Georgian Mathematical Journal
Volume 9 (2002), Number 3, 431–448

QUASILINEARIZATION METHODS FOR NONLINEAR PARABOLIC EQUATIONS WITH FUNCTIONAL DEPENDENCE

A. BYCHOWSKA

Abstract. We consider a Cauchy problem for nonlinear parabolic equations with functional dependence. We prove convergence theorems for a general quasilinearization method in two cases: (i) the Hale functional acting only on the unknown function, (ii) including partial derivatives of the unknown function.

2000 Mathematics Subject Classification: 35K10, 35K15, 35R10.
Key words and phrases: Parabolic, differential functional, quasilinearization, iterative method, Cauchy problem.

1. Introduction

In the present paper we generalize some results of [9] concerning fast convergence of non-monotone quasilinearization methods in two main direction: (i) the Laplacian is replaced by a general leading differential term with Hölder continuous coefficients, (ii) the continuity and Lipschitz conditions on a given function are weakened to Carathéodory-type conditions and generalized Lipschitz conditions with Lipschitz constants replaced by integrable functions. We study three cases:

(a) differential equations with functional dependence on the unknown function, (Section 2, Theorem 2.1);
(b) differential equations with functional dependence on the unknown function and on its spatial derivatives, (Section 3, Theorem 3.1);
(c) differential equations with functional dependence on the unknown function and on its spatial derivatives at the same point as leading differential terms; in this case we do not impose continuous differentiability condition on the initial data, (Section 3, Theorem 3.2).

Our results are related to the previous existence results for a Cauchy problem with functional dependence [10, 11] because the existence theorems are proved by means of iterative methods, in particular, by the method of direct iterations. Nonlinear comparison conditions have been inspired by the Ważewski's fruitful idea of the so-called comparison method [15]. Another important influence on all iterative methods (also on quasilinearization methods) comes from the use of weighted norms [12, 14] and differential inequalities [13, 14]. A monotone

ISSN 1072-947X / $8.00 /

version of the direct iteration method provides sequences of functions slightly faster convergent than those provided by its non-monotone version to an exact solution [2, 5]. Among the monotone iterative methods one should distinguish the Chaplygin method [1, 3], which produces significantly faster monotone sequences. The Chaplygin method starts from a pair $(u^{(0)}, \bar{u}^{(0)})$ of upper and lower solutions of the differential-functional problem (1), (2) below and provides monotone sequences $(u^{(\nu)}, \bar{u}^{(\nu)})$ which tend uniformly to an exact solution of this problem. One of these sequences fulfills similar differential-functional equations as the quasilinearization method. We admit that a general theory of monotone iterative techniques for various differential problems was given in [7].

There is yet another significant difference between our results and the results of [1, 2, 3, 5, 12, 16], namely: we study differential-functional problems in unbounded domains, and therefore miss any compactness arguments. It turns out that in unbounded domains too we derive suitable integral comparison inequalities from which we deduce fast uniform convergence of the sequences of successive approximation. By the wide use of suitable weighted norms the obtained convergence statements are global in t, i.e., there is no additional restriction on the convergence interval.

We simplify some proofs of the theorems even when we restrict our results to heat equations with a nonlinear and functional reaction term, cf. [9].

Note that differential-functional problems play an important role in many applications. For numerous examples arising in biology, ecology, physics, engineering we refer the reader to the monograph [16].

1.1. Formulation of the problem. We recall the basic properties of fundamental solutions and their applications to the existence and uniqueness theory for the differential equations, see [4] to clarify the background of our main results.

Let $E = (0, a] \times R^n$, $E_0 = [-\tau_0, 0] \times R^n$, $\widetilde{E} = E_0 \cup E$, $B = [-\tau_0, 0] \times [-\tau, \tau]$, where $a > 0$, $\tau_0, \tau_1, \dots, \tau_n \in [0, +\infty)$, and

$$\tau = (\tau_1, \dots, \tau_n), \quad [-\tau, \tau] = [-\tau_1, \tau_1] \times \cdots \times [-\tau_n, \tau_n].$$

If $u : E_0 \cup E \longrightarrow R$ and $(t, x) \in E$, then the Hale-type functional $u_{(t,x)} : B \longrightarrow R$ is defined by

$$u_{(t,x)}(s, y) = u(t + s, x + y) \text{ for } (s, y) \in B.$$

Since the present paper concerns bounded solutions (unbounded solutions must be handled more carefully), we can replace the above domain B by any unbounded subset of E_0, in particular all the results carry over to the case $B := E_0$.

Let $C(X)$ be the class of all continuous functions from a metric space X into R, and $CB(X)$ $(CB(X)^n)$ be the class of all continuous and bounded functions from X into R (R^n). Denote by $\partial_0, \partial_1, \dots, \partial_n$ the operators of partial derivatives with respect to $t, x_1, \dots, x_n$, respectively. Let $\partial = (\partial_1, \dots, \partial_n)$ and

$\partial_{jl} = \partial_j \partial_l \ (j,l = 1,2,\ldots,n)$. The differential operator $\mathcal{P}$ is defined by

$$\mathcal{P}u(t,x) = \partial_0 u(t,x) - \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{jl}u(t,x).$$

We consider a Cauchy problem

$$\mathcal{P}u(t,x) = f(t,x,u_{(t,x)}), \tag{1}$$

$$u(t,x) = \varphi(t,x) \ \text{ on } E_0. \tag{2}$$

The Cauchy problem (1), (2) reduces to the following integral equation:

$$u(t,x) = \int_{R^n} \Gamma(t,x,0,y)\varphi(0,y)dy + \int_0^t \int_{R^n} \Gamma(t,x,s,y) f(s,y,u_{(s,y)})dyds, \tag{3}$$

where $\Gamma(t,x,s,y)$ is a fundamental solution of the above parabolic problem. A function $u \in C(\widetilde{E})$ is called a classical solution of problem (1), (2) (in other words, a $C^{1,2}$ solution). If $\partial_0 u$, $\partial_j u$, $\partial_{jl}u \in C(E)$, then u satisfies equation (1) on E and the initial condition (2) on E_0. A function $u \in C(\widetilde{E})$ is called a C^0 solution if u coincides with φ on E_0, and it satisfies (3) on E. Any C^0 solution u whose derivatives $\partial_j u$ $(j = 1,\ldots,n)$ are continuous on E is called a $C^{0,1}$ solution of problem (1), (2). The notion of C^0, $C^{0,1}$, $C^{1,2}$ weak solutions requires only the existence of partial derivatives, being not necessarily continuous.

1.2. Existence and uniqueness. The supremum norm will be denoted by $\|\cdot\|_0$. The symbol $\|\cdot\|$ stands for the Euclidean norm in R^n.

Assumption 1.1. Suppose that $a_{jl} \in CB(E)$ for $j,l = 1,\ldots,n$, the operator $\mathcal{P}$ is parabolic, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq c'\|\xi\|^2 \quad \text{for all} \ \ (t,x) \in E, \ \ \xi \in R^n,$$

and the coefficients a_{jl} satisfy the Hölder condition

$$|a_{jl}(t,x) - a_{jl}(\tilde{t},\tilde{x})| \leq c''(|t-\tilde{t}|^{\frac{\alpha}{2}} + \|x-\tilde{x}\|^{\alpha}) \quad \text{for} \ \ j,l = 1,\ldots,n,$$

where c', $c'' > 0$.

Lemma 1.1. *If Assumption* 1.1 *is fulfilled, then there are* $k_0, c_0, c_1, c_2 > 0$ *such that*

$$|\Gamma(t,x,s,y)| \leq c_0(t-s)^{-\frac{n}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right),$$

$$|\partial_j\Gamma(t,x,s,y)| \leq c_1(t-s)^{-\frac{(n+1)}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right),$$

$$|\partial_0\Gamma(t,x,s,y)|, \ |\partial_{jl}\Gamma(t,x,s,y)| \leq c_2(t-s)^{-\frac{(n+2)}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right),$$

for all $0 \leq s < t \leq a$ *and* $x,y \in R^n$, $j,l = 1,\ldots,n$.

These estimates can be found in [6, 8].

Remark 1.1. Under Assumption 1.1, one obtains more general Hölder-type estimates for the fundamental solution with any Hölder exponent $\delta \in (0,1]$

$$\begin{aligned}
&|\Gamma(t,x,s,y) - \Gamma(\bar{t},\bar{x},s,y)| \\
&\quad \le c_{0+\delta}\,(t-s)^{-\frac{n+\delta}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right)[|t-\bar{t}|^{\frac{\delta}{2}} + \|x-\bar{x}\|^{\delta}], \\
&|\partial_j\Gamma(t,x,s,y) - \partial_j\Gamma(\bar{t},\bar{x},s,y)| \\
&\quad \le c_{1+\delta}\,(t-s)^{-\frac{(n+1+\delta)}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right)[|t-\bar{t}|^{\frac{\delta}{2}} + \|x-\bar{x}\|^{\delta}]
\end{aligned}$$

for $0 \le s < t \le \bar{t} \le a$ and $x, \bar{x}, y \in R^n$, $j, l = 1, \ldots, n$.

From now on we assume that Assumption 1.1 holds. We write after [6, 8] and [4] a basic existence result for a Cauchy problem without functional dependence, $f \equiv 0$.

Lemma 1.2 ([4, Lemma 1.2])**.** *If $\varphi \in CB(E_0)$, then there exists a classical solution $\widetilde{\varphi} \in CB(\widetilde{E})$ of the initial-value problem*

$$\mathcal{P}u = 0, \quad u \succ \varphi,$$

where the symbol $u \succ \varphi$ means the same as $u(t,x) = \varphi(t,x)$ for $(t,x) \in E_0$.

We cite after [4] the following existence and uniqueness theorems. Let $L^1[0,a]$ denote the set of all real Lebesgue integrable functions on $[0,a]$.

Theorem 1.1 ([4, Theorem 2.1])**.** *Let $\varphi \in CB(E_0)$, λ, m_f, $f(\cdot,x,0) \in L^1[0,a]$, and $f(t,\cdot,0) \in C(R^n)$. Assume that $|f(t,x,0)| \le m_f(t)$ and*

$$|f(t,x,w) - f(t,x,\bar{w})| \le \lambda(t)\,\|w-\bar{w}\|_0 \quad \text{on } E \times C(B). \tag{4}$$

Then there exists a unique bounded C^0 solution of problem (1), (2).

We omit the proof of this existence result. The detailed proof of this theorem, as well as of the next one, is provided in [4]. The main idea of the proof can be summarized as follows. Define the integral operator $\mathcal{T}$: if $u \succ \varphi$; then $\mathcal{T}u \succ \varphi$ is determined on E by the right-hand side of the integral equation (3). Then the unique fixed-point $u = \mathcal{T}u$ is obtained by means of the Banach contraction principle in suitable function spaces.

We formulate now the following assumption

Assumption 1.2. Suppose that

1) the functions $\lambda_1, m_{f,\varphi} \in L^1[0,a]$,
2) $|f(t,x,\widetilde{\varphi}_{(t,x)})| \le m_{f,\varphi}(t)$,

3) the functions

$$t \longmapsto \int_0^t \frac{1}{\sqrt{t-s}}\, \lambda_1(s)\, ds \qquad \text{and} \qquad t \longmapsto \int_0^t \frac{1}{\sqrt{t-s}}\, m_{f,\varphi}(s)\, ds$$

are bounded.

Further, we consider equation (1) with functionals at the derivatives.

Theorem 1.2 ([4, Theorem 2.3]). *Let $\varphi \in CB(E_0)$, $\partial\varphi \in CB(E_0)^n$, $\lambda \in L^1[0,a]$. Suppose that Assumption 1.2 and the inequality*

$$|f(t,x,w) - f(t,x,\bar{w})| \le \lambda(t)\, \|w - \bar{w}\|_0 + \lambda_1(t)\, \|\partial(w - \bar{w})\|_0$$

are satisfied. Assume that the condition

$$(t-s)^{1/2} \int_s^t \frac{c_0}{c_1}\, \lambda_1(\zeta)\, (t-\zeta)^{-1/2} (\zeta - s)^{-1/2} d\zeta \le \theta_1 < 1 \quad \textit{for } t > s$$

is satisfied. Then problem (1), (2) has a unique $C^{0,1}$ solution.

2. The Quasilinearization Method

The theorems of Section 1 are the reference point for the consideration concerning non-monotone iterative techniques which provide fast convergent sequences of approximate functions. In fact, the Banach contraction principle is based on the direct iteration method, where the convergence of function sequences of successive approximations is measured by a geometric sequence. Sequences in the quasilinearization method converge much faster.

We construct the sequence of successive approximations in the following way. Suppose that $u^{(0)} \in C(\widetilde{E}, R)$ is a given function. Having $u^{(\nu)} \in C(\widetilde{E}, R)$ already defined, the next function $u^{(\nu+1)}$ is a solution of the initial-value problem

$$\mathcal{P}u(t,x) = f\Big(t,x,u^{(\nu)}_{(t,x)}\Big) + \partial_w f\Big(t,x,u^{(\nu)}_{(t,x)}\Big) \cdot (u - u^{(\nu)})_{(t,x)}, \tag{5}$$

$$u(t,x) = \varphi(t,x) \ \text{ on } E_0. \tag{6}$$

Note that equation (5) is still differential-functional, but its right-hand side is linear with respect to u. The convergence of the sequence $\{u^{(\nu)}\}$ depends on the initial function $u^{(0)}$ and on the domain and regularity of the linear operator $\partial_w f(t,x,u^{(\nu)}_{(t,x)})$. Based on the integral formula (3) with the function f replaced by the right-hand side of (5), we can represent the $C^{0,1}$ solution $u^{(\nu+1)}$ of problem (5), (6) as follows:

$$\begin{aligned} u^{(\nu+1)}(t,x) \;=\; & \widetilde{\varphi}(t,x) + \int_0^t \int_{R^n} \Gamma(t,x,s,y) \\ & \times \Big\{ f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) + \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) \cdot (u^{(\nu+1)} - u^{(\nu)})_{(s,y)} \Big\}\, dy\, ds. \end{aligned} \tag{7}$$

Denote

$$\|u\|_t = \sup \{ |u(s,y)| : \ (s,y) \in \widetilde{E}, \ s \le t \}$$

for $u \in CB(\widetilde{E})$ and $t \in (0,a]$. If $F : C(B) \longrightarrow R\,(\text{or}\, R^k)$ is a bounded linear operator, then its norm is defined by

$$\|F\|_{C(B)} = \sup\{\|Fu\|_0 \ : \ u \in C(B), \ \|u\|_0 \le 1\}.$$

Now, we give sufficient conditions for the convergence of the sequence $\{u^{(\nu)}\}$.

Theorem 2.1. *Let $\varphi \in CB(E_0)$, $m_f, f(\cdot,x,0) \in L^1[0,a]$, $f(t,\cdot,0) \in C(R^n)$ and $|f(t,x,0)| \le m_f(t)$. Assume that:*

1) *there is a function $\lambda \in L^1[0,a]$ such that*

$$\|\partial_w f(t,x,w)\|_{C(B)} \le \lambda(t);$$

2) *there is a function $\sigma : [0,a]\times[0,+\infty) \longrightarrow [0,+\infty)$, integrable with respect to the first variable, continuous and nondecreasing with respect to the last variable, such that $\sigma(t,0) = 0$ and*

$$\|\partial_w f(t,x,w) - \partial_w f(t,x,\bar{w})\|_{C(B)} \le \sigma(t, \|w-\bar{w}\|_0) \quad \text{on } E \times C(B);$$

3) *there exists a nondecreasing and continuous function $\psi_0 : [0,a] \longrightarrow [0,+\infty)$ which satisfies the inequalities*

$$\psi_0(t) \ge |u^{(1)}(t,x) - u^{(0)}(t,x)| \tag{8}$$

and

$$\psi_0(t) \ge \widetilde{c}_0 \int_0^t \psi_0(s)\,\sigma(s,\psi_0(s))\,\exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta)d\zeta\right) ds \tag{9}$$

where $\widetilde{c}_0 = c_0\Big(4\pi/k_0\Big)^{n/2}$.

Then the sequence $\{u^{(\nu)}\}$ of solutions of (5), (6) *is well defined and uniformly fast convergent to u^*, where u^* is the unique $C^{0,1}$ solution of problem* (1), (2). *The convergence rate is characterized by the condition*

$$\frac{\|u^{(\nu+1)} - u^*\|_t}{\|u^{(\nu)} - u^*\|_t} \longrightarrow 0 \qquad \text{as} \qquad \nu \longrightarrow \infty \tag{10}$$

for $t \in [0,a]$.

Proof. The existence and uniqueness of the $C^{0,1}$ solution $u^{(\nu+1)}$ of problem (5), (6) follows from Theorem 1.1. We estimate the differences $u^{(\nu+1)} - u^{(\nu)}$ for $\nu = 0,1,\dots$. Put $\omega^{(\nu)} = u^{(\nu+1)} - u^{(\nu)}$. Since the function $u^{(\nu+1)}$ satisfies the integral identity (7), which applies also to $u^{(\nu+2)}$, we have the integral error equation

$$\omega^{(\nu+1)}(t,x) = \int_0^t \int_{R^n} \Gamma(t,x,s,y) \Big\{ f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big) - f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) + \partial_w f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big)\,\omega^{(\nu+1)}_{(s,y)} - \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)}\Big)\,\omega^{(\nu)}_{(s,y)} \Big\}\,dyds.$$

By the Hadamard mean-value theorem, we get

$$f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big) - f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) = \int_0^1 \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)} + \zeta\,(u^{(\nu+1)}_{(s,y)} - u^{(\nu)}_{(s,y)})\Big)\,d\zeta\;(u^{(\nu+1)}_{(s,y)} - u^{(\nu)}_{(s,y)}).$$

Hence we rewrite the error equation as follows:

$$\omega^{(\nu+1)}(t,x) = \int_0^t \int_{R^n} \Gamma(t,x,s,y)\,\Big\{ \partial_w f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big)\,\omega^{(\nu+1)}_{(s,y)} + \int_0^1 \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)} + \zeta\,\omega^{(\nu)}_{(s,y)}\Big)\,\omega^{(\nu)}_{(s,y)} d\zeta - \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)}\Big)\,\omega^{(\nu)}_{(s,y)} \Big\}\,dyds.$$

From the above error equation, based on the assumptions 1) and 2) of our theorem, we derive the inequality

$$|\omega^{(\nu+1)}(t,x)| \le \int_0^t \int_{R^n} |\Gamma(t,x,s,y)|\,\Big\{ \|\partial_w f\Big(t,x,u^{(\nu+1)}_{(s,y)}\Big)\|_{C(B)}\,\|\omega^{(\nu+1)}_{(s,y)}\|_0 + \|\omega^{(\nu)}_{(s,y)}\|_0 \int_0^1 \Big\|\partial_w f\Big(s,y,u^{(\nu)}_{(s,y)} + \zeta\omega^{(\nu)}_{(s,y)}\Big) - \partial_w f\Big(t,x,u^{(\nu)}_{(s,y)}\Big)\,\Big\|_{C(B)} d\zeta \Big\}\,dy\,ds$$
$$\le \int_0^t \int_{R^n} |\Gamma(t,x,s,y)|\,\Big\{\lambda(s)\|\omega^{(\nu+1)}_{(s,y)}\|_0 + \|\omega^{(\nu)}_{(s,y)}\|_0 \int_0^1 \sigma(s,\|\zeta\omega^{(\nu)}_{(s,y)}\|_0)d\zeta\Big\}dy\,ds.$$

It follows from Lemma 1.1 that

$$\widetilde{c}_0 \ge \int_{R^n} |\Gamma(t,x,s,y)|\,dy \qquad \text{for } 0 \le s \le t \le a, \quad x \in R^n.$$

By the monotonicity condition for the function σ, we have

$$|\omega^{(\nu+1)}(t,x)| \le \widetilde{c}_0 \int_0^t \Big\{\lambda(s)\,\|\omega^{(\nu+1)}\|_s + \|\omega^{(\nu)}\|_s\,\sigma(s,\|\omega^{(\nu)}\|_s)\,\Big\}ds,$$

which leads to the recurrent integral inequality

$$\|\omega^{(\nu+1)}\|_t \le \widetilde{c}_0 \int_0^t \Big\{\lambda(s)\,\|\omega^{(\nu+1)}\|_s + \|\omega^{(\nu)}\|_s\,\sigma(s, \|\omega^{(\nu)}\|_s)\,\Big\}ds.$$

Applying the Gronwall lemma to the above inequality, we get its explicit form

$$\|\omega^{(\nu+1)}\|_t \le \widetilde{c}_0 \int_0^t \|\omega^{(\nu)}\|_s\,\sigma(s, \|\omega^{(\nu)}\|_s)\,\exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta)d\zeta\right)ds. \tag{11}$$

We show that the sequence $\{\omega^{(\nu)}\}$ is uniformly convergent to 0. With the given function ψ_0, satisfying conditions (8) and (9), the functions $\psi_{\nu+1} : [0,a] \longrightarrow [0,+\infty)$ for $\nu = 0,1,\dots$ are defined by the recurrent formula

$$\psi_{\nu+1}(t) = \widetilde{c}_0 \int_0^t \psi_\nu(s)\,\sigma(s,\psi_\nu(s))\,\exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta)d\zeta\right)ds. \tag{12}$$

It is easy to verify that the sequence $\{\psi_\nu\}$ of continuous nondecreasing functions is nonincreasing as $\nu \longrightarrow \infty$. This can be can verified by induction on ν, applying (9) and (12). Furthermore, we have

$$\psi_\nu(t) \ge |\omega^{(\nu)}(t,x)| \qquad \text{on } E \tag{13}$$

for all $\nu = 0,1,\dots$. The proof of (13) can be done also by induction on ν. For $\nu = 0$ condition (13) coincides with (8). If inequality (13) holds for any fixed ν, then it carries over to $\nu+1$ by virtue of (11) and (12). In addition, the sequence $\{\psi_\nu\}$, being a nonincreasing and bounded from below, is convergent to a limit function $\bar{\psi}$, where $0 \le \bar{\psi}(t) \le \psi_0(t)$. Passing to the limit in equality (12) as $\nu \longrightarrow \infty$, we get

$$\bar{\psi}(t) = \widetilde{c}_0 \int_0^t \bar{\psi}(s)\sigma(s,\psi(s))\,\exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta)d\zeta\right)ds.$$

By Gronwall's lemma, we have $\bar{\psi} \equiv 0$. Since ψ_ν are nonidecreasing functions and equality (12) is fulfilled, we have

$$\frac{\psi_{\nu+1}(t)}{\psi_\nu(t)} \le \widetilde{c}_0 \int_0^t \sigma(s,\psi_\nu(s))\,\exp\left(\widetilde{c}_o \int_s^t \lambda(\zeta)d\zeta\right)ds. \tag{14}$$

Recalling that the function $\sigma(s,\cdot)$ is continuous and monotone, we observe that

$$\sigma(s,\psi_\nu(s)) \searrow 0 = \sigma(s,0) \quad \text{as } \nu \longrightarrow \infty.$$

Since $\psi_\nu \searrow 0$ as $\nu \longrightarrow \infty$, passing to the limit in (14) we get

$$\frac{\psi_{\nu+1}(t)}{\psi_\nu(t)} \longrightarrow 0 \quad \text{as } \nu \longrightarrow \infty.$$

Hence by d'Alembert's criterion, the series $\sum_{\nu=0}^\infty \psi_\nu(t)$ is uniformly convergent. Since $\|\omega^{(\nu)}\|_t \le \psi_\nu(t)$, the sequence $u^{(\nu)}$ is fundamental. Indeed, the norms of the differences $u^{(\nu)} - u^{(\nu+k)}$ can be estimated by a partial sum of the series $\sum_{\nu=0}^{\infty} \psi_\nu(t)$ as follows:

$$\begin{aligned}\|u^{(\nu)} - u^{(\nu+k)}\|_t &\le \|u^{(\nu)} - u^{(\nu+1)}\|_t + \cdots + \|u^{(\nu+k-1)} - u^{(\nu+k)}\|_t \\ &\le \psi_\nu(t) + \cdots + \psi_{\nu+k}(t).\end{aligned}$$

Consequently, the sequence $\{u^{(\nu)}\}$ is uniformly convergent to a continuous function u^*. We prove that the function u^* satisfies equation (1). The initial condition (2), i.e., $u^* \succ \varphi$ is fulfilled, because $u^{(\nu)} \succ \varphi$ and $u^{(\nu)} \longrightarrow u^*$ as $\nu \longrightarrow \infty$. It suffices to make the following observation. The integral equation (7) for the functions $u^{(\nu)}$ and $u^{(\nu+1)}$ is equivalent of problem (5), (6). Then, passing to the limit as $\nu \to \infty$ in equation (7) we obtain the integral equality (3) with $u = u^*$. By Theorem 1.1, the function u^* is a unique solution of problem (1), (2). The convergence rate is determined by estimate (13) and condition (14). This convergence is faster than a geometric convergence.

Now, we show that condition (10) is satisfied. Subtracting (3) with $u = u^*$ from the integral equation (7) and performing similar estimations as in the case of $\omega^{(\nu+1)}$, we get the inequality

$$\begin{aligned}&\|u^{(\nu+1)} - u^*\|_t \\ &\le \tilde{c}_0 \int_0^t \Big\{ \lambda(s)\, \|u^{(\nu+1)} - u^*\|_s + \|u^{(\nu)} - u^*\|_s\, \sigma(s, \|u^{(\nu)} - u^*\|_s) \Big\}\, ds.\end{aligned}$$

By Gronwall's lemma we have

$$\|u^{(\nu+1)} - u^*\|_t \;\le\; \tilde{c}_0 \int_0^t \|u^{(\nu)} - u^*\|_s\, \sigma(s, \|u^{(\nu)} - u^*\|_s)\, \exp\left(\tilde{c}_0 \int_s^t \lambda(\zeta) d\zeta\right) ds.$$

Since the semi-norm scale $\|\cdot\|_t$ is nondecreasing in t, we get the estimate

$$\|u^{(\nu+1)} - u^*\|_t \;\le\; \tilde{c}_0\, \|u^{(\nu)} - u^*\|_t \int_0^t \sigma(s, \|u^{(\nu)} - u^*\|_s)\, \exp\left(\tilde{c}_0 \int_s^t \lambda(\zeta) d\zeta\right) ds.$$

Thus we obtain the desired assertion

$$\frac{\|u^{(\nu+1)} - u^*\|_t}{\|u^{(\nu)} - u^*\|_t} \le \tilde{c}_0 \int_0^t \sigma(s, \|u^{(\nu)} - u^*\|_s)\, \exp\left(\tilde{c}_0 \int_s^t \lambda(\zeta) d\zeta\right) ds \longrightarrow 0$$

as $\nu \longrightarrow \infty$. This completes the proof of assertion (10) and Theorem 2.1. $\square$

In condition (8) we apply the unknown function $u^{(1)}$, which ought to have been obtained in Theorem 2.1. This assumption is made only for the sake of simplicity. A priori estimates of $|u^{(1)}(t,x) - u^{(0)}(t,x)|$ can be derived in the following way.

Remark 2.1. As a particular case of (7), we have the integral equation

$$u^{(1)}(t,x) = \widetilde{\varphi}(t,x) + \int_0^t \int_{R^n} \Gamma(t,x,s,y) \times \left\{ f\left(s,y,u^{(0)}_{(s,y)}\right) + \partial_w f\left(s,y,u^{(0)}_{(s,y)}\right) \left(u^{(1)} - u^{(0)}\right)_{(s,y)} \right\} dy\, ds. \quad (15)$$

Thus under the assumption 1) of Theorem 2.1 we obtain the integral inequality

$$\|u^{(1)} - u^{(0)}\|_t \le \|\widetilde{\varphi} - u^{(0)}\|_t + \widetilde{c}_0 \int_0^t \left\{ m_f(s) + \lambda(s)\Big[\|u^{(0)}\|_s + \|u^{(1)} - u^{(0)}\|_s \Big] \right\} ds. \quad (16)$$

Using the Gronwall lemma, we get the estimate

$$\|u^{(1)} - u^{(0)}\|_t \le \|\widetilde{\varphi} - u^{(0)}\|_t + \widetilde{c}_0 \int_0^t \left\{ m_f(s) + \lambda(s)\Big[\|u^{(0)}\|_s + \|\widetilde{\varphi} - u^{(0)}\|_s \Big] \right\} \exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta) d\zeta \right) ds.$$

In particular, if we put $u^{(0)} = \widetilde{\varphi}$ (the most natural initial conjecture), then

$$\|u^{(1)} - u^{(0)}\|_t \le \widetilde{c}_0 \int_0^t \left\{ m_f(s) + \lambda(s) \|\widetilde{\varphi}\|_s \right\} \exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta) d\zeta \right) ds.$$

However, one can get more precise estimates by making use of the modulus of continuity of the initial function φ. Since

$$|\widetilde{\varphi}(t,x) - u^{(0)}(t,x)| = |\widetilde{\varphi}(t,x) - \varphi(0,x)| = \left| \int_{R^n} \Gamma(t,x,0,y) \Big\{ \varphi(0,y) - \varphi(0,x) \Big\} dy \right|$$
$$\le \widetilde{c}_0 \int_{R^n} e^{-\|\eta\|^2} \min \Big\{ 2\|\varphi\|_0, \max_{|\xi| \le 2\|\eta\|\sqrt{t}/\sqrt{k_0}} |\varphi(0,x+\xi) - \varphi(0,x)| \Big\} d\eta := \Phi(t,x),$$

we get

$$\|u^{(1)} - u^{(0)}\|_t \le \|\Phi(t,\cdot)\|_0 + \widetilde{c}_0 \int_0^t \left\{ m_f(s) + \lambda(s) \Big[\|\varphi\|_0 + \|\Phi(s,\cdot)\|_0 \Big] \right\} \exp\left(\widetilde{c}_0 \int_s^t \lambda(\zeta) d\zeta \right) ds.$$

Now, we discuss two main cases (Lipschitz and Hölder), generated by particular functions σ satisfying the condition 2) of Theorem 2.1.

Example 2.1 (the Lipschitz case). Assume a generalized Lipschitz condition for $\partial_\omega f(t,x,\cdot)$, i.e., define $\sigma(t,r) = \lambda_1(t)\,r$, where $\lambda_1 \in L^1[0,a]$. If the function ψ_0 is satisfies the integral equation

$$\psi_0(t) = \widetilde{c_0}\int_0^t \psi_0^2(s)\lambda_1(s)\exp\left(\widetilde{c_0}\int_s^t \lambda(\zeta)d\zeta\right)ds + \|u^{(1)} - u^{(0)}\|_a \exp\left(\widetilde{c_0}\int_0^t \lambda(s)ds\right),$$

then both inequalities (8) and (9) are fulfilled. This integral equation easily reduces to the equation

$$\widetilde{\psi}_0(t) = \int_0^t \widetilde{\psi}_0^2(s)\widetilde{\lambda}_1(s)\,ds + \|u^{(1)} - u^{(0)}\|_a,$$

where

$$\widetilde{\psi}_0(t) = \psi_0(t)\exp\left(-\widetilde{c_0}\int_0^t \lambda(s)ds\right), \quad \widetilde{\lambda}_1(t) = \widetilde{c}_o\lambda_1(t)\exp\left(\widetilde{c_0}\int_0^t \lambda(s)ds\right).$$

The solution $\widetilde{\psi}_0$ of this equation is given by

$$\widetilde{\psi}_0(t) = \left[\frac{1}{c} - \int_0^t \widetilde{\lambda}_1(s)ds\right]^{-1},$$

where $c = \|u^{(1)} - u^{(0)}\|_a$. Therefore the function ψ_0 defined by

$$\psi_0(t) = \left[\frac{1}{c} - \widetilde{c_0}\int_0^t \lambda_1(s)\exp\left(\widetilde{c_0}\int_0^s \lambda(\zeta)d\zeta\right)ds\right]^{-1}\exp\left(\widetilde{c_0}\int_0^t \lambda(s)ds\right)$$

satisfies conditions (8) and (9).

In this case we obtain the so-called Newton rate of convergence, which we formulate as follows.

Corollary 2.1 (the Lipschitz case). *If all assumptions of Theorem* 2.1 *are satisfied with* $\sigma(t,r) = \lambda_1(t)\,r$, *then*

$$\|u^{(\nu)} - u^*\|_t \longrightarrow 0 \quad \textit{and} \quad \frac{\|u^{(\nu+1)} - u^*\|_t}{\|u^{(\nu)} - u^*\|_t^2} \le C_0 < +\infty$$

as $\nu \longrightarrow +\infty$ (*with some constant* $C_0 \ge 0$).

Proof. Arguing as in the proof of Theorem 2.1, we obtain the recurrence inequality

$$\|u^{(\nu+1)} - u^*\|_t \le C_0\|u^{(\nu)} - u^*\|_t^2 \int_0^t \lambda_1(s)\exp\left(\widetilde{c_0}\int_0^t \lambda(s)ds\right)ds,$$

which completes the proof. □

Example 2.2 (the Hölder case). This is a generalization of Example 2.1. Let $\sigma(t,r) = \lambda_2(t)\, r^\delta$ for $\delta \in (0,1]$. Then the inequality from the assumption 2) of Theorem 2.1 is the Hölder condition of derivative. If we formulate the integral equation

$$\psi_0(t) = \widetilde{c_0} \int_0^t \psi_0^{\delta+1}(s)\lambda_2(s) \exp\left(\widetilde{c_0} \int_s^t \lambda(\zeta) d\zeta\right) ds$$
$$+ \|u^{(1)} - u^{(0)}\|_a \exp\left(\widetilde{c_0} \int_0^t \lambda(s) ds\right),$$

then the solution is given by

$$\psi_0(t) = \exp\left(-\widetilde{c_0} \int_0^t \lambda(\zeta) d\zeta\right) \left[\frac{1}{c^\delta} - \delta\widetilde{c_0} \int_0^t \lambda_2(s) \exp\left(\delta\widetilde{c_0} \int_0^s \lambda(\zeta) d\zeta\right) ds\right]^{-1/\delta},$$

where $c = \|u^{(1)} - u^{(0)}\|_a$. In this case the convergence is of order $1+\delta$, i.e.,

$$\|u^{(\nu)} - u^*\|_t \longrightarrow 0 \qquad \text{as} \quad \nu \longrightarrow +\infty$$

and

$$\frac{\|u^{(\nu+1)} - u^*\|_t}{\|u^{(\nu)} - u^*\|_t^{1+\delta}} \le C_0 < +\infty, \qquad \text{where} \quad C_0 \ge 0.$$

3. The Quasilinearization Method – A Nonlinear Term Dependent on Derivatives

Consider a more complicated functional dependence: a Hale-type functional $u_{(t,x)}$ acting on partial derivatives of the unknown function. This situation, in general, imposes additional assumptions on the initial function φ, namely: $\partial\varphi$ has to be continuous on B. However, the convergence of the sequence $\{u^\nu\}$ is much stronger because we get the convergence with respect to a stronger norm $\|\cdot\|_0'$, where

$$\|w\|_0' = \|w\|_0 + \|\partial w\|_0$$

for $w \in C^{0,1}(B)$, the class of all $w \in C(B)$ such that $\partial w \in C(B)^n$. Similarly, we define the seminorm $\|\cdot\|_t'$ on $C^{0,1}(\widetilde{E})$:

$$\|u\|_t' = \|u\|_t + \|\partial u\|_t \qquad \text{for} \quad u \in C^{0,1}(\widetilde{E}).$$

The operator norm $\|\cdot\|_{C^{0,1}(B)}$ is defined in the space of all linear and bounded operators acting on $C^{0,1}(B)$ by the formula

$$\|F\|_{C^{0,1}(B)} = \sup\left\{\|Fw\|_0' : \ \|w\|_0' \le 1\right\}.$$

Theorem 3.1. *Let $\varphi \in CB(E_0)$, $\partial\varphi \in CB(E_0)^n$ and $f(t,\cdot,\widetilde{\varphi}_{(t,x)}) \in C(R^n)$. Suppose that:*

1) *Assumption 1.2 and the inequality $\|\partial_w f(t,x,w)\|_{C^{0,1}(B)} \le \lambda_1(t)$ are satisfied;*

2) *there is a function* $\sigma : [0,a]\times[0+\infty) \longrightarrow [0,+\infty)$, *integrable with respect to the first variable, continuous and nondecreasing with respect to the last variable, and*

$$\|\partial_w f(t,x,w) - \partial_w f(t,x,\bar{w})\|_{C^{0,1}(B)} \le \sigma(t, \|w-\bar{w}\|_0^{'});$$

3) *the conditions*

$$(t-s)^{1/2}\int\limits_s^t \frac{c_0}{c_1}\,\lambda_1(\zeta)\,(t-\zeta)^{-1/2}(\zeta-s)^{-1/2}d\zeta \le \theta_1\,,$$

$$(t-s)^{1/2}\int\limits_s^t \frac{c_0}{c_1}\,\sigma(\zeta,\psi_0(\zeta))\,(t-\zeta)^{-1/2}(\zeta-s)^{-1/2}d\zeta \le \theta_2$$

hold true for $0 \le s < t \le a$, *where* $\theta_1, \theta_2 \in (0,1)$ *and* $\theta_1+\theta_2<1$;

4) *there exists a nondecreasing, continuous function* $\psi_0 : [0,a] \longrightarrow [0,+\infty)$ *which satisfies the inequalities*

$$\psi_0(t) \ge \|u^{(1)} - u^{(0)}\|_t + \|\partial(u^{(1)}-u^{(0)})\|_t\,,$$

$$\psi_0(t) \ge \int\limits_0^t \Big\{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\Big\}\Big\{\,\lambda_1(s) + \sigma(s,\psi_0(s))\,\Big\}\,\psi_0(s)\,ds\,,$$

where $\widetilde{c}_1 = c_1\left(4\pi/k_0\right)^{n/2}$.

Then the sequence $\{u^{(\nu)}\}$ *of solutions of* (5), (6) *is well defined and uniformly fast convergent to* u^* *in* $\|\cdot\|_t^{'}$, *where* u^* *is a unique* $C^{0,1}$ *solution of problem* (1), (2). *Moreover,*

$$\frac{\|u^{(\nu+1)} - u^*\|_t^{'}}{\|u^{(\nu)} - u^*\|_t^{'}} \longrightarrow 0 \qquad as \qquad \nu \longrightarrow \infty$$

for $t \in (0,a]$.

Proof. The method of proving is similar to that used in Theorem 2.1. Applying the Hadamard mean-value theorem and the assumptions 1), 3), we get

$$\begin{aligned} |\omega^{(\nu+1)}(t,x)| \;\le\; & \int\limits_0^t\int\limits_{R^n} |\Gamma(t,x,s,y)|\,\bigg|\, f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big) - f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) \\ & + \partial_w f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big)\,\omega^{(\nu+1)}_{(s,y)} - \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)}\Big)\,\omega^{(\nu)}_{(s,y)}\,\bigg|\,dyds \\ \le\; & \widetilde{c}_0\int\limits_0^t \Big\{\lambda_1(s)||\omega^{(\nu+1)}||_s^{'} + ||\omega^{(\nu)}||_s^{'}\sigma(s,||\omega^{(\nu)}||_s^{'})\Big\}\,ds. \end{aligned}$$

In an analogous way we obtain the estimates

$$
\begin{aligned}
|\partial_j \omega^{(\nu+1)}(t,x)| \;\le\; & \int_0^t \int_{R^n} |\partial_j \Gamma(t,x,s,y)| \bigg| f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big) - f\Big(s,y,u^{(\nu)}_{(s,y)}\Big) \\
& + \partial_w f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big)\, \omega^{(\nu+1)}_{(s,y)} - \partial_w f\Big(s,y,u^{(\nu)}_{(s,y)}\Big)\, \omega^{(\nu)}_{(s,y)} \bigg| dyds \\
\le & \int_0^t \int_{R^n} |\partial_j \Gamma(t,x,s,y)| \bigg\{ \|\partial_w f\Big(s,y,u^{(\nu+1)}_{(s,y)}\Big)\|_{C^{0,1}(B)} \|\omega^{(\nu+1)}_{(s,y)}\|_0' \\
& + \int_0^1 \|\partial_w f\Big(s,y,u^{(\nu)}_{(s,y)} + \zeta \omega^{(\nu)}_{(s,y)}\Big) - \partial_w f\Big(t,x,u^{(\nu)}_{(s,y)}\Big)\|_{C^{0,1}(B)}\, d\zeta \|\omega^{(\nu)}_{(s,y)}\|_0' \bigg\} dyds \\
\le & \int_0^t \int_{R^n} |\partial_j \Gamma(t,x,s,y)| \Big\{ \lambda_1(s)\, \|\omega^{(\nu+1)}\|_s' + \|\omega^{(\nu)}\|_s'\, \sigma(s, \|\omega^{(\nu)}\|_s') \Big\} dy\, ds.
\end{aligned}
$$

On account of the estimate of Lemma 1.1 we obtain

$$
|\partial_j \omega^{(\nu+1)}(t,x)| \le \widetilde{c}_1 \int_0^t (t-s)^{-1/2} \Big\{ \lambda_1(s)\, \|\omega^{(\nu+1)}\|_s' + \|\omega^{(\nu)}\|_s'\, \sigma(s, \|\omega^{(\nu)}\|_s') \Big\} ds.
$$

Summing the estimates for $\|\omega^{(\nu+1)}\|_t$ and $\|\partial\omega^{(\nu+1)}\|_t$, we arrive at the integral error inequality

$$
\|\omega^{(\nu+1)}\|_t' \le \int_0^t \Big\{ \widetilde{c}_0 + \widetilde{c}_1\, (t-s)^{-1/2} \Big\} \Big\{ \lambda_1(s) \|\omega^{(\nu+1)}\|_s' + \|\omega^{(\nu)}\|_s' \sigma(s, \|\omega^{(\nu)}\|_s') \Big\}\, ds.
$$

Now, we derive explicit estimates of $\|\omega^{(\nu+1)}\|_t'$ and show that the sequence $\{\omega^{(\nu)}\}$ tends to 0. Define the sequence $\{\psi_\nu\}$ by

$$
\psi_{\nu+1}(t) = \int_0^t \Big\{ \widetilde{c}_0 + \widetilde{c}_1\, (t-s)^{-1/2} \Big\} \Big\{ \lambda_1(s)\, \psi_{\nu+1}(s) + \psi_\nu(s)\, \sigma(s, \psi_\nu(s)) \Big\}\, ds.
$$

Under the assumptions 3) and 4), the sequence $\{\psi_\nu\}$ is nondecreasing and convergent to $\bar\psi \equiv 0$ as $\nu \longrightarrow \infty$, which is a unique solution of the limit integral equation

$$
\bar\psi(t) = \int_0^t \Big\{ \widetilde{c}_0 + \widetilde{c}_1\, (t-s)^{-1/2} \Big\} \Big\{ \lambda_1(s) + \sigma(s, \bar\psi(s)) \Big\} \bar\psi(s)\, ds.
$$

Since the series $\sum_{\nu=0}^{\infty} \psi_\nu(t)$ is uniformly convergent, the sequence $\{u^{(\nu)}\}$ is fundamental with respect to $\|\cdot\|_t'$. This follows by arguments similar to those applied in the proof of Theorem 2.1. Hence this sequence is fast convergent to the $C^{0,1}$ solution of problem (1), (2). $\square$

Remark 3.1. In Remark 2.1, the estimate of the difference $|u^{(1)}(t,x) - u^{(0)}(t,x)|$ is based on the integral inequality (16). Observe that under the assumptions of Theorem 3.1 the integral equation (15) implies the estimates for $u^{(1)} - u^{(0)}$ and $\partial(u^{(1)} - u^{(0)})$, the summing of which leads to

$$\|u^{(1)} - u^{(0)}\|_t' \leq \|\widetilde{\varphi} - u^{(0)}\|_t' + \int_0^t \Big\{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\Big\}\Big\{m_{f,\varphi}(s) + \lambda_1(s)\Big[\|u^{(0)} - \widetilde{\varphi}\|_s' + \|u^{(1)} - u^{(0)}\|_s'\Big]\Big\}ds.$$

Therefore we have the estimate

$$\|u^{(1)} - u^{(0)}\|_t' \leq \psi_0(t),$$

where ψ_0 is a solution of the integral equation

$$\psi_0(t) = \|\widetilde{\varphi} - u^{(0)}\|_t' + \int_0^t \Big\{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\Big\}\Big\{m_{f,\varphi}(s) + \lambda_1(s)\Big[\|u^{(0)} - \widetilde{\varphi}\|_s' + \psi_0(s)\Big]\Big\}ds. \quad (17)$$

Taking $u^{(0)} = \widetilde{\varphi}$, we get an optimal error estimate. If, in addition, we assume that $m_{f,\varphi}(t) = \lambda_1(t) = K_0 t^\kappa$ and $\|\widetilde{\varphi} - u^{(0)}\|_t' \leq K_1 t^\kappa$, with given $\kappa > -1/2$ and $K_0, K_1 \geq 0$, then instead of equation (17) it is convenient to write another integral equation for ψ_0 :

$$\psi_0(t) = K_1 t^\kappa + \int_0^t \Big\{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\Big\}\Big(1 + \psi_0(s)\Big)K_0 s^\kappa\,ds,$$

which has a unique solution on $[0,a]$. There is $\widetilde{K} \geq K_1 \geq 0$ such that $\psi_0(t) \leq \widetilde{K}\,t^\kappa$.

Example 3.1. We assume as in Remark 3.1 that $m_{f,\varphi}(t) = \lambda_1(t) = K_0 t^\kappa$, where $\kappa > -1/2$. Let $\sigma(t,r) = \lambda_2(t) r^\delta$, where $\lambda_2(t) = K_2 t^\kappa$. If $\kappa > -\frac{1}{2}\cdot\frac{1}{1+\delta}$, then the assumption 4) of Theorem 3.1 is fulfilled for sufficiently small t when we put $\psi_0(t) = C t^\kappa e^{\sqrt{t}}$ with a sufficiently large positive constant C.

Now we study equation (1) without functional dependence on the derivatives, including $\partial u(t,x)$. In fact, we consider a differential-functional equation of the form

$$\mathcal{P}u(t,x) = \widetilde{f}(t,x,u_{(t,x)},\partial u(t,x)).$$

This is possible in the functional model (1) by a suitable choice of the right-hand side f. It can be defined as follows:

$$f(t,x,w) = \widetilde{f}(t,x,w,\partial w(0,0)).$$

Define the semi-norms

$$\|u\|_t'' = \|u\|_t + \sup_{0<s\leq t} \sqrt{s}\,\|\partial u(s,\cdot)\|_0 \qquad \text{for} \quad t \in (0,a].$$

Theorem 3.2. *Let* $\varphi \in CB(E_0)$, $f(t,\cdot,\widetilde{\varphi}_{(t,x)}) \in C(R^n)$, $f(\cdot,x,\widetilde{\varphi}_{(t,x)}), \lambda \in L^1[0,a]$ *and* $\lambda_1(t) = \lambda(t)\sqrt{t}$. *Assume that:*

1) *Assumption* 1.2 *and the inequality*
$$|f(t,x,w) - f(t,x,\bar{w})| \le \lambda(t)\,\|w-\bar{w}\|_0 + \lambda_1(t)\,\|\partial(w-\bar{w})(0,0)\|_0$$
are satisfied;
2) *there is a function* $\sigma : [0,a]\times[0+\infty) \longrightarrow [0,+\infty)$, *integrable with respect to the first variable, continuous and nondecreasing with respect to the last variable, and*
$$\begin{aligned}&|\,[\partial_w f(t,x,w) - \partial_w f(t,x,\bar{w})]\,h| \\ &\quad\le \Big(\|h\|_0 + \sqrt{t}\,\|\partial h(0,0)\|_0\Big)\,\sigma\Big(t, \|w-\bar{w}\|_0 + \sqrt{t}\,\|\partial(w-\bar{w})(0,0)\|_0\Big)\end{aligned}$$
for $h, w, \bar{w} \in C(B)$, $\partial h(0,\cdot), \partial w(0,\cdot), \partial\bar{w}(0,\cdot) \in C([-\tau,\tau])$;
3) *the inequalities*
$$(t-s)^{1/2}\int_s^t \frac{c_0}{c_1}\,\lambda_1(\zeta)\,(t-\zeta)^{-1/2}(\zeta-s)^{-1/2}d\zeta \le \theta_1$$
$$(t-s)^{1/2}\int_s^t \frac{c_0}{c_1}\,\sqrt{\zeta}\,\sigma(\zeta,\psi_0(\zeta))\,(t-\zeta)^{-1/2}(\zeta-s)^{-1/2}d\zeta \le \theta_2$$
are satisfied for $0 \le s < t \le a$, *where* θ_1, $\theta_2 \in (0,1)$ *and* $\theta_1+\theta_2 < 1$;
4) *there exists a nondecreasing, continuous function* $\psi_0 : [0,a] \longrightarrow [0,+\infty)$ *which satisfies the inequalities*
$$\psi_0(t) \ge \|u^{(1)} - u^{(0)}\|''_t\,,$$
$$\psi_0(t) \ge \int_0^t \Big\{\widetilde{c}_0 + \sqrt{t}\,\widetilde{c}_1(t-s)^{-1/2}\Big\}\Big\{\,\lambda(s) + \sigma(s,\psi_0(s))\,\Big\}\,\psi_0(s)\,ds\,,$$
where $\widetilde{c}_1 = c_1\Big(4\pi/k_0\Big)^{n/2}$.

Then the sequence $\{u^{(\nu)}\}$ *of solutions of* (5), (6) *is well defined and uniformly fast convergent to* u^* *in* $\|\cdot\|''_t$, *where* u^* *is a unique* $C^{0,1}$ *solution of problem* (1), (2). *Moreover,*
$$\frac{\|u^{(\nu+1)} - u^*\|''_t}{\|u^{(\nu)} - u^*\|''_t} \longrightarrow 0 \qquad as \qquad \nu \longrightarrow \infty$$
for $t \in (0,a]$.

Proof. The assertions are proved by arguments similar to those used for Theorem 3.1. The most important are the following estimates of $|\omega^{(\nu+1)}(t,x)|$ and

$\sqrt{t}\,|\partial_j\omega^{(\nu+1)}(t,x)|$:

$$|\omega^{(\nu+1)}(t,x)| \le \tilde{c}_0 \int_0^t \Big\{ \lambda(s)\,\|\omega^{(\nu+1)}\|''_s + \|\omega^{(\nu)}\|''_s\,\sigma(s,\|\omega^{(\nu)}\|''_s) \Big\}\,ds$$

and

$$\sqrt{t}\;|\partial_j\omega^{(\nu+1)}(t,x)|$$
$$\le \tilde{c}_1\sqrt{t}\int_0^t (t-s)^{-1/2}\Big\{ \lambda(s)\,\|\omega^{(\nu+1)}\|''_s + \|\omega^{(\nu)}\|''_s\,\sigma(s,\|\omega^{(\nu)}\|''_s) \Big\}\,ds.$$

Taking supremum norms in the left-hand sides of the above inequalities, we get the recurrence integral inequalities

$$\|\omega^{(\nu+1)}\|''_t$$
$$\le \int_0^t \Big\{ \tilde{c}_0 + \sqrt{t}\,\tilde{c}_1(t-s)^{-1/2}\Big\}\Big\{ \lambda(s)\,\|\omega^{(\nu+1)}\|''_s + \|\omega^{(\nu)}\|''_s\,\sigma(s,\|\omega^{(\nu)}\|''_s) \Big\}\,ds.$$

It is seen that

$$\|\omega^{(\nu+1)}\|''_t \le \psi_\nu(t) \longrightarrow 0 \qquad \text{as} \quad \nu\to\infty,$$

where

$$\psi_{\nu+1}(t) = \int_0^t \Big\{ \tilde{c}_0 + \sqrt{t}\,\tilde{c}_1(t-s)^{-1/2}\Big\}\Big\{ \lambda(s)\,\psi_{\nu+1}(s) + \psi_\nu(s)\,\sigma(s,\psi_\nu(s)) \Big\}\,ds.$$

The remaining part of the proof runs in the same way as in Theorem 3.1. □

Remark 3.2. Theorem 3.2 concerns the equation with various types of the Volterra functional dependence on the unknown function. The derivatives have the classical form (without functional dependence). Similarly, the sufficient conditions of convergence for the quasilinearization sequence in Theorems 2.1 and 3.1 are based on the respective existence statements in Theorems 1.1 and 1.2; also, Theorem 3.2 is based on Theorem 2.2 from [4].

References

1. S. Brzychczy, Chaplygin's method for infinite systems of parabolic differential-functional equations. *Univ. Iagellon. Acta Math.* **38**(2000), 153–162.

2. S. Brzychczy, Existence of solutions and monotone iterative method for infinite systems of parabolic differential-functional equations. *Ann. Polon. Math.* **72**(1999), 15–24.

3. S. Brzychczy, On estimation of the speed of the convergence of Chaplygin's succesive approximations for a parabolic system of differential-functional equations. (Russian) *Differentsial'nye Uravneniya* **25**(1989), No. 6, 1050–1052.

4. A. Bychowska and H. Leszczyński, Parabolic equations with functional dependence. *Z. Anal. Anwendungen* **20**(2001), 115–130

5. L. Byszewski, Monotone iterative method for a system of nonlocal initial-boundary parabolic problems. *J. Math. Anal. Appl.* **177**(1993), No. 2, 445–458.
6. A. Friedman, Partial differential equations of parabolic type. *Prentice-Hall, Inc., Englewood Cliffs, N.J.,* 1964.
7. G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, Monotone iterative techniques for nonlinear differential equations. *Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics,* 27. *Pitman (Advanced Publishing Program), Boston, MA; distributed by John Wiley & Sons, Inc., New York,* 1985.
8. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, Linear and quasilinear equations of parabolic type. (Russian) *Nauka, Moscow,* 1967; English translation: *Translation of Mathematical Monographs,* 23. *American Mathematical Society, Providence, R.I.,* 1968.
9. H. Leszczyński, Quasi-linearization methods for a non-linear heat equation with functional dependence. *Georgian Math. J.* **1**(2000), 97–116.
10. H. Leszczyński, Parabolic equations with Hale-type functionals. *Funct. Differ. Equ.* **6**(1999), No. 1–2, 169–175.
11. H. Leszczyński, A new existence result for a non-linear heat equation with functional dependence. *Ann. Soc. Polon. Math.* **37**(1997), 155–181.
12. R. Redlinger, Existence theorem for semilinear parabolic systems with functionals. *Nonlinear Anal.* **8**(1984), No. 6, 667–682.
13. W. Walter, The parabolic Cauchy problem and quenching. *Dynamics of Continuous, Discrete and Impulsive Systems, special issue "Advances in Quenching"* (to appear).
14. W. Walter, The parabolic Cauchy problem. II. *Lakshmikantham's legacy: a tribute on his* 75*th birthday. Nonlinear Anal.* **40**(2000), No. 1-8, *Ser. A: Theory Methods,* 617–627.
15. T. Ważewski, Sur une procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison. *Bull. Acad. Polon. Sci.* **8**(1960), 45–52.
16. J. Wu, Theory and applications of partial functional equations. *Springer-Verlag, New York, Berlin, Heidelberg, Tokyo,* 1996.

(Received 12.07.2001; revised 28.06.2002)

Author's address:
Technical University of Gdańsk
Dept. of Techn. Physics and Applied Mathematics
ul. Narutowicza 11/12, 80-952 Gdańsk
Poland
E-mail: agnes@mifgate.mif.pg.gda.pl