Georgian Mathematical Journal
Volume 9 (2002), Number 2, 325–330

ON COMMON FIXED POINTS

ZEQING LIU, M. S. KHAN, AND H. K. PATHAK

Abstract. Some fixed point theorems based on an asymptotic regularity condition have been obtained, which generalize the previously well-known results.

2000 Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: Metric spaces, asymptotically regular map, common fixed point.

1. Introduction

Throughout this paper, let $\mathfrak{R}^+$ denote the set of nonnegative real numbers, $W : \mathfrak{R}^+ \to \mathfrak{R}^+$ be a continuous function such that $0 < W(t) < t$ for all $t \in \mathfrak{R}^+ \backslash \{0\}$, f, g and h be selfmaps on a metric space (X, d). For a point $x_0 \in X$, if there exits a sequence $\{x_n\}$ in X such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \ldots$, then $O(f, g, h, x_0) = \{hx_n : n = 1, 2, \ldots\}$ is called the orbit of (f, g, h) at x_0. h is said to be orbitally continuous at x_0 if and only if it is continuous on $O(f, g, h, x_0)$. X is called orbitally complete at x_0 if and only if every Cauchy sequence in $O(f, g, h, x_0)$ converges in X. The pair (f, g) is said to be asymptotically regular (a.r.) with respect to h at x_0 if there exists a sequence $\{x_n\}$ in X such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \ldots$, and $d(hx_n, hx_{n+1}) \to 0$ as $n \to \infty$. For $x, y \in X$, define

$$M(x, y) = \max\Big\{d(hx, hy), d(hx, fx), d(hy, gy), d(hx, gy), d(hy, fx)\Big\},$$
$$N(x, y) = \max\Big\{d(hx, hy), d(hx, fx), d(hy, gy), \min\{d(hx, gy), d(hy, fx)\}\Big\}.$$

Sastry et al.[4] proved the following theorem.

Theorem 1.1. *Let f, g and h be selfmaps on a metric spaces (X, d) and $fh = hf$ or $gh = hg$. Suppose that there exits a point $x_0 \in X$ such that (f, g) is a.r. with respect to h at x_0, X is orbitally complete at x_0 and h is orbitally continuous at x_0. If for all $x, y \in X$,*

$$d(fx, gy) \leq rM(x, y) \tag{1.1}$$

holds, where $r \in (0, 1)$, then f, g and h have a unique common fixed point in X.

Ray [3] established the following result.

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

Theorem 1.2. *Let f, g and h be selfmaps on a complete metric space (X, d) and $fh = hf$, $gh = hg$, $f(X) \cup g(X) \subseteq h(X)$, and*

$$d(fx, gy) \leq d(hx, hy) - W(d(hx, hy)) \tag{1.2}$$

for all $x, y \in X$. If h is continuous, then f, g and h have a unique common fixed point in X.

The purpose of this paper is to extend Theorems 1.1 and 1.2 to a more general case. To this end, we need the following result due to Chang [1].

Theorem 1.3. *If $\varphi : fR^+ \to \mathfrak{R}^+$ is an upper semicontinuous function with $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, then there exists a strictly increasing, continuous function $\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $\varphi(t) \leq \psi(t) < t$ for $t > 0$.*

2. Main Results

Now we present our main theorems.

Theorem 2.1. *Let f, g and h be selfmaps on a metric space (X, d) and $fh = hf$ or $gh = hg$. Suppose that there exits a point $x_0 \in X$ such that (f, g) is said a.r. with respect to h at x_0, X is orbitally complete at x_0, and h is orbitally continuous at x_0. If*

$$d(fx, gy) \leq M(x, y) - W(M(x, y)) \tag{2.1}$$

holds for all $x, y \in X$, then f, g and h have a unique common fixed point in X.

Proof. Since $\{f, g\}$ is a.r. with respect to h at x_0, there exists a sequence $\{x_n\}$ in X such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \ldots$, and

$$dn = d(hx_n, hx_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.2}$$

In order to show that $\{hx_n\}$ is a Cauchy sequence, it is sufficient to show that $\{hx_{2n}\}$ is a Cauchy sequence. Suppose that the result is not true. Then there is be a positive number ε such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > 2k$ and

$$d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon. \tag{2.3}$$

For each integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (2.3) so that

$$d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \varepsilon. \tag{2.4}$$

Then for each even integer $2k$,

$$d(hx_{2m(k)}, hx_{2n(k)}) \leq d(hx_{2m(k)-2}, hx_{2n(k)}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

From (2.2), (2.3), (2.4) and the above inequality we have

$$d(hx_{2m(k)}, hx_{2n(k)}) \to \varepsilon \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Using the triangular inequality, we obtain

$$\begin{aligned}
&\left|d(hx_{2m(k)+1}, hx_{2n(k)}) - d(hx_{2m(k)}, hx_{2n(k)})\right| \le d_{2m(k)},\\
&\left|d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)+1}, hx_{2n(k)})\right| \le d_{2n(k)},\\
&\left|d(hx_{2m(k)+2}, hx_{2n(k)+1}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})\right| \le d_{2m(k)+1},
\end{aligned}$$

and

$$\left|d(hx_{2m(k)+2}, hx_{2n(k)}) - d(hx_{2m(k)+1}, hx_{2n(k)})\right| \le d_{2m(k)+1}.$$

From (2.2), (2.5) and the above inequalities, we get

$$\begin{aligned}
\varepsilon &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)}) = \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1})\\
&= \lim_{k\to\infty} d(hx_{2m(k)+2}, hx_{2n(k)+1}) = \lim_{k\to\infty} d(hx_{2m(k)+2}, hx_{2n(k)}).
\end{aligned}$$

Now, it follows from (2.1) that

$$\begin{aligned}
&d(hx_{2n(k)+1}, hx_{2m(k)+2}) = d(fx_{2n(k)}, gx_{2m(k)+1})\\
&\quad \le \max\Big\{d(hx_{2n(k)}, hx_{2m(k)+1}), d_{2n(k)}, d_{2m(k)+1},\\
&d(hx_{2n(k)}, hx_{2m(k)+2}), d(hx_{2m(k)+1}, hx_{2n(k)+1})\Big\}\\
&\quad - W\Big(\max\Big\{d(hx_{2n(k)}, hx_{2m(k)+1}), d_{2n(k)}, d_{2m(k)+1},\\
&d(hx_{2n(k)}, hx_{2m(k)+2}), d(hx_{2m(k)+1}, hx_{2n(k)+1})\Big\}\Big).
\end{aligned}$$

As $k \to \infty$, we have

$$\varepsilon \le \max\{\varepsilon, 0, 0, \varepsilon, \varepsilon\} - W\Big(\max\{\varepsilon, 0, 0, \varepsilon, \varepsilon\}\Big).$$

That is, $W(\varepsilon) \le 0$, which implies $\varepsilon = 0$, a contradiction. Hence $\{hx_n\}$ is a Cauchy sequence. Since X is (f, g, h)-orbitally complete at x_0, there exists $z \in X$ such that $hx_n \to z$ as $n \to \infty$. Now, applying (2.1) to $d(hx_{2n}, gz)$ and $d(fz, gx_{2n+1})$ and letting $n \to \infty$, we have

$$\begin{aligned}
d(z, gz) &\le \max\Big\{d(z, hz), d(hz, gz), d(z, gz)\Big\}\\
&\quad - W\Big(\max\Big\{d(z, hz), d(hz, gz), d(z, gz)\Big\}\Big)
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
d(fz, z) &\le \max\Big\{d(hz, z), d(hz, fz), d(z, fz)\Big\}\\
&\quad - W\Big(\max\Big\{d(hz, z), d(hz, fz), d(z, fz)\Big\}\Big)
\end{aligned} \tag{2.7}$$

respectively.

If $fh = hf$, then $fhx_{2n} = hfx_{2n} \to hz$ since h is orbitally continuous at x_0. By (2.1)

$$\begin{aligned} d(fhx_{2n}, gx_{2n+1}) \le \max\Big\{& d(hhx_{2n}, hx_{2n+1}), d(hhx_{2n}, fhx_{2n}), \\ & d_{2n+1}, d(hhx_{2n}, gx_{2n+1}), d(hx_{2n+1}, fhx_{2n})\Big\} \\ & - W\Big(\max\Big\{d(hhx_{2n}, hx_{2n+1}), d(hhx_{2n}, fhx_{2n}), \\ & d_{2n+1}, d(hhx_{2n}, gx_{2n+1}), d(hx_{2n+1}, fhx_{2n})\Big\}\Big). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(hz, z) \le d(hz, z) - W(d(hz, z)).$$

which implies $z = hz$.

If $gh = hg$, then $ghx_{2n+1} = hgx_{2n+1} \to hz$. Applying (2.1) to $d(x_{2n}, ghx_{2n+1})$ and letting $n \to \infty$, we obtain $z = hz$. By (2.6) and (2.7), we have $fz = gz = hz = z$. So z is a common fixed point of f, g and h. It is easy to show that z is a unique common fixed point. This completes the proof. □

Remark 2.1. (a) Taking $W(t) = (1 - r)t$, $r \in (0, 1)$, in Theorem 2.1, we obtain Theorem 1 of Sastry et al. [4].

(b) Liu [2] proved that there exist mappings $f, g, h : X \to X$ such that

$$d(fx, gy) \le \varphi(M(x, y))$$

for all $x, y \in X$, but they do not satisfy (1.1) for any $r \in (0, 1)$, where $f = g$, h is the identity mapping and $\varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is nondecreasing, right continuous and $vf(t) < t$ for $t > 0$. Wong [5] noted that the function φ is upper semicontinuous from the right. Observe that φ is nondecreasing. Then

$$\limsup_{s\to t} \varphi(s) = \lim_{\varepsilon\to 0} \sup_{s\in(t-\varepsilon,t+\varepsilon)} \varphi(s) = \lim_{\varepsilon\to 0} \sup_{s\in(t,t+\varepsilon)} \varphi(s) \le \lim_{\varepsilon\to 0} \varphi(t + \varepsilon) = \varphi(t)$$

for $t > 0$. In view of Theorem 1.3, we conclude that there exists a strictly increasing continuous function $\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $\varphi(t) \le \psi(t) < t$ for $t > 0$. Put $W(t) = t - \psi(t)$ for $t \ge 0$.Then W is continuous, $W(0) = 0$, $W(t) < t$ for $t > 0$ and

$$d(fx, gy) \le \varphi(M(x, y)) \le \psi(M(x, y)) = M(x, y) - W(M(x, y))$$

for $x, y \in X$. Thus we have shown that the class of mappings which satisfy (1.1) is a proper subclass of the class of mappings which satisfy (2.1). Hence Theorem 2.1 extends substantially the theorem of Sastry et al.[4].

Theorem 2.2. *Let f, g and h be selfmaps on a metric space (X, d) and $fh = hf$ or $gh = hg$. Suppose that there exists a point $x_0 \in X$ such that X is orbitally* *complete at x_0 and h is orbitally continuous at x_0. If at least one of the following inequalities*

$$d(fx, gy) \leq N(x, y) - W(N(x, y)), \tag{2.8}$$

$$\begin{aligned} d(fx, gy) \leq{} & \max\Big\{d(hx, hy), d(hx, fx), d(hy, gy)\Big\} \\ & - W\Big(\max\Big\{d(hx, hy), d(hx, fx), d(hy, gy)\Big\}\Big), \end{aligned} \tag{2.9}$$

$$d(fx, gy) \leq d(hx, hy) - W(d(hx, hy)) \tag{2.10}$$

holds for all $x, y \in X$, then f, g and h have a unique common fixed point in X.

Proof. Let $d_n = d(hx_n, hx_{n+1})$, $n = 0, 1, 2, \ldots$. Now suppose that (2.8) is true for the pair x_{2n}, x_{2n+1}, then

$$d_{2n+1} = d\Big(fx_{2n}, gx_{2n+1}) \leq \max\{d_{2n}, d_{2n+1}\} - W(\max\{d_{2n}, d_{2n+1}\}\Big)$$

which implies $d_{2n+1} \leq d_{2n}$. Similarly, if (2.9) or (2.10) is true, then, correspondingly, we obtain $d_{2n+1} \leq d_{2n}$. Similarly, we can show that $d_{2n} \leq d_{2n-1}$. Thus $\{d_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to a nonnegative real number, say b. It is easy to show that $d_n \leq d_n - W(d_n)$ for all n. Letting $n \to \infty$, we have $b \leq b - W(b)$, which implies $b = 0$. Thus $d_n \leq 0$ as $n \to \infty$, i.e., (f, g) is a.r. with respect to h at $x_0 \in X$. Now, as in the proof of Theorem 2.1, we can show that f, g and h have a unique common fixed point. This completes the proof. □

The following is an immediate consequence of Theorem 2.2.

Corollary 2.1. *Let f, g and h be selfmaps on a complete metric space (X, d), h be continuous, $f(X) \cup g(X) \subseteq h(X)$, $fh = hf$, $gh = hg$. If at least one of the conditions* (2.8), (2.9) *and* (2.10) *is satisfied, then f, g and h have a unique common fixed point.*

Remark 2.2. Corollary 2.1 extends Theorem 1 of Ray [3].

The following example shows that Theorem 2.2 extends properly Theorem 1 of Ray [3].

Example 2.1. Let $X = [0, 1) \cup \{2\}$ with the usual metric $|\cdot|$. Define $f, g, h : X \to X$ by $fx = gx = \frac{1}{3}x$ and $hx = x$ for $x \in X$. Take $x_0 = 2$ and $W(t) = \frac{1}{4}t$ for $t \geq 0$. Then $O(x_0, f, g, h) = \{\frac{2}{3^n} : n \geq 0\}$, X is orbitally complete at x_0, h is orbitally continuous at x_0 and

$$d(fx, gy) = \frac{1}{3}|x - y| \leq \frac{1}{4}|x - y| = d(hx, hy) - W(d(hx, hy))$$

for all $x, y \in X$. Therefore the conditions of Theorem 2.2 are satisfied. But Theorem 1 of Ray [3] is not applicable since X is not complete.

Remark 2.3. In recent years, several generalizations of the concept of commutativity such as weak commutative, compatibility and of baised maps have been introduced by different authors. We do not know if our results can be extended to these settings or not.

References

1. T. H. Chang, Fixed point theorems for contractive type set-valued mappings. *Math. Japon.* **38**(1993), 675–690.
2. Z. Liu, On Park's open questions and some fixed point theorems for general contractive type mappings. *J. Math. Anal. Appl.* **234**(1999), 165–182.
3. B. K. Ray, On common fixed points in metric spaces. *Indian J. Pure Appl. Math.* **19**(1988), 960–962.
4. K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao. and K. P. R. Rao, Common fixed points for asymptotically regular mappings. *Indian J. Pure Appl. Math.* **15**(1984), 849–854.
5. C. S. Wong, Maps of contractive type. *Fixed point theory and its applications* (*Proc. Sem., Dalhousie Univ., Halifax, N.S.,* 1975), 197–207. *Academic Press, New York,* 1976.

(Received 12.06.2001)

Author's addresses:

Zeqeing Liu
Department of Mathematics
Liaoning Normal University
Dalion
Liaoning 116022
P. R. China

M. S. Khan
Sultan Qaboos University, College of Science
Department of Mathematics and Statistics
P.O.Box 36, Al-Khod P.C. 123
Muscat, Sultanate of Oman
E-mail: mohammad@squ.edu.om

H. K. Pathak
Department of Mathematics
Kalyan Mahavidyalaya
Bhilai Nagar (CG) 490006
India
E-mail: sycomp@satyam.net.in