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GEOMETRY OF MODULUS SPACES

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Abstract. Let ϕ be a modulus function, i.e., continuous strictly increasing function on $[0, \infty)$, such that $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(x + y) \leq \phi(x) + \phi(y)$ for all x, y in $[0, \infty)$. It is the object of this paper to characterize, for any Banach space X, extreme points, exposed points, and smooth points of the unit ball of the metric linear space $\ell^\phi(X)$, the space of all sequences (x_n), $x_n \in X$, $n = 1, 2, \dots$, for which $\sum \phi(\|x_n\|) < \infty$. Further, extreme, exposed, and smooth points of the unit ball of the space of bounded linear operators on ℓ^p, $0 < p < 1$, are characterized.

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0. Introduction. Let $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a continuous function. We call ϕ a modulus function if:

(i) $\phi(x) = 0$ if and only if $x = 0$;
(ii) ϕ is increasing;
(iii) $\phi(x + y) \leq \phi(x) + \phi(y)$.

The functions $\phi(x) = x^p$, $p \in (0, 1)$, and $\phi(x) = \ln(1 + x)$ are modulus functions.

For a modulus function ϕ, we let ℓ^ϕ denote the space of all real-valued sequences (x_n) for which $\sum \phi(|x_n|) < \infty$. For $x, y \in \ell^\phi$, $d(x, y) = \sum \phi(|x_n - y_n|)$ is a metric on ℓ^ϕ. For $x \in \ell^\phi$ we let $\|x\|_\phi$ denote $d(x, 0)$. The space $(\ell^\phi, \|\|_\phi)$ is a metric linear space. These spaces were initiated by Ruckle [4].

Throughout this paper, R denotes the set of real numbers. If X is a Banach space, X^* will denote the dual of X. If $x^* \in X^*$ and $x \in X$, we let $\langle x^*, x\rangle$ denote the value of x^* at x. We let ℓ^p denote the space of all (real) sequences (x_n) for which $\sum |x_n|^p < \infty$, $0 < p < \infty$. For $x \in \ell^p$, we let

$$\|x\|_p = \begin{cases} (\sum |x_i|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sum |x_i|^p & \text{if } 0 < p < 1. \end{cases}$$

So $\|x\|_1$ is the 1-norm of x in ℓ^1. For $p = \infty$, ℓ^∞ is the space of all bounded (real) sequences. If $x \in \ell^\infty$, we let $\|x\|_\infty = \sup_i |x_i|$.

Let us summarize the basic properties of $(\ell^\phi, \| \|_\phi)$ in

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Theorem A. *Let ϕ be any modulus function. Then:*
(1) *$(\ell^\phi, \| \|_\phi)$ is a complete metric linear space.*
(2) *If $\|x\|_\phi \leq \phi(a)$, then $\|x\|_1 \leq a$.*
(3) *$\ell^\phi \subseteq \ell^1$, and the inclusion map $I : \ell^\phi \longrightarrow \ell^1$ is continuous.*
(4) *If $\phi(1) = 1$, then for every $x \in \ell^\phi$ there exists $r > 0$ such that $\|rx\|_\phi = 1$.*
(5) *There exist α and a in $[0, \infty)$ such that $\phi(x) > \alpha x$ for all $x \in [0, a)$.*

Proof. The proof of (1) is in [4]. Statements (2) and (3) are easy to handle. Statement (5) is in [5]. So we prove only (4).

There are two cases:either $\|x\|_\phi < 1$ or $\|x\|_\phi > 1$. If $\|x\|_\phi > 1$, define $F : [0, 1] \longrightarrow [0, \infty)$ by $F(t) = \|tx\|_\phi$. Then F is continuous with $F(0) = 0$ and $F(1) > 1$. By the intermediate value theorem there is $r \in (0, 1)$ such that $F(r) = 1$. Hence $\|rx\|_\phi = 1$. The other case follows from statement (2) and the assumption $\phi(1) = 1$. $\square$

Let X be a Banach space. A linear mapping T:$\ell^\phi \longrightarrow X$ is called bounded if there exists $\lambda > 0$ such that $\|Tx\| \leq \lambda$ for all x in ℓ^ϕ for which $\|x\|_\phi \leq 1$. We let $L(\ell^\phi, X)$ denote the space of all bounded linear operators on ℓ^ϕ with values in X. We let $(\ell^\phi)^*$ denote $L(\ell^\phi, R)$. For $T \in L(\ell^\phi, X)$ we set $\|T\| = \sup\{\|Tx\| : \|x\|_\phi \leq 1\}$. For the case $0 < p < 1$ we let $B(\ell^p, \ell^p)$ denote the space of linear operators on ℓ^p for which $\|Tx\|_p \leq \lambda \|x\|_p$ for all $x \in \ell^p$ with some λ depending on T. Since $a|b|^p = \left|a^{\frac{1}{p}} b\right|^p$ for $a > 0$, it follows that $\sup\{\|Tx\|_p : \|x\|_p \leq 1\} = \inf\{\lambda : \|Tx\|_p \leq \lambda \|x\|_p$ for all $x \in \ell^p\}$. Hence $B(\ell^p, \ell^p) = L(\ell^p, \ell^p)$.

For a modulus function ϕ and a Banach space X, we set $\ell^\phi(X) = \{(x_n) : x_n \in X$ and $\sum \phi(\|x_n\|) < \infty\}$.If $x = (x_n) \in \ell^\phi(X)$, then we define $\|x\|_\phi = \sum \phi(\|x_n\|)$. It is easy to check that $(\ell^\phi(X), \|\|_\phi)$ is a complete metric linear space.

Extreme points of the unit ball of $L(\ell^p, \ell^p)$, $1 < p < \infty$, have been studied extensively by many authors ([6]–[10] and others). A full characterization of extreme points of the unit ball of $L(\ell^p, \ell^p)$, $1 < p < \infty$, is still an open problem.

In this paper we characterize extreme,exposed, and smooth points of the unit balls of ℓ^ϕ, $\ell^\phi(X)$ and $L(\ell^p, \ell^p)$, $0 < p < 1$.

1. Basic Structure of Spaces $\ell^\phi(X)$. Throughout this paper we will assume that:

(i) ϕ is strictly increasing;
(ii) $\phi(1) = 1$.

Let M denote the class of all modulus functions satisfying **(i)** and **(ii)**. We set $(\ell^\phi(X))^* = L(\ell^\phi, R)$, where R is the set of real numbers.

Theorem 1.1. *Let $\phi \in M$ and X be any Banach space. Then $[\ell^\phi(X)]^*$ is isometrically isomorphic to $\ell^\infty(X^*)$.*

Proof. Let $F \in \ell^\infty(X^*)$. So $F = (x_1^*, x_2^*, \dots)$ with $x_i^* \in X^*$ and $\sup_i \|x_i^*\| < \infty$. Define $\widetilde{F} : \ell^\phi(X) \longrightarrow R$ such that for $x = (x_i) \in \ell^\phi(X), \widetilde{F}(x) = \sum \langle x_i, x_i^* \rangle$.

Hence $\left|\widetilde{F}(x)\right| \le \sum \|x_i\| \, \|x_i^*\| \le \|F\| \sum \|x_i\|$. Now for any function ϕ in M one can easily show that $\ell^\phi(X) \subseteq \ell^1(X)$. Further, if $\|f\|_\phi = 1$, then $\|f\|_1 \le 1$. Thus

$$\left\|\widetilde{F}\right\| \le \|F\| \tag{$*$}$$

On the other hand, if $\widetilde{F} \in [\ell^\phi(X)]^*$, then we define x_i^* in X^* as : $x_i^*(x) = \widetilde{F}(0, 0, \ldots, 0, x, 0, \ldots)$ where x appears in the ith coordinate. Set $F = (x_1^*, x_2^*, \ldots)$. Then since $\sup\limits_i \|x_i^*\| \le \left\|\widetilde{F}\right\|$, we obtain $F \in \ell^\infty(X^*)$ and $\|F\|_\infty \le \left\|\widetilde{F}\right\|$. This together with $(*)$ gives $\|F\|_\infty = \left\|\widetilde{F}\right\|$. Thus the mapping $J : \ell^\infty(X^*) \longrightarrow [\ell^\phi(X)]^*$, $J(F) = \widetilde{F}$ is linear onto and an isometry. This ends the proof. $\square$

As a consequence we get

Corollary 1.2. $(\ell^\phi)^* = \ell^\infty$.

Remark 1. If $\phi(x+y) < \phi(x)+\phi(y)$ for any $x > 0$, $y > 0$, then there are some elements x of ℓ^ϕ such that there is no x^*in ℓ^∞ for which $\langle x, x^* \rangle = \|x\| \, \|x^*\|$. Indeed, if $\|x\|_\phi = 1$, then the continuity of ϕ, being strictly increasing and $\phi(1) = 1$, implies that $\|x\|_1 = 1$ unless x has only one nonzero coordinate. So for x with more than one nonzero terms there cannot exist x^* in ℓ^∞ which attains its norm at x. However, if x has only one nonzero coordinate, then $\|x\|_1 = \|x\|_\phi$, if $\|x\|_\phi = 1$ and such x^* exists.

2. Geometry of $B_1(\ell^\phi(X))$**.** A point x of a set K of a metric linear space E is called extreme if there exist no y and z in K such that $y \ne z$ and $x = \frac{1}{2}(y+z)$. The point x in $B_1(E)$ is called exposed if there exists $f \in B_1(E^*)$ such that $f(x) = d(x, 0)$, and $f(y) < d(y, 0)$ for all y in $B_1(E)$, $y \ne x$. We call x a smooth point of $B_1(E)$ if there exists a unique $f \in B_1(E^*)$ such that $f(x) = d(x, 0)$.

In this section we will characterize extreme, exposed, and smooth points of $B_1(\ell^\phi(X))$ for any Banach space X.

Theorem 2.1. *Let $\phi \in M$. The following statements are equivalent:*

(i) *f is an extreme point of $B_1(\ell^\phi(X))$.*

(ii) *$f(n) = 0$ for all n except for one coordinate, say, $f(n_0)$, and $f(n_0)$ is an extreme point of $B_1(X)$.*

Proof. **(i)$\longrightarrow$(ii)**. Let f be extreme and, if possible, assume that f does not vanish at n_1 and n_2. Define

$$g(n) = \begin{cases} f(n), & n \ne n_1, n_2, \\ \frac{\|f(n_1)\| + \|f(n_2)\|}{\|f(n_1)\|} f(n_1), & n = n_1, \\ 0, & n = n_2, \end{cases}$$

$$h(n) = \begin{cases} f(n), & n \ne n_1, n_2, \\ \frac{\|f(n_1)\| + \|f(n_2)\|}{\|f(n_2)\|} f(n_2), & n = n_2, \\ 0, & n = n_1. \end{cases}$$

Then $g \neq h$. Further,

$$\|g\|_\phi = \sum \phi(\|g(n)\|) \leq \sum \phi \|f(n)\| \leq 1.$$

Similarly, $\|h\|_\phi \leq 1$. Now

$$f = \frac{\|f(n_1)\|}{\|f(n_1)\| + \|f(n_2)\|}\, g + \frac{\|f(n_2)\|}{\|f(n_1)\| + \|f(n_2)\|}\, h = tg + (1-t)h, \quad 0 < t < 1,$$

where $t = \frac{\|f(n_1)\|}{\|f(n_1)\| + \|f(n_2)\|}$.

Hence f is not an extreme point. Thus f must be of the form

$$f(n) = \delta_{nn_0} \cdot x_0,$$

where δ_{ij} stands for the Kronecker's delta.

Now we claim that x_0 is an extreme point of $B_1(X)$. Indeed, $\|f\|_\phi = 1 = \phi(\|x_0\|)$. Since ϕ is strictly increasing, we have $\|x_0\| = 1$. If x_0 is not an extreme point, then $x_0 = \frac{1}{2}(y+z)$ for some y and z in $B_1(X)$. Then one can construct f_1 and f_2 in $B_1(\ell^\phi(X))$ such that $f = \frac{1}{2}(f_1 + f_2)$. Hence x_0 must be extreme.

Conversely: **(ii)⟶(i)**. Let $f(n) = \delta_{nn_0} \cdot x$ with x an extreme point of $B_1(X)$. If f is not extreme, then there exist g and h in $B_1(\ell^\phi(X))$ such that $f = \frac{1}{2}(g+h)$. But then $g(n_0) = h(n_0) = x$ since x is an extreme point. Since $\|x\| = 1$ and ϕ is strictly increasing and $\phi(1) = 1$, we have $g(n) = h(n) = 0$ for all $n \neq n_0$. But this implies that $f = g = h$, and f is extreme. This ends the proof of the theorem. □

As a corollary, we get

Theorem 2.2. *A point x is an extreme point of $B_1(\ell^\phi)$ if and only if $x_n = 0$ for all n except for one n, say, n_0, and $|x_{n_0}| = 1$.*

Proof. Take R for X. □

As for the exposed points we have

Theorem 2.3. *Let $f \in B_1(\ell^\phi(X))$. The following statements are equivalent:*
(i) *f is an exposed point.*
(ii) *$f(n) = \delta_{nn_0} \cdot x$ and x is an exposed point of $B_1(X)$.*

Proof. **(i)⟶(ii)**. Let f be exposed. Then f is an extreme point. Hence $f(n)\delta_{nn_0} \cdot x$ with x an extreme point of $B_1(X)$. If x is not exposed, then for every $x^* \in B_1(X^*)$ with $x^*(x) = 1$, there exists $z \in B_1(X)$ such that $x^*(z) = 1$ and $z \neq x$. Now let $F \in [\ell^\phi(X)]^* = \ell^\infty(X^*)$ such that $\|F\| = 1$, and $F(f) = 1$. In that case, if $F = (x_1^*, x_2^*, \dots)$, then $F(f) = x_{n_0}^*(x) = 1$. Since x is not exposed, there exists $z \neq x$ in $B_1(X)$ such that $x_{n_0}^*(z) = 1$. But then $F(g) = 1$, where $g(n) = \delta_{nn_0} \cdot z$ and f is not exposed. Hence x must be exposed in $B_1(X)$.

Conversely: **(ii)⟶(i)**. Let $f = \delta_{nn_0} \cdot x$ with x exposed in $B_1(X)$. If x^* is the functional that exposes x, then one can easily see that $F(n) = \delta_{nn_0} \cdot x^*$ is the functional that exposes f. This ends the proof. □

Theorem 2.3 readily implies

Theorem 2.4. *An element f is an exposed point of $B_1(\ell^\phi)$ if and only if f is extreme.*

As for smooth points we have

Theorem 2.5. *$B_1(\ell^\phi(X))$ has no smooth points for any Banach space X.*

Proof. Let $f \in B_1(\ell^\phi(X))$. If there exists $F \in B_1(\ell^\infty(X^*))$ such that $F(f) = 1$, then by Remark 1 f must have only one nonzero coordinate, say, $f(n_0) = x_{n_0}$. Since $\phi(1) = 1$, it follows that $\|x_{n_0}\| = 1$. Consider the functionals:

$$F_1(n) = \delta_{nn_0} \cdot x^* \text{with } \; x^*(x_{n_0}) = 1,$$
$$F_2(n) = \delta_{nn_0} \cdot x^* + \delta_{n,n_0+1} \cdot z^* \quad \text{with} \quad \|z^*\| = 1.$$

Then, F_1 and F_2 are two different elements in $B_2(\ell^\phi(X))$ such that $F_1(f) = F_2(f) = 1$. Thus f is not smooth. This ends the proof. □

It follows that $B_1(\ell^\phi)$ has no smooth points.

3. Geometry of $B_1(L(\ell^p))$, $0 < p < 1$. The characterization of the extreme points of $B_1(L(\ell^p))$, $1 < p < \infty$, is still an open difficult problem [1], [3]. In this section we give a complete description of the extreme points and the exposed points of the unit ball of $L(\ell^p)$ for $0 < p < 1$. We remark that Kalton, [2], studied isomorphisms of and some classes of operators on ℓ^p, $0 < p < 1$.

Theorem 3.1. *Let $T \in B_1(L(\ell^p))$, $0 < p < 1$. The following statements are equivalent:*

(i) *T is an extreme point.*

(ii) *T is a permutation on the basis elements.*

Proof. **(ii)$\longrightarrow$(i)**. Let T be a permutation of the basis elements $e_1, e_2, \ldots$. If T is not extreme, then there exists $S \in B_1(L(\ell^p))$ such that $S \neq 0$ and $\|S \pm T\| \leq 1$. Thus $\|(S \pm T)x\| \leq 1$ for all x in $B_1(\ell^p)$. Thus, in particular, $\|Se_n \pm Te_n\| \leq 1$ for all n. Since $\|S\| \leq 1$, it follows that Te_n is not extreme for those n for which $Se_n \neq 0$. Since $S \neq 0$, we get a contradiction, noting that $\pm e_n$ are the extreme points of ℓ^p. Thus T must be extreme.

Conversely: **(i)$\longrightarrow$(ii)**. Let T be an extreme element of $B_1(L(\ell^p))$, but, if it is possible, assume there exists k_0 such that Te_{k_0} is not a basis element and hence not an extreme element of $B_1(\ell^p)$. Thus there exists z in $B_1(\ell^p)$ such that $\|Te_{k_0} \pm z\| \leq 1$. Define the operator S on ℓ^p as $S = e_{k_0} \otimes z$, so $Sx = x_{k_0}z$. Then

$$\begin{aligned} \|(S \pm T)x\|_p &= \left\|(S \pm T)(\textstyle\sum x_i e_i)\right\|_p = \left\|\textstyle\sum x_i (S \pm T)e_i\right\|_p \\ &\leq \textstyle\sum |x_i|^p \, \|(S \pm T)e_i\|_p \, . \end{aligned}$$

But

$$(S \pm T)e_i = \begin{cases} Te_0, & i \neq k_0, \\ z \pm Te_{k_0}, & i = k_0. \end{cases}$$

Thus in either case we have $\|(S \pm T)e_i\| \leq 1$ for all i. So $\|(S \pm T)x\| \leq \sum |x_i|^p$. It follows that $\|S \pm T\| \leq 1$, and T is not extreme, which contradicts the assumption. So T must be a permutation. This ends the proof. $\square$

To characterize the exposed points, we need

Theorem 3.2. *$L(\ell^p)$ is isometrically isomorphic to $\ell^\infty(\ell^p)$.*

Proof. Let $f \in \ell^\infty(\ell^p)$. Then $f : N \longrightarrow \ell^p$ with $\sup_n \|f(n)\|_p < \infty$. Define $T : \ell^p \longrightarrow \ell^p$, by $Tx = \sum x_k f(k)$. Then $\|Tx\|_p \leq \sum \|x_p f(k)\|_p \leq \sum |x_k|^p \|f(k)\|_p \leq \|f\|_\infty \|x\|_p$. Thus $\|T\| \leq \|f\|_\infty$. But $Te_k = f(k)$. So $\|f(k)\|_p = \|Te_k\|_p \leq \|T\|$. It follows that $\|f\|_\infty \leq \|T\|$ .Hence $\|f\|_\infty$=$\|T\|$.

On the other hand, let $T \in L(\ell^p)$. Define $f(n) = Te_p$. Then one can easily show that $f \in \ell^\infty(\ell^p)$ and $\|f\|_\infty$=$\|T\|$. This ends the proof. $\square$

Now for the exposed points we have

Theorem 3.3. *Let $T \in B_1(L(\ell^p))$. The following statements are equivalent:*
(i) *T is exposed.*
(ii) *T is extreme.*

Proof. That **(i)**$\longrightarrow$**(ii)** is immediate.

For the converse, let T be an extreme point. By Theorem 3.1, T is a permutation of the basis elements. Let f be the function corresponding to T as in Theorem 3.2. Thus $f(n) = \pm e_{k(n)}$. Define $G : L(\ell^p) \longrightarrow R$, $G(S) = \sum t_n \langle f(n), g(n)\rangle$, where $0 < t_n$, $\sum t_n = 1$, and g is the element in $\ell^\infty(\ell^p)$ that represents S as in Theorem 3.2. Then, G is bounded and $\|G\| \leq 1$. Further $G(T) = 1$. Now, if it is possible, assume there exists some S in $B_1(L(\ell^p))$ such that $G(S) = 1$. Then $\sum t_n \langle f(n), g(n)\rangle = 1$. This implies that $\langle f(n), g(n)\rangle = 1$. Since $f(n) = e_{k(n)}$, it follows that $g(n) = f(n)$, and so $S = T$. Hence T is exposed. This ends the proof. $\square$

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References

1. W. Deeb and R. Khalil, Exposed and smooth points of some classes of operators in $L(\ell^p)$, $1 \leq p < \infty$. *J. Funct. Anal.* **103**(1992), 217–228.
2. N. J. Kalton, Isomorphisms between L_p-functions. *J. Funct. Anal.* **42**(1981), 299–337.
3. R. Khalil, Smooth points of unit balls of operator and function spaces. *Demonstr. Math.* **29**(1996), 723–732.
4. W. Ruckle, FK spaces in which the sequence of coordinate vector is bounded. *Canadian J. Math.* **25**(1973), 973–978.
5. W. Deeb and R. Younis, Extreme points of a class of non-locally convex topological vector spaces. *Math. Rep. Toyama Univ.* **6**(1983), 95–103.

6. W. Deeb and R. Khalil, Exposed and smooth points of some classes of operators in $L(\ell^p)$. *J. Funct. Anal.* **103**(1992), 217–228.
7. R. Grzaslewicz, Extreme operators on 2-dimensional ℓ^p-spaces. *Coll. Math.* **44**(1981), 209–215.
8. R. Grzaslewicz, A note on extreme contractions on ℓ^p-spaces. *Portugal. Math.* **40**(1981), 413–419.
9. R. Khalil, A class of extreme contractions in $L(\ell^p)$. *Annali di Mat. Pura ed Applic.* **145**(1988), 1–5.
10. C.-H. Kan, A class of extreme L^p contractions, $p \neq 1, 2, \infty$. *Illinois J. Math.* **30**(1986), 612–635.

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