 Napoli
Italy
E-mail: fiorenza@unina.it

M. Krbec
Institute of Mathematics
Academy of Sciences of the Czech Republic,
Žitná 25, 115 67 Prague 1
Czech Republic
E-mail: krbecm@matsrv.math.cas.cz