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ON A PROBLEM OF LINEAR CONJUGATION IN THE CASE OF NONSMOOTH LINES AND SOME MEASURABLE COEFFICIENTS

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Abstract. A boundary value problem of linear conjugation is considered for more general curves than those studied previously. A condition on the coefficient is found, under which the classical results are valid for these curves.

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1^0. Let Γ be a rectifiable closed Jordan curve dividing the plane into to domains D^+_Γ and D^-_Γ (it is assumed that $\infty \in D^-_\Gamma$). The direction that leaves the domain D^+_Γ lie on the left is assumed to be the positive direction on Γ. The line directed in this direction is denoted by Γ^+, whereas the line directed in the opposite direction is denoted by Γ^-. Occasionally, we will write Γ instead of Γ^+.

In the sequel, we will need classes of analytic functions $E_p(D^\pm_\Gamma)$ which are usually called Hardy–Smirnov classes. For boundary value problems the following definition of such classes is convenient.

We say that $\Phi(z) \in E_p(D_\Gamma)$, where $p > 0$ and D_Γ denotes either D^+_Γ or D^-_Γ, if:

a) $\Phi(z)$ is analytic in D_Γ;

b) $\Phi(\infty) = 0$ if $D_\Gamma = D^-_\Gamma$;

c) there exists a sequence of curves $\{\Gamma_n\}_{n=1}^\infty \in D_\Gamma$ such that $\infty \overline{\in} \Gamma_n$, $\Gamma_n \to \Gamma$ as $n \to \infty$ and

$$\sup_n \int_{\Gamma_n} |\Phi(z)|^p ds < \infty \tag{1.1}$$

By the convergence $\Gamma_n \to \Gamma$ we understand the same convergence as in [1], p. 203.

For $p \geq 1$, the above-defined class $E_p(D^-_\Gamma)$ coincides with the class which is often denoted by $\overset{\circ}{E}_p(D^-_\Gamma)$. We will not use this notation because we will need only functions $\Phi(z)$ vanishing at infinity.

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As usual, we denote by $(S_\Gamma\,\varphi(t))(\tau)$ and $(K_\Gamma\,\varphi(t))(z)$ the integrals (whose contour can be both closed and open):

$$(S_\Gamma\,\varphi(t))(\tau)=\frac{1}{\pi i}\int\limits_\Gamma \varphi(t)(t-\tau)^{-1}dt,\quad \tau\in\Gamma,$$

$$(K_\Gamma\,\varphi(t))(z)=\frac{1}{2\pi i}\int\limits_\Gamma \varphi(t)(t-z)^{-1}dt,\quad t\overline{\in}\Gamma.$$

It is assumed that $\varphi\in L(\Gamma)$. The first integral is understood in the sense of the Cauchy principal value (see, e.g., [2], p. 13). For these integrals we sometimes use the notations $S_\Gamma\,\varphi$ and $K_\Gamma\,\varphi$ or $S\varphi$ and $K\varphi$.

We say that $\Gamma\in R$ if for any $\varphi\in L_p(\Gamma)$, $p>1$, we have

$$\|S_\Gamma\,\varphi\|_{L_p(\Gamma)}\le M_p\|\varphi\|_{L_p(\Gamma)}.$$

If $\Gamma\in R$, $\varphi\in L_p(\Gamma)$, $\psi\in L_q(\Gamma)$, $q=p(p-1)^{-1}$, then

$$\int\limits_\Gamma (S_\Gamma\,\varphi)(t)\psi(t)dt=-\int\limits_\Gamma \varphi(t)(S_\Gamma\,\psi)(t)dt\quad\text{(Riesz equality)},\tag{1.2}$$

and if, in addition, Γ is a closed line, then

$$S_\Gamma^2\,\varphi=\varphi\tag{1.3}$$

(with respect to (1.2) and (1.3) for $\Gamma\in R$, see [3], [4], [5]).

In the sequel we will need one more well-known definition. We say that the positive measurable function $\rho(t)$ is a weight and write $\rho\in W_p(\Gamma)$, $p>1$, if

$$\left\|\rho S_\Gamma\,\rho^{-1}\varphi\right\|_{L_p(\Gamma)}\le M_p\|\varphi\|_{L_p(\Gamma)},\quad\forall\ \varphi\in L_p(\Gamma).$$

We will need the following properties of functions of the class $E_p(D_\Gamma)$, where $\Gamma\in R$:

(i_1) $\Big\{\Phi(z)=(K_{\Gamma^\pm}\Phi^\pm)(z),\ \ \Phi^\pm\in L_1(\Gamma)\Big\}\Longleftrightarrow\Big\{\Phi(z)\in E_1(D_\Gamma^\pm)\Big\}$.

(i_2) $\Big\{\Phi(z)=E_\delta(D_\Gamma^\pm),\ \ \delta>0,\ \ \Phi^\pm\in L_p(\Gamma),\ \ p>\delta\Big\}\Longleftrightarrow\Big\{\Phi(z)\in E_p(D_\Gamma)\Big\}$.

(i_3) $\Big\{\varphi\in L_\infty\Big\}\Longrightarrow\Big\{\big(\exp(K\varphi)(z)\big)-1\in E_\delta(D_\Gamma)\text{ for some }\delta>0\Big\}$.

(i_4) In (1.1) we can take as Γ_n the images of the circumferences $|w|=r_n$, $r_n\to 1$ for the conformal transformation of the circle $|w|<1$ to D_Γ.

If $D_\Gamma=D_\Gamma^+$, then (i_1) and (i_4) are valid for all rectifiable simple closed lines (see, e.g., [1], p. 208 and p. 203); (i_2) follows from Smirnov's theorem ([1], p. 264) and Havin's theorem ([6], p. 512); (i_3) is proved in [7], p. 68.

For $D_\Gamma=D_\Gamma^-$, these statements or their close analogs are used by various authors, but $E_p(D_\Gamma^-)$ is defined in different ways, which, for $p<1$, gives different classes. Therefore here we present the proofs the more so that they are very simple.

Take the conformal transformation $z=\omega_0(\zeta)=(\zeta-a)^{-1}+a$ and the inverse transformation $\zeta=(z-a)^{-1}+a$, where $a\in D_\Gamma^+$, $z\in D_\Gamma^-$, $\zeta\in D_{\Gamma_0}^+$, where Γ_0

is the image of Γ. (An analogous reasoning can be found in [5], p. 32.) We obviously obtain the equality

$$(K_{\Gamma^\pm}\varphi)(z)=(\zeta-a)\Big(K_{\Gamma_0^\mp}(\varphi(\omega_0(\tau)(\tau-a)^{-1})\Big)(\zeta) \tag{1.4}$$

and the inclusion

$$\Phi(z)\in E_p(D_\Gamma^-),\ p>0 \Longleftrightarrow \frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\in E_p(D_{\Gamma_0}^+). \tag{1.5}$$

Indeed,

$$\begin{aligned}(K_{\Gamma^\pm}\varphi)(\omega_0(\zeta))&=(2\pi i)^{-1}\int\limits_{\Gamma^\pm}\varphi(t)(t-\omega_0(\zeta))^{-1}dt\\ &=-(2\pi i)^{-1}\int\limits_{\Gamma_0^\mp}\Big(\frac{1}{\tau-a}+a-\frac{1}{\zeta-a}-a\Big)^{-1}(\tau-a)^{-2}\varphi(\omega_0(\tau))d\tau\\ &=(\zeta-a)\Big(K_{\Gamma_0^\mp}\varphi(\omega_0(\tau))(\tau-a)^{-1}\Big)(\zeta).\end{aligned}$$

We will prove (1.5). Since $a\overline{\in}\Gamma_n^0$, where Γ_n^0 are the images of the lines $\Gamma_n\in D_\Gamma^-$, from (1.1), for the mapping $\zeta=\omega_0^{-1}(z)$, we have for any analytic function $\Phi(z)$ in D_Γ^-

$$\begin{aligned}&\Big\{\Phi(z)\in E_p(D_\Gamma^-),\,p>0\Big\}\\ &\Longleftrightarrow\Big\{\sup_n\int\limits_{\Gamma_n^0}\Big|\Phi(\omega_0(\zeta))\Big|^p|d\zeta|<\infty,\Gamma_n^0\in D_{\Gamma_0^+},\ a\overline{\in}\Gamma_n^0,\ \Phi(\omega_0(a))=0\Big\}\\ &\Longleftrightarrow\Big\{\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\in E_p(D_{\Gamma_0^+})\Big\}.\end{aligned}$$

Thus (1.4) and (1.5) are proved. Hence we easily obtain (i_1), (i_2) and (i_3) for $D_\Gamma=D_\Gamma^-$. Indeed,

$$\begin{aligned}&\Big\{\Phi(z)\in E_1(D_\Gamma^-)\Big\}\Longleftrightarrow\Big\{\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\in E_1(D_{\Gamma_0}^+)\Big\}\\ &\Longleftrightarrow\Big\{\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))=\Big(K_{\Gamma_0^+}\Big(\frac{1}{\tau-a}\varphi(\omega_0(\tau)\Big)\Big)(\zeta)\Big\}\\ &\Longleftrightarrow\Big\{\Phi(z)=(\zeta-a)\Big(K_{\Gamma_0^+}(\tau-a)^{-1}\varphi(\omega_0(\zeta))\Big)(\zeta)=(K_{\Gamma^-}\varphi)(z)\Big\}.\end{aligned}$$

We have obtained (i_1). Let us show (i_2) for $D_\Gamma=D_\Gamma^-$:

$$\begin{aligned}&\Big\{\Phi(z)\in E_\delta(D_\Gamma^-),\ \Phi^-\in L_p(\Gamma),\ p>\delta\Big\}\\ &\Longleftrightarrow\Big\{\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\in E_\delta(D_{\Gamma_0}^+),\frac{1}{\tau-a}\Phi^+(\omega_0(\tau))\in L_p(\Gamma_0),\,p>\delta\Big\}\\ &\Longleftrightarrow\Big\{\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\in E_p(D_{\Gamma_0}^+)\Big\}\Longleftrightarrow\Big\{\Phi(z)\in E_p(D_\Gamma^-)\Big\}.\end{aligned}$$

Let us show (i_3) for $D_\Gamma = D_\Gamma^-$:

$$\Big\{\Phi(z) = (\exp K_{\Gamma^-}\varphi)(z) - 1,\ z\in D_\Gamma^-,\ \varphi\in L_\infty(\Gamma)\Big\}$$
$$\Longleftrightarrow \Big\{\Phi(\omega_0(\zeta)) = \Big(\exp(\zeta-a)\Big(K_{\Gamma_0^+}\varphi(\omega_0(\tau)(\tau-a)^{-1}\Big)(\zeta)\Big) - 1,\ a\in D_{\Gamma_0}^+,$$
$$(\tau-a)^{-1}\varphi(\omega_0(\tau))\in L_\infty(\Gamma)\Big\} \Longrightarrow \Big\{\Phi(\omega_0(\zeta))\in E_\delta(D_{\Gamma_0}^+),\ \varphi(\omega_0(a))=0\Big\}$$
$$\Longleftrightarrow (\zeta-a)^{-1}\Phi(\omega_0(\zeta))\in E_\delta(D_{\Gamma_0}^+) \Longleftrightarrow \Phi(z)\in E_\delta(D_\Gamma^-).$$

Next let us show (i_4) for $D_\Gamma = D_\Gamma^-$. Let Γ_0 be as above and Γ_n^0 be the images of the circumferences $|w| = r_n$, $r_n\to 1$, for the conformal transformation $\zeta = \omega_1(w)$ of the unit circle $|w|<1$ in $D_{\Gamma_0}^+$. Let further Γ_n be the images of the circumferences $|w| = r_n$ for the transformation $z = \omega_0(\omega_1(w))$, where $\omega_0(\zeta)$ is the same as above. It is obvious that

$$\sup_n \int_{\Gamma_n} |\Phi(z)|^p|dz| = \sup_n \int_{\Gamma_n^0} |\Phi(\omega_0(\zeta))|^p|\omega_0'(\zeta)||d\zeta|$$
$$= \sup_n \int_{\Gamma_n} |\Phi(\omega_0(\zeta))|^p|\zeta-a|^{-2}|d\zeta| \le \mathrm{const}\int_{\Gamma_n^0}\Big|\frac{1}{\zeta-a}\Phi(\omega_0(\zeta))\Big|^p||d\zeta| < \infty$$

(here we take into account that $a\overline{\in}\Gamma_n^0$).

2^0. Our boundary value problem of linear conjugation is formulated as follows:

Find a function $\Phi(z)\in E_p(D_\Gamma^\pm)$ whose angular boundary values satisfy the condition

$$\Phi^+(t) = G(t)\Phi^-(t) + f(t),\quad t\in\Gamma, \tag{2.1}$$

where $f\in L_p(\Gamma)$, $p>1$, and $G\in L_\infty(\Gamma)$ are the known functions.

In the case, in which $G(t)$ is piecewise-continuous and Γ is a Lyapunov curve, this problem was solved for the first time by B. Khvedelidze [9]. The results obtained subsequently are presented in [2] and other works of various authors. We will be concerned with the case of a measurable coefficient which was investigated by I. Simonenko [10] for Lyapunov lines.

The conditions of [10] can be written as followsin the form

$$(S_1)\quad 0 < \operatorname*{vrai\,inf}_{t\in\Gamma}|G(t)| < \operatorname*{vrai\,sup}_{t\in\Gamma}|G(t)| < \infty \tag{2.2}$$

(S_2) the function $\arg G(t) = \varphi(t)$ can be chosen so that $\varphi(t) = \varphi_1(t)+\varphi_2(t)$, where $\varphi_1(t)$ is continuous on Γ except perhaps for one point at which it has a first order discontinuity $\varphi_1(t_0-) - \varphi_1(t_0+) = 2\pi\varkappa$, where $\varkappa$ is an integer and $\varphi_1(t_0\mp)$ denotes the unilateral limits along the negative and the positive direction on Γ, respectively; $\varphi_2(t)$ satisfies the inequality

$$\operatorname*{vrai\,sup}_{t\in\Gamma}|\varphi_2(t)| < \frac{\pi}{\max(p,q)}, \quad q = p(p-1)^{-1}, \quad p > 1. \tag{2.3}$$

The function, whose argument satisfies (2.3) and modulus (2.2), is called sectorial. In the sequel, for p and q it will always be assumed that $p > 1$ and $q = p(p-1)^{-1}$.

For the briefness of our exposition we introduce yet another definition:

The statement $(A_\varkappa)$ or, simply, (A) is valid for problem (2.1) if:

1) all solutions of this problems, when they exist, are written in the traditional form

$$\Phi(z) = X(z)\Big(K_{\Gamma^+}(f/X^+)\Big)(z) + X(z)P_{\varkappa-1}(z), \tag{2.4}$$

where $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa-1$ and, in addition to this, it assumed that $P_k(z) \equiv 0$ for $k < 0$. By $X(z)$ we mean the following:

$$X(z) = \begin{cases} X_0(z) & \text{for} \quad z \in D_\Gamma^+, \\ (z-a)^{-\varkappa}X_0(z) & \text{for} \quad z \in D_\Gamma^-, \end{cases} \tag{2.5}$$

$$\begin{gathered} X_0(z) = \exp\Big(K_\Gamma \ln(t-a)^{-\varkappa}G(t)\Big)(z), \\ X_0(z) - 1 \in E_p(D_\Gamma^\pm), \quad X_0^{-1}(z) - 1 \in E_q(D_\Gamma^\pm), \end{gathered} \tag{2.6}$$

where $a \in D_\Gamma^+$, $\varkappa$ is the integer depending on $G(t)$ and called the index of the problem;

2) for $\varkappa \geq 0$ the problem is solvable unconditionally, while for $\varkappa < 0$ for the problem to be solvable it is necessary and sufficient that the condition

$$\int_\Gamma t^k f(t)(X^+(t))^{-1}dt = 0, \quad k = 0, 1, \dots, -\varkappa - 1. \tag{2.7}$$

be fulfilled.

I. Simonenko showed [10] that for Lyapunov lines, when the conditions (S_1) and (S_2) are fulfilled, the statement $(A_\varkappa)$ is true for problem (2.1). Later, other authors extended this result to sufficiently general lines (see, for example, [7], [8], [11]), but in that case the conditions imposed on the lines exclude the existence of cusps.

Our objective here is to modify condition (2.3) so that the classical result be valid for the lines with cusps.

Denote by Γ_{ab} a simple arc with ends a, b and directed from a to b. Denote by $\chi(\Gamma_{ab})$ the characteristic function of the set $\{t : t \in \Gamma_{ab}\}$.

We say that $\Gamma_{ab} \in R_A$ if it can be complemented to a closed contour $\Gamma^0_{ab} \in R$, $\Gamma_{ab} \subset \Gamma^0_{ab}$, so that the statement (A_0) be true for any function $[G(t)]^{\chi(\Gamma_{ab})}$ satisfying, on Γ^0_{ab}, the conditions (S_1) and (S_2) with the zero index.

Main Result. *If in problem* (2.1) *the simple closed curve* Γ *can be represented as* $\Gamma = \bigcup_{k=1}^{n} \Gamma_{a_k a_{k+1}}$, $a_{k+1} = a_1$, *where the direction on* $\Gamma_{a_k a_{k+1}}$ *coincides* *with the positive direction on* Γ, $\Gamma_{a_k a_{k+1}} \in R_A$ $(k=1,2,\dots,n)$ *and the function* $G(t)$ *satisfies, for* $t\in\Gamma$, *the conditions* (S_1) *and* (S_2) *and, moreover, for the points* a_k $(k=1,2,\dots,n)$ *there exist, on* Γ, *angular neighborhoods* $\Gamma_{b_k a_k}\subset\Gamma$, $\Gamma_{a_k c_k}\subset\Gamma$, *where the conditions*

$$\begin{aligned}
&\operatorname*{vrai\,sup}_{t\in\Gamma_{b_k a_k}}|\varphi_2(t)| < \alpha_k\frac{\pi}{\max(p,q)}\,, \\
&\operatorname*{vrai\,sup}_{t\in\Gamma_{a_k c_k}}|\varphi_2(t)| < (1-\alpha_k)\frac{\pi}{\max(p,q)}
\end{aligned} \tag{2.8}$$

are fulfilled for some $\alpha\in[0,1]$, *then the statement* $(A_\varkappa)$ *is valid for problem* (2.1).

It is obvious that in these conditions both cusps and other cases may occur at the points a_k.

3^0. The solution of a boundary value problem of linear conjugation is usually closely connected with a factorization problem. The factorization of the function $G(t)$ in $E_p(D_\Gamma^\pm)$ implies that it can be represented as

$$\begin{aligned}
&G(t)=X^+(t)(t-a)^\varkappa/X^-(t), \quad t\in\Gamma, \quad a\in D_\Gamma^+, \\
&X(z)-1\in E_p(D_\Gamma^\pm), \quad X^{-1}(z)-1\in E_q(D_\Gamma^\pm),
\end{aligned} \tag{3.1}$$

where $\varkappa$ is an integer, $X^\pm(t)$ are boundary values of the function $X(z)$.

It is obvious that (3.1) can be rewritten as

$$\begin{aligned}
&X^+(z)=(t-a)^{-\varkappa}\cdot G(t)\cdot X^-(t), \quad t\in\Gamma, \quad a\in D_\Gamma^+, \\
&X(z)-1\in E_p(D_\Gamma^\pm), \quad X^{-1}(z)-1\in E_q(D_\Gamma^\pm),
\end{aligned} \tag{3.1$'$}$$

The function $X(z)$ is called the factor function, and the number $\varkappa$ the index of the problem.

Lemma 1. *If the function* $G(t)$ *admits a factorization in* $E_p(D_\Gamma)$, $p>1$, *then to it there corresponds the unique number* $\varkappa$ *which can be put in* (3.1) *or, which is the same, in* (3.1$'$) *and for fixed* $a\in D_\Gamma^+$ *the factorization is unique.*

Proof. Suppose there are two factorizations

$$\begin{aligned}
G(t)&=(t-a_1)^{\varkappa_1}\cdot X_1^+(t)\cdot(X_1^-(t))^{-1} \\
&=(t-a_2)^{\varkappa_2}\cdot X_2^+(t)\cdot(X_2^-(t))^{-1}, \ \ a_1,a_2\in D_\Gamma^+.
\end{aligned} \tag{3.2}$$

Let $\varkappa_1>\varkappa_2$. From (3.2) we have

$$X_1^+(t)\cdot(X_2^+(t))^{-1}=(t-a_2)^{\varkappa_2-\varkappa_1}\Big(\frac{t-a_2}{t-a_1}\Big)^{\varkappa_1}X_1^-(t)\cdot(X_2^-(t))^{-1}. \tag{3.3}$$

We introduce the function

$$X(z)=\begin{cases} X_1(t)\cdot(X_2(t))^{-1} & \text{for } z\in D_\Gamma^+, \\ (z-a_2)^{\varkappa_2-\varkappa_1}\Big((z-a_2)/(z-a_1)\Big)^{\varkappa_1}X_1(z)\cdot(X_2(z))^{-1} & \text{for } z\in D_\Gamma^-. \end{cases}$$

By the definition of classes, the property (i_4) and the Hölder inequality it obviously follows that $X(z) \in E_1(D_\Gamma^\pm)$. (3.3) clearly implies that $X^+(t) = X^-(t)$ for $t \in \Gamma$. Therefore $X(z) \equiv 0$, which is impossible. Thus $\varkappa_1 = \varkappa_2$. If now, along with $\varkappa_1 = \varkappa_2$, it is assumed that $a_1 = a_2$, then we obtain $X(z) \equiv 1$, i.e., $X_1(z) = X_2(z)$. $\square$

Remark. Representation (3.1) and formula (2.6) clearly imply that if the condition $(A_\varkappa)$ is fulfilled, then the function $X_0(z) = \exp(K_\Gamma \ln(t-a)^{-\varkappa}G(t))(z)$ is the factor function of the function $(t-a)^{-\varkappa}G(t)$ with the zero index and the factor function of the function $G(t)$ with the index $\varkappa$.

In connection with Lemma 1 see also [2], p. 110.

4^0. We will give some auxiliary propositions.

Lemma 2. *If $f \in L_p(\Gamma)$, $\Gamma \in R$, $p > 1$, $G(t)$ satisfies the condition (S_1) and $\Phi(z) \in E_p(D_\Gamma^\pm)$, then the following statements are equivalent:*

(1) *the boundary value problem* (2.1) *has a unique solution for any $f \in L_p(\Gamma)$;*

(2) *The operator $N_G\,\varphi = \dfrac{1}{2}(1+G)\varphi + \dfrac{1}{2}(1-G)S\varphi$ is invertible in $L_p(\Gamma)$;*

(3) *The function $G(t)$ admits a factorization with the index $\varkappa = 0$ and the function $\Phi(z)$ given by the equality*

$$\Phi(z) = X_0(z)\Big(K(f/X_0^+)\Big)(z), \tag{4.1}$$

where $X_0(z)$ is the factorization of the function $G(t)$, is a solution of problem (2.1)*;*

(4) *The invertible operator N_G in $L_p(\Gamma)$ is written in the form*

$$N_G^{-1} f = \frac{1}{2}\,(1+G^{-1})f + \frac{1}{2}\,(1-G^{-1})X_0^+\, S_\Gamma(f/X_0^+),$$

where $X_0(z)$ is the same as in (3)*;*

(5) *$|X_0^+| \in W_p(\Gamma)$, where $X_0(z)$ is the same as in* (3) *and* (4)*.*

The statements of the lemma follow from the classical schemes and are encountered in this form or another in the works of many authors. Nevertheless, for the completeness of our exposition, we will give the proofs, the more so they are simple and some of their arguments make them different from the well known proofs.

By virtue of (i_1) and (i_2) the solution of the problem is $\Phi(z) = K_\Gamma\,\varphi$, $\varphi \in L_p(\Gamma)$. Applying Privalov's basic lemma ([1], p. 184), we obviously obtain $\Phi^+ - G\Phi^- = \frac{1}{2}(1+G)\varphi + \frac{1}{2}(1-G)S\varphi$. Hence it clearly follows that (1) is equivalent to (2).

Let us show now that (2) implies (3).

Note preliminarily that if (1) holds for some $G(t)$, then the same is true for $G_1(t) = AG(t)$, where $A = \mathrm{const}$, $A \neq 0$. Therefore it can be assumed without loss of generality that $|G(t)| \neq 1$ for $t \in \Gamma$.

Taking into account the invertibility of the operator $N_{_G}$ in $L_p(\Gamma)$ and using (1.2), we conclude that the operator $\frac{1}{2}(1+G)\psi - \frac{1}{2}S(1-G)\psi$ is invertible in $L_q(\Gamma)$. Using the notation $(1-G)\psi = \psi_1$, we have

$$\begin{aligned}
&\frac{1}{2}\,(1+G)\psi - \frac{1}{2}\,S(1-G)\psi = \frac{1}{2}\,\frac{1+G}{1-G}\psi_1 - \frac{1}{2}\,S\psi_1 = \frac{1}{2}\,\frac{G^{-1}+1}{G^{-1}-1}\psi_1 - \frac{1}{2}\,S\psi_1\\
&\quad = (G^{-1}-1)^{-1}\Big[\frac{1}{2}\,(1+G^{-1})\psi_1 - \frac{1}{2}\,(G^{-1}-1)S\psi_1\Big]\\
&\quad = (G^{-1}-1)^{-1}\Big[\frac{1}{2}\,(1+G^{-1})\psi_1 + \frac{1}{2}\,(1-G^{-1})S\psi_1\Big] = (G^{-1}-1)^{-1}N_{_{G^{-1}}}\psi_1.
\end{aligned}$$

Therefore $N_{_{G^{-1}}}\psi$ is invertible in $L_q(\Gamma)$ and the problem

$$X_1^+(t) = G^{-1}(t)X_1^-(t) + G^{-1}(t), \quad X_1 \in E_q(D_{_\Gamma}^\pm) \tag{4.2}$$

has a unique solution.

If to this we add that the problem

$$X^+(t) = G(t)X^-(t) + G(t), \quad X \in E_p(D_{_\Gamma}^\pm), \tag{4.3}$$

has a unique solution by virtue of (1), then we obtain

$$X_1^+X^+ = G^{-1}(X_1^{-1}+1)\cdot G(X^{-1}+1) = (X_1^{-1}+1)(X^{-1}+1).$$

Therefore

$$X_1^+X^+ - 1 = (X_1^- + 1)(X^- + 1) - 1. \tag{4.4}$$

Now since $X_1X \in E_1(D_{_\Gamma}^+)$ and $(X_1+1)(X+1)-1 \in E_1(D_{_\Gamma}^-)$, from (4.4) we have

$$\begin{aligned}
X_1(z)X(z) - 1 = 0 \quad &\text{for} \quad z \in D_{_\Gamma}^+,\\
(X_1(z)+1)(X(z)+1) - 1 = 0 \quad &\text{for} \quad z \in D_{_\Gamma}^-,
\end{aligned}$$

which implies

$$\begin{gathered}
(X(z))^{-1} = X_1(z) \in E_q(D_{_\Gamma}^+) \ \text{ and } \ (X(z)+1)^{-1} = X_1(z)+1,\\
(X(z)+1)^{-1} - 1 \in E_q(D_{_\Gamma}^-).
\end{gathered} \tag{4.5}$$

Using (4.5) together with (4.3), we conclude that the function

$$X_0(z) = \begin{cases} X(z) & \text{for} \quad z \in D_{_\Gamma}^+,\\ X(z)+1 & \text{for} \quad z \in D_{_\Gamma}^-, \end{cases}$$

is the factor function of the function $G(t)$ with the index $\varkappa = 0$.

By the classical scheme it further clearly follows that if $\Phi(z)$ is a unique solution of problem (2.1) and $X_0(z)$ is the facor function, then

$$\Phi^+/X_0^+ = \Phi^-/X_0^- + f/X_0^+, \quad \Phi/X_0 \in E_1(D_{_\Gamma}^\pm).$$

From this we obtain (4.1). Thus (1) implies (3).

To prove that the converse statement is true, we must show only that the solution is unique. Assume that this not so. Then

$$\Phi_1^+ = G\Phi_1^- + f, \quad \Phi_2^+ = G\Phi_2^- + f \quad \Phi_1, \Phi_2 \in E_p(D_\Gamma^\pm).$$

From this and (3.1) we have

$$(\Phi_1^+ - \Phi_2^+)/X_0^+ = (\Phi_1^- - \Phi_2^-)/X_0^-.$$

But as above $(\Phi_1(z) - \Phi_2(z))/X_0(z) \in E_1(D_\Gamma^\pm)$ and therefore $\Phi_1(z) = \Phi_2(z)$. Thus the statements (1), (2) and (3) are equivalent.

Concurrently, we have shown that for $\varkappa = 0$ the solution, if it exists for given (not necessarily arbitrary) f, is unique.

Let us next show that (3) implies (4). Since (3) is equivalent to (1), there exists a unique solution in $E_p(D_\Gamma^\pm)$. By the definition of classes for $p > 1$ we have $E_p \subset E_1$ and by (i_1) and (i_2) the solution is $\Phi(z) = K\varphi$, $\varphi \in L_p(\Gamma)$. Now it is clear that

$$\begin{aligned}\varphi = N_0 f = \Phi^+ - \Phi^- &= \frac{1}{2}X_0^+\Big(f/X_0^+ + S(f/X_0^+)\Big) - \frac{X_0^-}{2}\Big(-f/X_0^+ + S(f/X_0^+)\Big)\\ &= \frac{1}{2}(1+G^{-1})f + \frac{1}{2}(1-G^{-1})X_0^+ S(f/X_0^+) = N_G^{-1} f.\end{aligned} \tag{4.6}$$

Thus we have found that (3) implies (4). It is obvious that (4) implies (2) and therefore (1), (2), (3), (4) are equivalent.

Let us show that (4) implies (5). As said above, it can always be assumed that $G(t) \neq 1$ for $t \in \Gamma$. Therefore (4) clearly implies $|X_0^+| \in W_p(\Gamma)$, i.e., (5) is true.

Assume now that (5) is fulfilled. If f is continuous, then by (i_4), the definition of classes and the Hölder inequality we obtain $\Phi(z) = X_0(z)(K(f/X_0))(z) \in E_1(D_\Gamma^\pm)$ and $\Phi(z) = K\varphi$. It is obvious that $\Phi(z)$ satisfies the boundary condition (2.1). Now on account of (4.6), for continuous f_n we have that $\varphi_n = N_0 f_n$ and $K\varphi_n$ is a solution. But since $|X_0^+| \in W_p(\Gamma)$, we can pass to the limit and thus obtain $\varphi = N_0 f$ for any $f \in L_p$, $p > 1$. Moreover, $\varphi \in L_p(\Gamma)$, $\Phi^\pm = \frac{1}{2}(\pm\varphi + S\varphi) \in L_p(\Gamma)$ and because of the property of classes (i_2) we have $\Phi(z) \in E_p(D_\Gamma^\pm)$. The solution is unique by virtue of $\varkappa = 0$. Therefore $N_0 = N_G^{-1}$. The statements (1), (2), (3), (4) and (5) are equivalent.

Lemma 3. *If one of the statements of Lemma* 2 *is fulfilled for the function* $G_0(t)$*, then for the problem*

$$\Phi^+(t) = (t-a)^\varkappa G_0(t)\Phi^-(t) + f(t), \quad f \in L_p(\Gamma), \quad p > 1,$$

the statements ($A_\varkappa$) *hold except formula* (2.6)*, where* $X_0(z)$ *means in a general case the factor function* $G_0(t)$*.*

Proof. The statement (1) of Lemma 2 is fulfilled in the conditions of the lemma. The rest of the proof follows from the well verified arguments of many authors (see, for example, [10], p. 291). True, they consider the Lyapunov lines but here it is sufficient to require of the lines that the properties of the classes $E_p(D^{\pm}_{\Gamma})$, which we have given for $\Gamma \in R$ in the beginning of the paper, take place. $\square$

5^0. Let us consider a simple closed contour $\Gamma \in R$. If $\Gamma_{ab} \subset \Gamma$, then the direction from a to b is assumed to coincide with the positice direction on Γ. Here and in the sequel it is assumed that $\Gamma = \bigcup_{k=1}^{n} \Gamma_{a_k a_{k+1}}$, where $a_{n+1} = a_1$, $\Gamma_{a_k a_{k+1}} \in R_A$, $k = 1, \dots, n$, and do not have pairwise the common internal points. Occasionally, instead of $\Gamma_{a_k a_{k+1}}$, we write Γ_k. Also, we denote by Γ^0_k the contour participating the definition of $R_{_A}$, i.e., $\Gamma^0_k \equiv \Gamma^0_{a_k a_{k+1}}$, and by $\chi_k(t)$ the characteristic function of the set $\{t : t \in \Gamma_k\}$.

Lemma 4. *If $G(t)$ is a measurable function satisfying condition* (2.2) *and $G_k(t) \equiv [G(t)]^{\chi_k(t)}$, $t \in \Gamma \cup \Gamma^0_k$, satisfies on Γ^0_k the condition (S_2) with $\varkappa = 0$, then:*

1) $X_k(z) = \exp\Big(K_{\Gamma}\chi_{_k} \ln G\Big)(z)$ *is the factor function of the function $G_k(t)$ in $E_p(D^{\pm}_{\Gamma})$;*

2) *The boundary value problem*

$$\Phi^+(t) = G_k(t)\Phi^-(t) + f(t), \quad t \in \Gamma, \quad f \in L_p(\Gamma), \tag{5.1}$$

has a unique solution in $E_p(D^{\pm}_{\Gamma})$;

$$3) \qquad \left|\exp\frac{1}{2}\Big(S_{\Gamma}(\chi_{_k} \ln G)\Big)(t)\right| \in W_p(\Gamma). \tag{5.2}$$

Proof. Since $\Gamma_k \in R_{_A}$, the factor function of the function $G_k(t)$ in $E_p(D^{\pm}_{\Gamma^0_k})$ is $X_k(z) = \exp\Big(K_{\Gamma^0_k}(\chi_{_k} \ln G)\Big)(z)$. But then $X_k(z) - 1 = K_{\Gamma^0_k}\psi$, $\psi \in L_p(\Gamma^0_k)$ and $X_k^{-1}(z) - 1 = (K_{\Gamma^0_k}\psi_1)(z), \psi_1 \in L_q(\Gamma^0_k)$. Obviously, $\psi = (K_{\Gamma^0_k}\psi)^+ - (K_{\Gamma^0_k})^- = X_k^+ - X_k^- = X_k^-(X_k^+/X_k^- - 1) = X_k^-(G_k(t) - 1) = \chi_{_k}(t)\psi(t)$. Analogously, $\psi_1 = \chi_{_k}(t)\psi_1(t)$. Therefore $K_{\Gamma^0_k}\psi = K_{\Gamma^0_k}\chi_{_k}\psi = K_{\Gamma}\chi_{_k}\psi \in E_p(D^{\pm}_{\Gamma})$ and $K_{\Gamma^0_k}\psi_1 = K_{\Gamma}\chi_{_k}\psi \in E_q(D^{\pm}_{\Gamma})$.

Thus

$$X_k(z) - 1 \in E_p(D^{\pm}_{\Gamma}) \quad \text{and} \quad X_k^{-1}(z) \in E_q(D^{\pm}_{\Gamma}). \tag{5.3}$$

Moreover, for $t \in \Gamma - \Gamma_k$ we have $(K_{\Gamma}\chi_{_k}\psi)^+/(K_{\Gamma}\chi_{_k}\psi)^- = 1$. Therefore

$$X_k^+(t)/X_k^-(t) = G_k(t) \quad \text{for} \quad t \in \Gamma \tag{5.4}$$

(5.3) and (5.4) prove the statement 1).

Now we will show the validity of the statment 2).

A solution of problem (5.1) is to be sought in the form

$$\Phi(z) = \Phi_1(z) + \Phi_2(z) \tag{5.5}$$

where $\Phi_1(z) = (K_\Gamma\psi)(z)$, $\psi \in L_p(\Gamma)$ and $\Phi_2(z) = \Big(K_\Gamma(\chi_{\Gamma-\Gamma_k} f)\Big)(z)$. Equality (5.1) then takes the form

$$\Phi_1^+ + \Phi_2^+ = G_k(t)(\Phi_1^- + \Phi_2^-) + f(t), \quad t \in \Gamma. \tag{5.6}$$

Denoting $f_0(t) = f(t) - \Phi_2^+(t) + G_k\Phi_2^-(t)$, we rewrite (5.6) as

$$\Phi_1^+(t) = G_k(t)\Phi_1^-(t) + f_0(t), \quad t \in \Gamma, \quad \Phi_1 \in E_p(D_\Gamma^\pm), \tag{5.7}$$

and (5.7) is equivalent to

$$\begin{cases} \phi_1^+ = G(t)\Phi_1^-(t) + f_0(t) & \text{for } t \in \Gamma_k, \quad \Phi_1 \in E_p(D_\Gamma^\pm) \\ \Phi_1^+ = \Phi_1^- & \text{for } t \in \Gamma - \Gamma_k. \end{cases} \tag{5.8}$$

From (5.8) it follows that if $\Phi_1(z)$ is a solution of (5.8), then

$$\Phi_1(z) = (K_\Gamma\psi)(z) = K_\Gamma(\Phi_1^+ - \Phi_1^-) = K_\Gamma\chi_k\psi = K_{\Gamma_k^0}\chi_k\psi \in E_p(D^\pm_{\Gamma_k^0}).$$

Therefore $\Phi_1(z)$ simultaneously belongs both to $E_p(D_\Gamma^\pm)$ and to $E_p(D^\pm_{\Gamma_k^0})$.

Now it is clear that between the solutions of problem (5.1) in $E_p(D_\Gamma^\pm)$ and the problem

$$\Phi_1^+(t) = G_k(t)\Phi_1^-(t) + f_0(t), \quad t \in \Gamma_k^0, \quad \Phi_1 \in E_p(D^\pm_{\Gamma_k^0}), \tag{5.9}$$

there exists a one-to-one correspondence carried out by formula (5.5). But (5.9) has a unique solution. Therefore (5.1), too, has a unique solution.

Now, by virtue of statement (5) of Lemma 2, 1) and 2) imply $|X_k^+(t)| \in W_p(\Gamma)$, from which it follows that

$$\begin{aligned} |X_k^+(t)| &= \Big|\exp(K_\Gamma\chi_k \ln G)^+\Big| = G(t)^{\frac{1}{2}\chi_k} \exp\frac{1}{2}\, S_\Gamma\chi_k \ln G| \in W_p(\Gamma) \\ &\Longrightarrow \Big|\exp\frac{1}{2} S_\Gamma\chi_k \ln G\Big| \in W_p(\Gamma). \qquad \square \end{aligned}$$

Corollary 1. *If we take $G_k(t) = |G(t)|^{\chi_k(t)}$, then by virtue of Lemma 4:*

1) *the function*

$$\exp K_{\Gamma_k} \ln |G| \tag{5.10}$$

is the factor function of the function $|G|^{\chi_k(t)}$ in $\bigcap\limits_{p>1} E_p(D_\Gamma^\pm)$;

2)
$$\Big|\exp\frac{1}{2} S_{\Gamma_k} \ln |G|\Big| \in \bigcap_{p>1} W_p(\Gamma). \tag{5.11}$$

Corollary 2. *If we take $G_k(t) = \exp i\chi_k\varphi_2$, where φ_2 satisfies condition* (2.3)*, then:*

1) $\exp(K_\Gamma i\chi_k\varphi_2)(z)$ *is the factor function for* $\exp i\chi_k\varphi_2$ *in* $E_p(D_\Gamma^\pm)$*, where* p *participates in condition* (2.3)*;*

$$\text{2)} \qquad \left|\exp\frac{i}{2}S_\Gamma\chi_{k}\varphi_2\right| \in W_p(\Gamma) \quad \textit{for the same} \quad p. \tag{5.12}$$

Lemma 5. *If Γ_{bc} is a simple arc on Γ, $\Gamma_{bc} \subset \Gamma$, $\Gamma_{bc} = \Gamma_{ba} + \Gamma_{ac}$, $\Gamma_{ba} \in R_A$, $\Gamma_{ac} \in R_A$ and $\varphi_2(t)$ is a measurable function on Γ which satisfies the conditions*

$$\begin{aligned} &\operatorname*{vrai\,sup}_{t\in\Gamma_{ba}}|\varphi_2(t)| < \alpha\frac{\pi}{\max(p,q)}, \\ &\operatorname*{vrai\,sup}_{t\in\Gamma_{ac}}|\varphi_2(t)| < (1-\alpha)\frac{\pi}{\max(p,q)}, \end{aligned} \tag{5.13}$$

where α is some number from $[0,1]$, *then*

$$\rho(t) = \left|\exp\frac{i}{2}(S_{\Gamma_{bc}}\varphi_2)(t)\right| \in W_p(\Gamma)\cap W_q(t). \tag{5.14}$$

Proof. We make use of the particular case of the well-known Stein theorem (see, for example [10], p. 288):

If $\|(S\varphi)\rho_k\|_p \le M_p\|\varphi\rho_k\|$, $k=1,2$, $p>1$, then

$$\|(S\varphi)\rho\| \le M_p\|\varphi\rho\|_p \quad \text{for} \quad \rho = \rho_1^\alpha\cdot\rho_2^{1-\alpha}, \quad \alpha\in[0,1].$$

This fact can also be written as follows:

If $\rho_k \in W_p(\Gamma)$, $k=1,2$, $p>1$ and $\alpha\in[0,1]$, then

$$\rho = \rho_1^\alpha\cdot\rho_2^{1-\alpha} \in W_p(\Gamma). \tag{5.15}$$

Denote by $\chi_1(t) = \chi(\Gamma_{ba})$ the characteristic function of the set $\{t : \Gamma_{ba}\}$, and by $\chi_2(t)$ that of the set $\{t : t\in\Gamma_{ac}\}$. On account of (5.12) and (5.13) we have

$$\rho_1 = \left|\exp\frac{i}{2}S_\Gamma\frac{1}{\alpha}\chi_1\varphi_2\right| \in W_p(\Gamma), \quad \rho_2 = \left|\exp\frac{i}{2}S_\Gamma\frac{1}{1-\alpha}\chi_2\varphi\right| \in W_p(\Gamma),$$

from which by virtue of (5.15) we obtain

$$\begin{aligned} \rho = \rho_1^\alpha\cdot\rho_2^{1-\alpha} &= \left|\exp\frac{i}{2}S_\Gamma\chi_1\varphi_2\right|\cdot\left|\exp\frac{i}{2}S_\Gamma\chi_2\varphi_2\right| \\ &= \left|\exp\frac{i}{2}S_\Gamma(\chi_1+\chi_2)\varphi_2\right| = \left|\exp\Big(\frac{i}{2}S_{\Gamma_{bc}}\varphi_2\Big)(t)\right| \in W_p(\Gamma). \end{aligned}$$

Obviously, in the conditions (5.13) p and q play the same role and therefore (5.14), too, is valid. □

Theorem 1. *If $\Gamma = \bigcup\limits_{k=1}^{n}\Gamma_{a_ka_{k+1}}$, $a_{n+1}=a_1$, $\Gamma_{a_ka_{k+1}} \subset R_A$, and, moreover, for each point a_k $(k=1,\dots,n)$ there exists an arc neighborhood $\Gamma_{b_ka_k}\cup\Gamma_{a_kc_k}\subset\Gamma$, where conditions* (2.8) *are fulfilled for the function G, and conditions* (2.2) *and* (2.3) *are fulfilled on the entire contour Γ, then*

$$\rho(t) = \left|\exp\frac{i}{2}(S_\Gamma\varphi_2)(t)\right| \in W_p(\Gamma)\cap W_q(t). \tag{5.16}$$

Proof. To prove the theorem, we make use of Theorem 4.2 from [13], p. 52, by virtue of which the necessary and sufficient condition for the inequality

$$\int\limits_{\Gamma} |S_{\Gamma} f|^{p_0} w \, d\nu \le c \int\limits_{\Gamma} |f|^{p_0} w \, d\nu, \tag{5.17}$$

to hold for $p_0 > 1$, is $w \in A_{p_0}$.

In terms of the above-mentioned work this means that

$$\Big\{\rho = w^{\frac{1}{p_0}} \in W_{p_0}(\Gamma),\ p_0 > 1,\ \Gamma \in R\Big\} \Longleftrightarrow \Big\{\rho^{p_0} = w \in A_{p_0}(\Gamma)\Big\}. \tag{5.18}$$

(for the definition of A_p, see [13], p. 42).

Let us write the condition A_{p_0} for $\rho = w^{\frac{1}{p_0}}$. Denote by ν the length of the arc $\Gamma(z,r)$ which is cut out from Γ by the circle with center $z \in \Gamma$ and radius r. (Here, like in (5.17), ν is the arc measure on Γ.) We obtain

$$\rho \in W_{p_0} \Longleftrightarrow \sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} \bigg(\frac{1}{\nu\Gamma(z;r)} \int\limits_{\Gamma(z;r)} \rho^{p_0} d\nu\bigg)^{\frac{1}{p_0}} \bigg(\frac{1}{\nu(z;r)} \int\limits_{\Gamma(z;r)} \rho^{-q_0} d\nu\bigg)^{\frac{1}{q_0}} < \infty,$$
$$q_0 = p_0(1-p_0)^{-1}. \tag{5.19}$$

For brevity, we denote by $B(\Gamma(z;r);\rho(t);p_0)$ the expression in (5.19) from which the supremum is taken. Now (5.19) can be rewritten as

$$\rho \in W_{p_0} \Longleftrightarrow \sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} B\Big(\Gamma(z;r);\rho(t);p_0\Big) < \infty. \tag{5.20}$$

Take now the points b_k and c_k $(k = 1,2,\dots,n)$ according to condition (2.8) and take the points $e_k \in \Gamma_{b_k a_k}$ and $d_k \in \Gamma_{a_k c_k}$ different from the end points (Fig. 1).

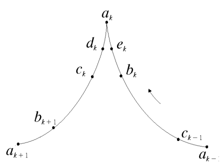

Fig. 1

Also assume that $d_{n+1} = d_1$, $e_{n+1} = e_1$, $b_{n+1} = b_1$, $c_{n+1} = c_1$ and $\Gamma_{b_k a_k}$ $(k = 1,\dots,n)$ do not intersect pairwise. Denote $\delta_k^{(1)} = \min\limits_{\substack{t\in\Gamma_{e_k d_k} \\ \tau\in\Gamma-\Gamma_{b_k c_k}}} |t-\tau|$,

$\delta_k^{(2)} = \min\limits_{\substack{t\in\Gamma_{d_k e_{k+1}} \\ t\in\Gamma-\Gamma_{a_k a_{k+1}}}} |t-\tau|$.

Also denote $\delta = \frac{1}{2} \min\limits_{\substack{i=1,2 \\ i\le k\le n}} \delta_k^{(i)}$. Assume that $z \in \Gamma_{e_k d_k}$. Then $\Gamma(z,\delta) \subset \Gamma_{b_k c_k}$. But by virtue of Lemma 5 we have $\left|\exp \frac{i}{2}(S_{\Gamma_{b_k c_k}}\varphi_2)(t)\right| \in W_p(\Gamma) \cap W_q(t)$. Then if p_0 is equal to p or q, we have

$$\sup_{\substack{z\in\Gamma_{e_k d_k} \\ 0<r<\delta}} \left(B(\Gamma(z;r); \left|\exp \frac{i}{2}(S_{\Gamma_{b_k c_k}}\varphi_2)(t)\right|, p_0\right) < \infty. \tag{5.21}$$

Meanwhile, if $z \in \Gamma_{e_k d_k}$ and $t \in \Gamma(z;\delta)$, then

$$\left|\exp \frac{i}{2}\,(S_{\Gamma-\Gamma_{b_k c_k}}\varphi_2)(t)\right| < \mathrm{const}$$

and therefore along with (5.21) we have

$$\sup_{\substack{z\in\Gamma_{e_k d_k} \\ 0<r<\delta}} \left(B(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma}\varphi_2)(t)\right|, p_0\right) < \infty. \tag{5.22}$$

Let now $z \in \Gamma_{d_k e_{k+1}}$. In view of (5.12) and (5.18) we obtain

$$\sup_{\substack{z\in\Gamma_{d_k e_{k+1}} \\ 0<r<\delta}} \left(B(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma_{a_k a_{k+1}}}\varphi_2)(t)\right|, p_0\right) < \infty. \tag{5.23}$$

But, as above, for $t\in\Gamma(z,r)$, $z\in\Gamma_{d_k e_{k+1}}$ we have $\left|\exp\frac{i}{2}(S_{\Gamma-\Gamma_{a_k a_{k+1}}}\varphi_2)(t)\right| <$ const, which, together with (5.23), gives

$$\sup_{\substack{z\in\Gamma_{d_k e_{k+1}} \\ 0<r<\delta}} \left(B(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma}\varphi_2)(t)\right|, p_0\right) < \infty. \tag{5.24}$$

Since (5.22) and (5.24) are fulfilled for $k = 1,\dots,n$, we obtain

$$\sup_{\substack{z\in\Gamma \\ 0<r<\delta}} B\left(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma}\varphi_2)(t)\right|, p_0\right) < \infty. \tag{5.25}$$

Since $\left|\exp \frac{i}{2}(S_{\Gamma}\varphi_2)(t)\right| \in L_{p_0}(\Gamma)$, it is obvious that

$$\sup_{\substack{z\in\Gamma \\ \delta<r<\operatorname{diam}\Gamma}} B\left(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma}\varphi_2)(t)\right|, p_0\right) < \infty,$$

which, together with (5.25), gives

$$\sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} B\left(\Gamma(z;r); \left|\exp \frac{i}{2}\,(S_{\Gamma}\varphi_2)(t)\right|, p_0\right) < \infty \tag{5.26}$$

which, together with (5.20), proves the theorem. □

Corollary. *If φ_2 satisfies conditions* (2.3) *and* (2.8), *then there exists a number* $0<\varepsilon<1$ *such that* $(1-\varepsilon)^{-1}\varphi_2$ *also satisfies these conditions. Hence from Theorem* 1 *it follows that*

$$\Big|\exp\frac{i}{2}\Big(S_\Gamma\,\frac{1}{1-\varepsilon}\,\varphi_2\Big)(t)\Big|\in W_p(\Gamma)\cap W_q(\Gamma) \tag{5.27}$$

Remark. By a reasoning similar to that used in proving Theorem 1 one can prove a more general statement.

Let Γ be a simple (not necessarily closed) continuous line and $\Gamma\in R$. Let Γ_t be an arc neighborhood of the point $\Gamma\in R$. For a more exact definition of Γ_t we use, along with the notation Γ_{ab}, the notations $\Gamma_{[a,b]}$, $\Gamma_{(a,b)}$, $\Gamma_{[a,b)}$, $\Gamma_{(a,b]}$ which indicate the belonging or not belonging of the end points to the arc Γ_{ab}.

Now if t is an internal point of the line Γ, then $\Gamma_t=\Gamma_{(t_1,t_2)}$, where $t\in\Gamma_{(t_1,t_2)}$ and t_1,t_2 are the points on Γ taken according to the positive direction on Γ. If however $\Gamma=\Gamma_{ab}$ and $t=a$ or $t=b$, then $\Gamma_a\equiv\Gamma_{[0,t_1)}$ and $\Gamma_b\equiv\Gamma_{(t_2,b]}$, $t_1,t_2\in\Gamma_{ab}$.

For $p>1$ the following statement holds true:

$$(\rho(t)\in W_p(\Gamma_t)\quad\text{for any}\quad t\in\Gamma)\Longrightarrow\rho\in W_p(\Gamma). \tag{5.28}$$

The proof is published in [12]. Like Theorem 1, this result is based on Theorem 4.2 from [13], p. 52.

Theorem 2. *If Γ is the same as in Theorem* 1, $G(t)=|G(t)|\exp i(\varphi_1+\varphi_2)$, *where $\varphi_1(t)$ is a continuous function, $|G(t)|$ and $\varphi_2(t)$ satisfy conditions* (2.2) *and* (2.3) *as well as the additional condition* (2.8), *then*

$$\rho(t)=\Big|\exp\frac{1}{2}\,(S_\Gamma\ln G)(t)\Big|\in W_p(\Gamma)\cap W_q(\Gamma). \tag{5.29}$$

Proof. Denote

$$\rho_1(t)\equiv\Big|\exp\frac{1}{2}(S_\Gamma\ln|G|)(t)\Big|,\quad \rho_2(t)\equiv\Big|\exp\frac{i}{2}(S_\Gamma\varphi_1)(t)\Big|,$$
$$\rho_3(t)\equiv\Big|\exp\frac{i}{2}(S_\Gamma\varphi_2)(t)\Big|.$$

It is obvious that $\rho_1(t)=\bigcup\limits_{k=1}^{n}\Big|\exp\frac{1}{2}(S_{\Gamma_k}\ln|G|)(t)\Big|$. For $0<\alpha<1$ we have $\Big|\exp\frac{1}{2}\Big(S_{\Gamma_1}\frac{1}{\alpha}\ln|G|\Big)(t)\Big|\in W_p(\Gamma)$ for any $p>1$, and $\Big|\exp\frac{1}{2}\Big(S_{\Gamma_2}\frac{1}{1-\alpha}\ln|G|\Big)(t)\Big|\in W_p(\Gamma)$ for any $p>1$. Hence, applying the Stein theorem (5.15), we find that $\Big|\exp\frac{1}{2}\Big(S_{\Gamma_1+\Gamma_2}\ln|G|\Big)(t)\Big|\in W_p(\Gamma)$ for any $p>1$, and for any $G(t)$ satisfying (2.2). Performing the same n times we obtain $\rho_1\in W_p(\Gamma)$ for any $p>1$ and $G(t)$ satisfying (2.2). Therefore

$$\rho_1^{\frac{1}{\alpha_0}}\in\bigcap_{p>1}W_p(\Gamma)\quad\text{for}\quad 0<\alpha_0<1. \tag{5.30}$$

ρ_2 also belongs to $\bigcap\limits_{p>1} W_p(\Gamma)$, which follows from the fact that the boundary value problem for $\Gamma \in R$ with a continuous coefficient is solved completely ([14], p. 547).

It is clear that

$$\rho_2^{\frac{1}{1-\alpha_0}} \in \bigcap_{p>1} W_p(\Gamma), \quad 0 < \alpha_0 < 1. \tag{5.31}$$

By the same Stein theorem, (5.30) and (5.31) imply that $\rho_1 \cdot \rho_2 \in \bigcap\limits_{p>1} W_p(\Gamma)$ for any $G(t)$ and $\varphi_1(t)$ satisfying the conditions of the theorem. Therefore

$$(\rho_1 \cdot \rho_2)^{\frac{1}{\varepsilon}} \in \bigcap_{p>1} W_p(\Gamma) \quad \text{for} \quad 0 < \varepsilon < 1. \tag{5.32}$$

Let p_o be any one of the numbers p and q participating in condition (2.8), and ε be the same as in (5.27). Then

$$(\rho_3)^{\frac{1}{1-\varepsilon}} \in W_{p_0}(\Gamma). \tag{5.33}$$

By virtue of the Stein theorem, (5.32) and (5.33) give

$$\Big((\rho_1 \cdot \rho_2)^{\frac{1}{\varepsilon}}\Big)^{\varepsilon} \cdot \Big(\rho_3^{\frac{1}{1-\varepsilon}}\Big)^{1-\varepsilon} = \rho_1 \cdot \rho_2 \cdot \rho_3 \in W_{p_0},$$

which is equivalent to (5.29). □

Theorem 3. *If Γ is the same as everywhere in Section 5^0, $G(t)$ is the measurable function $G(t) = |G(t)| \exp(i\varphi_1(t) + i\varphi_2(t))$ on Γ, conditions* (2.2), (2.3) *are fulfilled, the function $\varphi_1(t)$ is continuous except for one point at which it has a first kind jump equal to $2\pi\varkappa$, where $\varkappa$ and integer, and if, additionally,* (2.8) *is fulfilled φ_2, then the statement $A_{(\varkappa)}$ is valid for the boundary value problem* (2.1) *with the coefficient $G(t)$.*

Proof. First assume that $\varkappa = 0$. By virtue of Lemma 2 of [8], p. 1654, the boundedness of $\ln |G_0|$ and formula (5.29) imply that $X_0(z) = \exp(K_\Gamma \ln G_0)(z)$ is the factor function of the function $G_0(t)$. Since we have defined $E_\delta(D^-)$ for $\delta < 1$ and the factor function in a somewhat different way, in our case we will repeat with an insignificant modification the reasoning of [8], p. 1654.

From (i_3) it follows that $X_0(z) - 1 \in E_\delta(D_\Gamma^\pm)$ for some $\delta > 0$. Further, by (5.29) and (5.18) we obtaon $\rho^\pm \in L_p(\Gamma) \cap L_q(\Gamma)$, which, by ($i_2$), implies that $X_0^\pm(z) - 1 \in E_p(D_\Gamma^\pm) \cap E_q(D_\Gamma^\pm)$, where $X_0(z)$, which evidently satisfies conditions (3.1), is the factor function with $\varkappa = 0$.

Thus statement (5) is fulfilled and so are the other statements of Lemma 2 and problem (2.1) has a unique solution in $E_p(D_\Gamma^\pm)$.

If $\varkappa \neq 0$, then, taking $a \in D_\Gamma^+$, we can write

$$G_0(t) = (t-a)^{-\varkappa} G(t) = |G_0(t)| \exp(i\varphi_1^0 + i\varphi_2),$$

where $\varphi_1^0(t)$ is already continuous, and by virtue of the facts proved the problem with the coefficient $G_0(t)$ has a unique solution, while the factor function of the function $G_0(t)$ has form (2.6). Now due to Lemma 3 we conclude that the statement $(A_\varkappa)$ is true for problem (2.1) with the coefficient $G(t) = (t - a)^\varkappa G_0(t)$. $\square$

Remark. Theorem 3 is the main result of this paper. As an example of curves for which this theorem is valid we may name piecewise-smooth curves and curves with bounded rotation (Radon curves), where an additional condition is imposed on coefficient (2.8) at cusps. As Γ_k we can also take the curves considered in [8] and others. Cases may occur, in which unilateral tangent lines do not exist at the points a_k.

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