Georgian Mathematical Journal
Volume 9 (2002), Number 2, 287–294

ORDINARY DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

TADEUSZ JANKOWSKI

Abstract. The method of lower and upper solutions combined with the monotone iterative technique is used for ordinary differential equations with nonlinear boundary conditions. Some existence results are formulated for such problems.

2000 Mathematics Subject Classification: 34A45, 34B15, 34A40.
Key words and phrases: Lower and upper solutions, monotone iterations, convergence, existence of solutions.

1. Introduction

In this paper, we shall consider the following differential problem

$$\begin{cases} x'(t) = f\Big(t, x(t)\Big), \quad t \in J = [0, T], \quad T > 0, \\ x(0) = g(x(T)), \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$.

It is well known that the monotone iterative technique is a powerful method used to approximate solutions of several problems (see, for example [5]). The purpose of this paper is to show that it can be applied successfully to problems of type (1). Assuming one-sided Lipschitz condition on f (with respect to the second variable) combined with the corresponding monotonicity conditions on g, it is shown that linear iterations converge to a solution of problem (1). Some comparison results are also formulated.

In many papers, the monotone iterative technique was applied to problem (1) when $g(u) = \lambda u + k$. If $\lambda = 0$, then (1) reduces to the initial value problem for differential equations and this case is considered, for example, in [6], [7], [9], [10]. If $\lambda = 1$ and $k = 0$, then we have a periodic boundary problem considered, for example, in [8], [10], while if $\lambda = -1$ and $k = 0$, then we have an anti-periodic boundary problem, see [10], [11], [12]. A general case where $\lambda, k \in \mathbb{R}$ is discussed in [1]. Nonlinear problems, more general than (1), are studied, for example, in [2], [3], [4].

2. Some General Facts

From Theorem 1.1 proven in [2] follows

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

Theorem 1. *Let $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$. Moreover, we assume that there exist functions $y_0, z_0 \in C^1(J, \mathbb{R})$ such that*

$$y_0(t) \le z_0(t), \quad y_0'(t) \le f(t, y_0(t)), \quad z_0'(t) \ge f(t, z_0(t)), \;\; t \in J,$$
$$y_0(0) \le g(s) \le z_0(0) \;\; \text{for} \;\; y_0(T) \le s \le z_0(T).$$

Then problem (1) *has at least one solution in* $\Delta = \{w \in C^1(J, \mathbb{R}) : y_0(t) \le w(t) \le z_0(t),\ t \in J\}$.

On the basis of Theorem 1 one can easily prove

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied and, moreover, let the function g be nondecreasing. Then problem* (1) *has, in the set Δ, the minimal and the maximal solution.*

As for the uniqueness of the solution of problem (1), the following result holds.

Theorem 3 (Kiguradze). *Let the conditions of Theorem 1 be fulfilled and, moreover,*

$$f(t, v) - f(t, u) \le h(t)(v - u) \;\; \text{for} \;\; t \in J, \;\; y_0(t) \le u \le v \le z_0(t), \tag{2}$$
$$g(v) - g(u) \le L(v - u) \;\; \text{for} \;\; y_0(T) \le u \le v \le z_0(T), \tag{3}$$

where $h : J \to \mathbb{R}$ is an integrable function and L is a nonnegative constant such that

$$L \exp\Big(\int_0^T h(s)\, ds\Big) < 1. \tag{4}$$

Then problem (1) *has, in the set Δ, a unique solution.*

Proof. The existence of a solution of (1) follows from Theorem 1. Thus it remains to prove the uniqueness. Let $x, \overline{x} \in \Delta$ be arbitrary two solutions of (1). We distinguish two cases.

Case 1. $x(t) \neq \overline{x}(t)$ for all $t \in J$. Indeed, without the loss of generality, we can assume that $p(t) = x(t) - \overline{x}(t) > 0$ for $t \in J$. Hence

$$p'(t) = f(t, x(t)) - f(t, \overline{x}(t)) \le h(t)p(t), \;\; t \in J,$$
$$p(0) = g(x(T)) - g(\overline{x}(T)) \le Lp(T),$$

by assumptions (2) and (3). Therefore,

$$0 < p(0) \le Lp(T) \le L \exp\Big(\int_0^T h(s)\, ds\Big) p(0).$$

By condition (4), this inequality yields $p(0) = 0$ and thus $p(t) = 0$ on J, which is a contradiction.

Case 2. There exists $t_0 \in J$ such that $x(t_0) = \overline{x}(t_0)$. If $t_0 = T$ or $t_0 = 0$, then

$$x(0) = g(x(T)) = g(\overline{x}(T)) = \overline{x}(0).$$

This and condition (2) prove that $x(t) = \overline{x}(t)$ on J, which is a contradiction. If $t_0 \in (0, T)$, then $x(t) = \overline{x}(t)$ on $[t_0, T]$. Hence $x(T) = \overline{x}(T)$, so $x(0) = \overline{x}(0)$ showing that $x(t) = \overline{x}(t)$ on J. It is a contradiction. This proves that problem (1) has, in Δ, a unique solution. It ends the proof. □

In the next two sections we are going to construct the solution of problem (1).

3. Case where g is Nondecreasing

A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (1) if

$$\begin{cases} u'(t) \leq f(t, u(t)), & t \in J, \\ u(0) \leq g(u(T)), \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Let $\Omega = \{u : \ y_0(t) \leq u \leq z_0(t), \ t \in J\}$ be a nonempty set.

We introduce the following assumptions for later use.

- (H_1) $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$;
- (H_2) y_0, $z_0 \in C^1(J, \mathbb{R})$ are a lower and an upper solution of (1), respectively, such that $y_0(t) \leq z_0(t), \ t \in J$;
- (H_3) there exists $M \geq 0$ such that $f(t, u) - f(t, v) \leq M[v - u]$ for $t \in J, \ y_0(t) \leq u \leq v \leq z_0(t)$;
- (H_4) g is nondecreasing on the interval $[y_0(T), z_0(T)]$.

Lemma 1. *Let Assumption H_1 hold. Assume that $u, v \in \Delta$ are a lower and an upper solution of problem* (1)*, respectively, and $u(t) \leq v(t)$ on J. Let Assumptions H_3, H_4 hold. Let $y, z \in C^1(J, \mathbb{R})$ and*

$$y'(t) = \widetilde{f}(t, y(t)) \equiv f(t, u(t)) - M[y(t) - u(t)], \ \ t \in J, \ \ y(0) = g(u(T)), \quad (5)$$

$$z'(t) = \overline{f}(t, z(t)) \equiv f(t, v(t)) - M[z(t) - v(t)], \ \ t \in J, \ \ z(0) = g(v(T)). \quad (6)$$

Then

$$u(t) \leq y(t) \leq z(t) \leq v(t), \ \ t \in J, \quad (7)$$

and y, z are a lower and an upper solution of problem (1)*, respectively.*

Proof. It is easy to see that problems (5) and (6) have their unique solutions y, $z \in C^1(J, \mathbb{R})$. Put $p = u - y$, so $p(0) \leq g(u(T)) - g(u(T)) = 0$, and $p'(t) \leq f(t, u(t)) - \widetilde{f}(t, y(t)) = -Mp(t)$. It gives $p(t) \leq 0$ on J so $u(t) \leq y(t)$, $t \in J$. Similarly, we get $z(t) \leq v(t)$, $t \in J$. Now let $q = y - z$. Then $q(0) = g(u(T)) - g(v(T)) \leq 0$, by Assumption H_4. Moreover, Assumption H_3 yields

$$q'(t) = f(t, u(t)) - f(t, v(t)) - M\Big[q(t) - u(t) + v(t)\Big] \leq -Mq(t).$$

This and the condition for $q(0)$ prove that $q(t) \le 0$ on J, so (7) holds.

Now we need to show that y, z are a lower and an upper solution of (1), respectively. Indeed, in view of Assumptions H_3 and H_4, we have

$$\begin{aligned} y'(t) &= \widetilde{f}(t, y(t)) - f(t, y(t)) + f(t, y(t)) \\ &\le f(t, y(t)) + M[y(t) - u(t)] - M[y(t) - u(t)] = f(t, y(t)), \\ z'(t) &= \overline{f}(t, z(t)) - f(t, z(t)) + f(t, z(t)) \\ &\ge f(t, z(t)) - M[v(t) - z(t)] - M[z(t) - v(t)] = f(t, z(t)) \end{aligned}$$

and

$$y(0) = g(u(T)) \le g(y(T)), \quad z(0) = g(v(t)) \ge g(z(T)).$$

This ends the proof. □

Note that if Assumptions H_1 to H_4 are satisfied, then by Lemma 1 the sequences $y_n, z_n \in C^1(J, \mathbb{R})$ $(n = 0, 1, \cdots)$ are defined uniquely so that for every natural n we have

$$\begin{cases} y'_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], \quad t \in J, \quad y_{n+1}(0) = g(y_n(T)), \\ z'_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], \quad t \in J, \quad z_{n+1}(0) = g(z_n(T)). \end{cases}$$

Theorem 4. *Assume that Assumptions H_1, H_2, H_3 and H_4 hold. Then*

$$\begin{gathered} y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \\ t \in J, \quad n = 0, 1, \ldots, \end{gathered} \tag{8}$$

and uniformly on J we have

$$\lim_{n\to\infty} y_n(t) = y(t), \quad \lim_{n\to\infty} z_n(t) = z(t),$$

where y and z are the minimal and the maximal solution of problem (1) *in* Δ.

Proof. Using Lemma 1, by mathematical induction we can get (8). Indeed, the limits of sequences $\{y_n\}, \{z_n\}$ are solutions of problem (1). By Theorem 2, problem (1) has, in Δ, extremal solutions. To finish the proof, it is enough to show that y and z are the minimal and the maximal solution of (1). To do it, we need to show that if w is any solution of (1) such that $y_0(t) \le w(t) \le z_0(t)$, $t \in J$, then $y_0(t) \le y(t) \le w(t) \le z(t) \le z_0(t)$, $t \in J$. Assume that for some k, $y_k(t) \le w(t) \le z_k(t)$, $t \in J$. Put $p = y_{k+1} - w$, $q = w - z_{k+1}$, so $p(0) = g(y_k(T)) - g(w(T)) \le 0$, $q(0) = g(w(T)) - g(z_k(T)) \le 0$. Moreover,

$$\begin{aligned} p'(t) &= f(t, y_k(t)) - M[y_{k+1}(t) - y_k(t)] - f(t, w(t)) \\ &\le M[w(t) - y_k(t)] - M[y_{k+1}(t) - y_k(t)] = -Mp(t), \\ q'(t) &= f(t, w(t)) - f(t, z_k(t)) + M[z_{k+1}(t) - z_k(t)] \le -Mq(t), \end{aligned}$$

by Assumptions H_3. It shows $p(t) \le 0$, $q(t) \le 0$ on J, so $y_{k+1}(t) \le w(t) \le z_{k+1}(t)$ on J. Hence, by the method of mathematical induction, we have that $y_n(t) \le w(t) \le z_n(t)$ for $t \in J$ and for all natural n. Taking the limit $n \to \infty$, we get the assertion of Theorem 4. □

4. Case where g is Nonincreasing

Functions u, $v \in C^1(J, \mathbb{R})$ are called weakly coupled (w.c.) lower and upper solutions of problem (1) if

$$\begin{cases} u'(t) \le f(t, u(t)), \quad t \in J, \ \ u(0) \le g(v(T)), \\ v'(t) \ge f(t, v(t)), \quad t \in J, \ \ v(0) \ge g(u(T)). \end{cases}$$

Let us introduce the following assumptions.

(H_5) y_0, $z_0 \in C^1(J, \mathbb{R})$ are w.c. lower and upper solutions of problem (1) and $y_0(t) \le z_0(t)$, $t \in J$;

(H_6) g is nonincreasing on the interval $[y_0(T), z_0(T)]$;

(H_7) there exist nonnegative constants A, M and an integrable function $K : J \to \mathbb{R}$ such that

$$-M[v-u] \le f(t,v) - f(t,u) \le K(t)[v-u] \ \text{ for } \ t \in J, \qquad y_0(t) \le u \le v \le z_0(t), \tag{9}$$

$$-A[v-u] \le g(v) - g(u) \le 0 \ \text{ for } \ y_0(T) \le u \le v \le z_0(T), \tag{10}$$

and

$$A \exp\Big(\int_0^T K(s)\, ds\Big) < 1. \tag{11}$$

Lemma 2. *Assume that Assumption H_1 is satisfied. Let u, $v \in \Delta$ be w.c. lower and upper solutions of (1) and $u(t) \le v(t)$, $t \in J$. Let Assumptions H_3, H_6 hold. Let $y, z \in C^1(J, \mathbb{R})$ and*

$$\begin{cases} y'(t) = f(t, u(t)) - M[y(t) - u(t)], \quad t \in J, \ \ y(0) = g(v(T)), \\ z'(t) = f(t, v(t)) - M[z(t) - v(t)], \quad t \in J, \ \ z(0) = g(u(T)). \end{cases}$$

Then

$$u(t) \le y(t) \le z(t) \le v(t), \ \ t \in J, \tag{12}$$

and y, z are w.c. lower and upper solutions of problem (1).

Proof. Note that y and z are well defined. Put $p = u - y$, $q = z - v$. Then $p(0) \le g(v(T)) - g(v(T)) = 0$, $q(0) \le g(u(T)) - g(u(T)) = 0$, and

$$\begin{aligned} p'(t) &\le f(t, u(t)) - f(t, u(t)) + M[y(t) - u(t)] = -Mp(t), \\ q'(t) &\le f(t, v(t)) - M[z(t) - v(t)] - f(t, v(t)) = -Mq(t), \ \ t \in J. \end{aligned}$$

It yields $p(t) \le 0$, $q(t) \le 0$, $t \in J$, so $u(t) \le y(t)$, $z(t) \le v(t)$, $t \in J$. Now let $p = y - z$, so $p(0) = g(v(T)) - g(u(T)) \le 0$, by Assumption H_6. In view of Assumption H_3, we see that

$$p'(t) = f(t, u(t)) - f(t, v(t)) - M[y(t) - u(t) - z(t) + v(t)] \le -Mp(t).$$

Hence $p(t) \le 0$ on J, so (12) holds.

By Assumptions H_3 and H_6, we have

$$\begin{aligned} y'(t) &= f(t,u(t)) - M[y(t)-u(t)] - f(t,y(t)) + f(t,y(t)) \\ &\le f(t,y(t)) + M[y(t)-u(t)] - M[y(t)-u(t)] = f(t,y(t)), \\ z'(t) &= f(t,v(t)) - M[z(t)-v(t)] - f(t,z(t)) + f(t,z(t)) \ge f(t,z(t)), \end{aligned}$$

and

$$y(0) = g(v(T)) \le g(z(T)), \quad z(0) = g(u(T)) \ge g(y(T)).$$

It proves that y, z are w.c. lower and upper solutions of problem (1).

The proof is complete. $\square$

We see that if the assumptions of Lemma 2 are satisfied then the sequences y_n, $z_n \in C^1(J,\mathbb{R})$ $(n = 0, 1, \cdots)$ are defined uniquely so that for every natural n we have

$$\begin{cases} y'_{n+1}(t) = f(t,y_n(t)) - M[y_{n+1}(t) - y_n(t)], \quad t \in J, \quad y_{n+1}(0) = g(z_n(T)), \\ z'_{n+1}(t) = f(t,z_n(t)) - M[z_{n+1}(t) - z_n(t)], \quad t \in J, \quad z_{n+1}(0) = g(y_n(T)). \end{cases}$$

Theorem 5. *Assume that Assumptions H_1, H_5, H_7 hold. Then problem* (1) *has a unique solution $x \in \Delta$,*

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \qquad t \in J, \quad n = 0, 1, \dots, \tag{13}$$

and uniformly on J we have

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} z_n(t) = x(t).$$

Proof. It is easy to verify that all assumptions of Theorem 3 are satisfied, so problem (1) has a unique solution $x \in \Delta$. To finish the proof, it is enough to show that the sequences y_n, z_n converge to x. Using Lemma 2, we can prove (13), by mathematical induction. It follows from the standard argument that $\{y_n\}, \{z_n\}$ converge uniformly to their respective limit functions. Let $y(t) = \lim\limits_{n\to\infty} y_n(t)$, $z(t) = \lim\limits_{n\to\infty} z_n(t)$. Then

$$\begin{cases} y'(t) = f(t,y(t)), \quad t \in J, \quad y(0) = g(z(T)), \\ z'(t) = f(t,z(t)), \quad t \in J, \quad z(0) = g(y(T)) \end{cases} \tag{14}$$

and $y_0(t) \le y(t) \le z(t) \le z_0(t)$, $t \in J$. Put $p = z - y$, so $p(t) \ge 0$ on J. Hence $p(0) = g(y(T)) - g(z(T)) \le Ap(T)$, by Assumption H_6. Moreover,

$$p'(t) = f(t,z(t)) - f(t,y(t)) \le K(t)p(t), \quad t \in J.$$

This inequality yields

$$0 \le p(0) \le Ap(T) \le Ap(0) \exp\left(\int_0^T K(s)\, ds \right)$$

since

$$0 \le p(t) \le p(0) \exp\Big(\int\limits_0^t K(s)\,ds\Big), \ \ t \in J.$$

By condition (11), $p(0) = 0$, and thus $p(t) = 0$, $t \in J$. This proves that $y(t) = z(t)$ on J, so y and z are solutions of problem (1). Since problem (1) has a unique solution x, we have $y = z = x$.

It ends the proof. □

Remark. The results of this paper remain true if the constant $-M$ is replaced by an integrable on J function M.

Acknowledgment

The author expresses thanks to Prof. I. Kiguradze for useful comments and suggestions.

References

1. T. Jankowski, Boundary value problems for ODEs. (To appear).
2. I. Kiguradze and B. Půža, On some boundary value problems for a system of ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **12**(1976), 2139–2148; English transl.: *Differ. Equations* **12**(1976), 1493–1500.
3. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30**(1987), 3–103; English transl.: *J. Sov. Math.* **43**(1988), 2259–2339.
4. I. Kiguradze, On systems of ordinary differential equations and differential inequalities with multi–point boundary conditions. (Russian) *Differentsial'nye Uravneniya* **33**(1997), 646–652; English transl.: *Differ. Equations* **33**(1997), 649–655.
5. G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, Monotone iterative techniques for nonlinear differential equations. *Pitman, Boston*, 1985.
6. V. Lakshmikantham, Further improvements of generalized quasilinearization method. *Nonlinear Anal.* **27**(1996), 223–227.
7. V. Lakshmikantham, S. Leela, and S. Sivasundaram, Extensions of the method of quasilinearization. *J. Optim. Theory Appl.* **87**(1995), 379–401.
8. V. Lakshmikantham, N. Shahzad, and J. J. Nieto, Methods of generalized quasilinearization for periodic boundary value problems. *Nonlinear Anal.* **27**(1996), 143–151.
9. V. Lakshmikantham and N. Shahzad, Further generalization of generalized quasilinearization method. *J. Appl. Math. Stochastic Anal.* **7**(1994), No. 4, 545–552.
10. V. Lakshmikantham and A. S. Vatsala, Generalized quasilinearization for nonlinear problems. *Mathematics and its Applications,* 440. *Kluwer Academic Publishers, Dordrecht,* 1998.
11. Y. Yin, Remarks on first order differential equations with anti-periodic boundary conditions. *Nonlinear Times Digest* **2**(1995), No. 1, 83–94.
12. Y. Yin, Monotone iterative technique and quasilinearization for some anti-periodic problems. *Nonlinear World* **3**(1996), 253–266.

(Received 10.04.2001; revised 11.02.2002)

Author's address:
Technical University of Gdańsk
Department of Differential Equations
11/12 G. Narutowicz Str., 80–952 Gdańsk
Poland
E-mail: tjank@mifgate.mif.pg.gda.pl