Georgian Mathematical Journal
Volume 9 (2002), Number 3, 495–505

ON INTERSECTIONS OF SMALL PERFECT SETS

H. FAST

Abstract. For a not empty perfect subset of the unit circle $\mathcal{C}$ there is a perfect subset of $\mathcal{C}$ measure zero which being rotated to every position intersects the first set on a nonempty perfect set. This result may be stated in terms of set of distances between pairs of points from these two sets.

A generalization of this result to a product of tori is suggested.

2000 Mathematics Subject Classification: 26-XX, 28-XX.
Key words and phrases: Perfect set, circle, rotation, measure.

1. The fact that the set of distances between pairs of points of the ternary Cantor set fills the unit interval was noticed a long time ago [1]. This fact can be equivalently stated by saying that every line $\{(x, y) : y-x = t\}$, $t \in [-1, 1]$ in the (x, y)-plane meets the cross-product of the Cantor set with itself. Actually, except for a countable set of values of t every such line intersects that cross-product on a nonempty perfect set.

Seing the mentioned lines as 'light rays', the product of the Cantor set with itself appears as an 'impenetrable obstacle' to such light. Since such product is just a kind of 'dust-set' in the plane, its total light-blocking ability seems contrary to intuition.

Another form of the same property: placed on the circle $\mathcal{C}$ of length one the Cantor set intersects its own arbitrarily placed on $\mathcal{C}$ copy and except for a countable set of the copy's exceptional positions, their intersection is a nonempty perfect set.

In the sequel we refer to a not empty perfect set of measure zero as 'Cantor-like'. One may ask about Cantor-like pairs of sets which exhibit a property similar to the one just mentioned of two copies of the ternary Cantor set. We show in this paper that for every perfect subset $P \subset \mathcal{C}$ a Cantor-like set may be found such that the two placed arbitrarily on $\mathcal{C}$ intersect on a Cantor-like set (and unlike in the case of two copies of the Cantor set, wit no exceptional positions.). It needs to be emphasized that the original set P may be small according to any additional criteria of smallness (for instance, its Hausdorff dimension may as small as desired).

The result of this paper may be equivalently stated in terms of set of distances between pairs of points from the two sets. Our result is:

Proposition 1. *For every nonempty perfect subset $P \subset \mathcal{C}$ there is a Cantor-*

ISSN 1072-947X / \$8.00 /

like subset $Q \subset \mathcal{C}$ such that both sets arbitrarily placed on $\mathcal{C}$ intersect on a Cantor-like set.

In the references section we list a number of studies dealing with various aspects of quests similar to the one pursued in this paper.

Close in spirit to this paper is the construction in a general space of a 'tiny' set with total 'light obstructing' capability [2]. Sets of distances between pairs of points of a set or of points taken from two different sets were the subject of two monographs: [3] and [4] and partially of [5] (in the latter one finds among others an old published in [6] result and its subsequent generalizations). Close to the result of this paper is one found in [7], [9]. Finally we would like to mention the papers [9], [10] and a the monograph [11, Problem 23, p. 374; p. 509].

2. All the sets considered in this section are *elementary* subsets of the unit circle, by which we mean finite unions of disjoint closed arcs.

Introduce into considerations a class of numerical positive sequences $[t] = [t_k]_{k=1}^{\infty}$ referred to as *strongly decreasing sequences* which is defined by the conditions: $t_1 < 1$, $\frac{t_{m+1}}{t_m} < \frac{1}{10^n}$. Sequences of that class are subjected to *down adjustment*, an operation consisting in replacement of one sequence by another in the same class with smaller terms or with faster rate of decrease. For instance, in order to satisfy an extra condition a term is made smaller, which causes a necessary change in the part of the sequence following that term. Selecting a subsequence from a given strongly decreasing sequence is another example of down-adjusting.

In the course of our construction infinite iterations of down-adjustment operation take place. Thus created sequence of ever smaller strongly decreasing sequences converge to a limit sequence, also in that class.

An important role in this paper plays the class $\wp(t)$, $t > 0$ of subsets E of $\mathcal{C}$ defined by the condition that out of every couple of point $x, y \in \mathcal{C}$ at a mutual distance t at least one of the two points falls into E.

When $|E| < \frac{1}{2}$, the complement of the set $E \cup (E + t)$ is of positive measure, hence nonempty. Which means that a couple of points: $x \notin E \cup (E + t)$ and $y = x - t$ is not in E. Therefore, $E \in \wp(t)$ for even one number t implies: $|E| \geq \frac{1}{2}$.

(A trivial observation: $\mathcal{C} \in \wp(t)$ for every t. Also and for an $E \in \wp(t)$.)

When both sets A^1, A^2 of a pair $A = (A^1, A^2)$ are in $\wp(t)$ we write is simply $A \in \wp(t)$.

We write for simplicity: $\wp(t, t', t'', \dots)$ for the intersection $\wp(t) \cap \wp(t') \cap \wp(t'') \cdots$ of classes corresponding to individual numbers: t, $t', t'', \dots$.

And one more convenient description of a two-set relation: a set A is said to be at *most ϵ-half dense* within a set B when $\frac{|A \cap B|}{|B|} \leq \frac{1}{2} + \epsilon$.

The construction to which the remainder of this section is devoted aims at constructing pairs of nearly symmetrical in properties sets with possibly small measures and which belong to $\wp$ classes containing ever increasing number of entries.

Let a counterclockwise oriented circle $\mathcal{C}$ be placed in the (x, y) plane with its center at the origin. One of its intersection points which the coordinate axes mark as 0-point. (Let us mention here that the operations of sum or difference $x \pm y$ performed on points $x, y \in \mathcal{C}$ are understood as usual, as such operations modulo one. The $E+t = \{x+t : x \in E\}$ is the *rotational t-translate* of $E \subset \mathcal{C}$).

Lemma 2.1. *For a $t > 0$ there is a pair $A(t) = (A^1(t), A^2(t))$ of subsets of $\mathcal{C}$ such that $A(t) \in \wp(t)$ and $|A^i(t)|$ is at most t-half dense in $\mathcal{C}$, $i = 1, 2$. Moreover, $A^1(t) \cup A^2(t) = \mathcal{C}$ and $|A^1(t) \cap A^2(t)| = 2t$.*

Proof. Starting with the point $0 \in \mathcal{C}$ and moving counterclockwise partition $\mathcal{C}$ into arcs I_k of length t, (with exception of the last which is in general incomplete - shorter than t arc.). The number $s(t)$ of partition points equals the integer part of $\frac{1}{t}$.

Take the union of odd-numbered complete arcs I_k, adjoin to that union the closed t-neighborhood $[-t, t]$ of $0 \in \mathcal{C}$ and name the resulting set $A^1(t)$. Repeat the same, taking this time union of even-numbered I_k and name this set $A^2(t)$.

It is quite clear that the $A(t)$ satisfies conditions of the lemma. □

Introduce in turn an operation Φ acting on pairs of subsets of $\mathcal{C}$ and numbers, producing from an input pair $A = (A^1, A^2)$ and a number $t > 0$ an output pair $\widehat{A} = (\widehat{A}^1, \widehat{A}^2)$. The description how this is performed we present below in a number of consecutive steps. (The reader will recognize similarity to the process of formation a pair in Lemma 2.1.)

(i) Starting with the $0 \in \mathcal{C}$ and moving counterclockwise, partition $\mathcal{C}$ into arcs I_k of length t the same way as it was done in the proof of Lemma 2.1.

(ii) Select one particular complete I_k, for instance the I_1. Then in every odd-numbered complete I_k place a congruent copy of $I_1 \cap A^1$ and in every complete even-numbered I_k place the congruent copy of $I_1 \cap A^2$. Form the union of all these copies. As easily seen, the measure of each of the two unions is not larger than $\frac{1}{2} + t$.

Adjoin to that union the closed t-neighborhood $[-t, t]$ of the point $0 \in \mathcal{C}$. Name the resulting set $\widehat{A}^{1'}$.

(iii) Repeat the step (ii) switching A^1 with A^2, that is, place copies of $I_1 \cap A^1$ in even-numbered arcs and copies of A^2 in even numbered.

Name the resulting set $\widehat{A}^{2'}$. Clearly, about the two created sets we may say: $|A^{i'}| \leq \frac{1}{2} + 3t$, $i = 1, 2$, and $|A^{1'} \cap A^{2'}| = 2t$.

(Note the nearly $2t$-periodic character of both created sets: $A^{1'}$ and $A^{2'}$, their periodicity being disturbed only in the vicinity of $0 \in \mathcal{C}$.)

(iv) We are close to the completion of the construction but we make yet one more addition to the two constructed sets, namely, we adjoin to each of them the closed t'-neighborhood $U(t')$ of the set of endpoint of the I_k arcs, after which it becomes: $\widehat{A}^i = \widehat{A}^{i'} \cup U(t')$, $i = 1, 2$.

This is a good place to add one more useful observation: since $|U(t')| \le 2t's(t) \le 2t'/t$, adjoining it adds at most $2t'/t$ to measure, hence $|A^i| \le \frac{1}{2} + 3t + 2t'/t$, $i = 1, 2$, and $|A^1 \cap A^2| \le 2t + t'/t$.

The pair: $\widehat{A} = (\widehat{A}^1, \widehat{A}^2) = \Phi(A|t)$ is the result of Φ acting on A and t.

Lemma 2.2. *$s(E)$ denote the number of endpoints of arc-components of a set (elementary) $E \subset \mathcal{C}$. Let $\epsilon > 0$ and let $A = (A^1, A^2)$ be a pair of subsets of $\mathcal{C}$ and t, t' a couple of positive numbers: $t' < t$ About A assume that:*

(a) *$A^1 \cup A^2 = \mathcal{C}$;*

(b) *$|A^1 \cap A^2| \le 3t$;*

(c) *$A \in \wp(t')$.*

Then under additional appropriate conditions on smallness of t and t' the following holds for $\widehat{A} = \Phi(A, t)$:

(a′) *$\widehat{A}^1 \cup \widehat{A}^2 = \mathcal{C}$;*

(b′) *$\widehat{A} \in \wp(t, t')$;*

(c′) *For an $\epsilon > 0$ under t appropriately chosen small enough we achieve that $\widehat{A}^i$, $i = 1, 2$ are at most ϵ-half dense in E.*

Proof. (a′) From the assumption (a) follows $I_1 \cap (A^1 \cup A^2) = I_1$. From here, taking into account steps describing the construction of Φ operation follows (a′).

(b′) From the fact that $I_1 \cap (A^1 \cup A^2) = I_1$ it follows that a couple of points $x, x + t$ with $x \in I_1$ and $x + t \in I_2$ must have at least one point within $(I_1 \cap A^1) \cup (I_2 \cap A^2)$. Since as a result of (ii) in the Φ construction this set is $2t$-periodically repeated along $\mathcal{C}$ with the irregularity to its periodicity remaining near 0 point. When that part of $\mathcal{C}$ is sealed away by the adjoined $[-t, t]$, at least one of the two points of the couple $x, x+t$ falls within $\widehat{A}^1$ for any position of $x \in \mathcal{C}$. The same goes for $\widehat{A}^2$. Thus far this yields $\widehat{A} \in \wp(t)$ only.

But the assumption (c) implies already that a couple $x, x+t'$ remaining within the I_1 must have a point within each of the two sets $I_1 \cap A^i$, $i = 1, 2$, hence the same remains true for each set $A^{i'}$, $i = 1, 2$. when such couple remains within any other partition arc I_k (or $[-t, t]$). Due to adjoining the $U(t')$, however, this property holds for the enlarged set $\widehat{A}^i$, $i = 1, 2$. and for such a $x, x + t'$ couple in every position on $\mathcal{C}$ without exception. In other words, $\widehat{A} \in \wp(t')$ as well.

Combinely, we obtain: $\widehat{A} \in \wp(t, t')$, that is, (b′).

(c′). Selecting arc-pairs $I_k \cup I_{k+1}$ which cover E (that is, all those which have nonempty intersections with E) we see easily that the total length of such arc-pairs is bounded from above by the number $|E| + 4ts(E)$.

In the step (ii) of the Φ construction, within both arcs of a pair I_k, I_{k+1} one copy of $I_1 \cap A^1$ and one of $I_1 \cap A^2$ were respectively placed and these two copies become the $I_k \cup I_{k+1}$-portion of the set $\widehat{A}^{1'}$.

The assumption $|A^1 \cap A^2| < 3t$ of this lemma implies that among the I_k arcs there is (at least) one where $|I_k \cap A^1 \cap A^2| < 3\frac{t}{s(t)}$ and without loss of generality one may assume in the construction of Φ that I_1 is such an arc. From here follows $|(I_k \cup I_{k+1}) \cap \widehat{A}^1| \le \frac{1}{2}|I_k \cup I_{k+1}| + \frac{3t}{s(t)}$ $(= t + \frac{3t}{s(t)})$, in other words, $A^{1'}$ is at most $\frac{3t}{s(t)}$-half dense within an $I_k \cup I_{k+1}$.

Adding these estimates over all the $I_k \cup I_{k+1}$ which cover E we obtain that the measure of the portion of $\widehat{A}^1 \cap \widehat{A}^2$ contained within the union of all such arcs is bounded from above by $\frac{1}{2}(|E| + 4t\,s(E)) + 3t = \frac{1}{2}|E| + t\,(2s(E)+3)$.

By imposing on smallness of t an additional condition: $t\,|E|(2s(E)+3) < \epsilon$ we derive (c$'$).

Quite a similar answer for A^2 and $\widehat{A}^2$. Hence (d$'$).

$\nu(n)$, $n = 1, 2, \ldots$, denote for brevity the triangular naturals: $\nu(n) = \frac{n(n+1)}{2}$, $n = 1, 2, \ldots$. □

Lemma 2.3. *Let* $[t]$ *be a strongly decreasing sequence. Let* E_n, $n = 1, 2, \ldots$, *be a sequence of (elementary) subsets of* $\mathcal{C}$. *There is a subsequence* $[\widehat{t}_n]$ *of* $[t]$ *and a sequence of pairs* $A_n = (A_n^1, A_n^2)$, $n = 1, 2, \ldots$, *of subsets of* $\mathcal{C}$ *having the properties:*

(a) $A_n^1 \cup A_n^2 = \mathcal{C}$,

(b) $A_n \in \wp(\widehat{t}_{\nu(n)}, \ldots, \widehat{t}_{\nu(n+1)-1})$,

(c) A_n^i *is at most* $\frac{1}{n}$*-half dense in* E_n, $i = 1, 2$.

Proof. First a remark about the notation: the result of performing a number n of iterative operations: $\Phi(\ldots\Phi((\Phi(A|u_n)|u_{n-1})\ldots)|u_1)$ we shall write simply $A(u_1, \ldots, u_n)$.

Both sequences, $[\widehat{t}]$ and A_n, $n = 1, 2, \ldots$, shall be constructed inductively.

On two selected terms $\widehat{t}_1 < \widehat{t}_2$ from $[t]$ extra requirements will be imposed as needed.

Use the pair $A(\widehat{t}_2)$ produced in Lemma 2.1 (in which $t = \widehat{t}_2$) as the input pair in Lemma 2.2. It is easy to verify for $A(\widehat{t}_2)$ the (a),(b) (c) of Lemma 2.2: (a): $A^1(\widehat{t}_2) \cup A^2(\widehat{t}_2) = \mathcal{C}$, (b): $|A^1(\widehat{t}_2) \cap A^2(\widehat{t}_2)| \le 3t_1$ (note: this requires $2\widehat{t}_2 s(\widehat{t}_1) < 2\widehat{t}_2/\widehat{t}_1 < \widehat{t}_1$), (c): $A(\widehat{t}_2) \in \wp(\widehat{t}_2)$. The output pair $A_1 = \widehat{A}(\widehat{t}_2) = A(\widehat{t}_1, \widehat{t}_2) = \Phi(A(\widehat{t}_2)|\widehat{t}_1)$ has the properties (a$'$)–(c$'$). We have thus:

(a$'$): $A_1^1 \cup A_1^2 = \mathcal{C}$,

(b$'$): $A_1 \in \wp(\widehat{t}_1, \widehat{t}_2)$,

(c$'$): A_2^i are at most $\frac{1}{2}$-half dense within E_1.

These mean satisfaction of the (a)–(c) properties of this lemma for $n = 1$.

We have defined the first term of the sequence of pairs, the pair A_1.

In order to facilitate grasping the general inductive step we shall demonstrate here in detail the next step of passing from A_1 to A_2.

Let $\widehat{t}_3 > \widehat{t}_4 > \widehat{t}_5$ be the next three terms selected from $[t]$, smaller than $\widehat{t}_2$, which will be still subjected to additional smallness requirements, as needed.

Use this time the obtained in the previous step pair $A_1 = A(\widehat{t}_1, \widehat{t}_2)$ in which the subscripts of the two terms have been shifted forwards to produce an $A(\widehat{t}_4, \widehat{t}_5)$, a pair which is now used as the input along with the number $\widehat{t}_3$ of the Φ operation. The output is a pair $A_2 = A(\widehat{t}_3, \widehat{t}_4, \widehat{t}_5) = \Phi(A(\widehat{t}_4, \widehat{t}_5)|\widehat{t}_3)$.

Following the earlier pattern we have to verify first that this input satisfied the assumptions (a)–(c) of Lemma 2.2. This is so because that is implied by the verified already (a)–(c) properties of this lemma for A_1. The output conclusions (a$'$)–(c$'$) of Lemma 2.2 follow, of which the important for us are the two last: (b$'$): $A_2 \in \wp(\widehat{t}_3, \widehat{t}_4, \widehat{t}_5)$ and (c$'$): A_2^i, $i = 1, 2$ are at most 2-half dense in E_2.

The needed additional requirements on $\widehat{t}_3$ - $\widehat{t}_5$ are easy to figure out but clumsy to write explicitly. We leave this detail to the reader. $\square$

And finally the general inductive step.

We assume here that the initial segment $\{\widehat{t}_k,\ k = 1, \ldots, \nu(n)\}$ of the prospective subsequence of $[t]$ and n satisfying the (a) - (c) conditions of this lemma pairs $A_k = A(\widehat{t}_{\nu(k)+1}, \ldots, \widehat{t}_{\nu(k+1)})$, $k = 1, \ldots, n$, have been already constructed.

In order to extend further the subsequence select from $[t]$ next n terms: $\widehat{t}_k$, $k = \nu(n+1), \ldots, \nu(n+2) - 1$ and $t_{\nu(n)+1} < \frac{1}{2^n} < \epsilon$.

Set $A_{n+1} = \Phi(A(\widehat{t}_{\nu(n+1)+1}, \ldots, t_{\nu(n+2)-1}) \,|\, t_{\nu(n+1)})$.

Similarly as was done in the presented above exemplary case of passing from A_1 to A_2, having A_n satisfying the inductive assumptions of this lemma, under appropriate additional conditions on smallness of the terms $\widehat{t}_k$, $k = \nu(n+1), \ldots, \nu(n+2) - 1$, Lemma 2.2 permits us to conclude that also the A_{n+1} can be made to satisfy these conditions, in particular, the important for later applications conditions: $A_{n+1} \in \wp(\widehat{t}_{\nu(n)+2}, \ldots, t_{\nu(n+1)})$ and A_{n+1}^1 being at most $2 + \frac{1}{2^n}\epsilon$-half dense in E_{n+1}.

3. This section carries further the construction of set sequences of the previous section. Three sequences of elementary sets are introduced here: B_m, Q_m and $\widehat{Q}_m$, $m = 1, 2, \ldots$, the last of which is used directly to produce the figuring in Proposition 1 set Q.

The whole Section 2 of this paper in fact serves as a preparation to define the B_m sets.

In order to simplify the notation, the derived in the last section subsequence $[\widehat{t}]$ of the originally given sequence $[t]$ we shall write again as $[t]$.

From the sequence of pairs of sets constructed in the previous section select the subsequence A_{2^m}, $m = 1, 2, \ldots$. Actually, only the first set of every pair of sets A_{2^m} will be used in the sequel of this paper. Rename this set B_m, thus $B_m = A_{2^m}^1$.

Let us repeat two properties of B_m as following from Lemma 2.3:

$$B_m \in \wp(t_{2^{\nu(m)}}, \ldots, t_{2^{\nu(m+1)}-1})$$

and

$$\frac{|B_m \cap E_{2^m}|}{|E_{2^m}|} \leq \frac{1}{2} + \frac{1}{2^m}.$$

And again the notation: $[t_{2^{\nu(m)}}, \ldots, t_{2^{\nu(m+1)}-1}]$ for the 2^m-long segment is cumbersome and calls for simplification. We shall write for it $[t_1^m, \ldots, t_{2^m}^m]$ instead.

In what follows we adopt the convenient notation $E^{+\delta}$ for a closed δ-neighborhood of a set E.

We shall write for brevity $\delta_m = 3t_1^m$ for the triple of the initial term of a $[t_1^m, \ldots, t_{2^m}^m]$ segment.

Having defined based on a strongly decreasing sequence $[t]$ sets B_m, define recursively based on the same sequence another sequence of elementary sets, Q_m, $m = 1, 2, \dots$, as follows:

Set $Q_1 = \mathcal{C}$ and the general term Q_m define by:

$$Q_{m+1} = Q_m^{+\delta_n} \cap B_m.$$

(Thus, as an example, $Q_2 = B_1 = A_2^1$, $Q_3 = B_1^{+\delta_2} \cap B_2$, and so forth.)

Lemma 3.1. *Q_{m+1} is at most $\frac{1}{2^{m-1}}$-half dense in the Q_m, $m = 1, 2, \dots$.*

Proof. B_m is at most $\frac{1}{2^m}$-half dense within E_m. Since E_m is one of a sequence of arbitrarily chosen elementary subsets of $\mathcal{C}$, we may set: $E_m = Q^{+\delta_m}$. By this we achieve:

$$\frac{|Q_m^{+\delta_m} \cap B_m|}{|Q_m^{+\delta_m}|} = \frac{|Q_{m+1}|}{|Q_m^{+\delta_m}|} < \frac{1}{2} + \frac{1}{2^m}.$$

By making δ_m small enough (which, let us mention it, requires appropriate down-adjustment of $[t]$) we replace $|Q_m^{+\delta_m}|$ with $|Q_m|$ in the denominator of the last fraction at the price of increasing the right term in the inequality from $\frac{1}{2^m}$ to $\frac{1}{2^{n-1}}$. This yields the lemma. $\square$

The following is a helpful intrusion valid in a general metric space X. It is a lemma dealing with the operation of taking a neighborhood of a set.

Lemma 3.2. *Let $E \subset X$, and $r, r' > 0$. The set resulting from two consecutive r and r' neighborhood taking of a set results in a subset of the $r + r'$ neighborhood of the same set: $(E^{+r})^{+r'} \subset E^{+(r+r')}$.*

Proof. Indeed, let $x \in (E^{+r})^{+r'}$ and let x' be the closest to x point of E^{+r} and x'' be the closest ro x' point of $\bar{E}$. We have: $|x - x''| \le |x - x'| + |x' - x''|$ which proves the inclusion. $\square$

Introduce yet one more, this time the last one, sequence $\widehat{Q}_m$, $m = 1, 2, \dots$, of elementary sets.

Write $\rho(m)$ for the remainder of the series $\sum\limits_{k=1}^{\infty} 2\delta_m$, that is, $\rho(m) = 2\sum\limits_{k=m}^{\infty} \delta_m$. Set: $\widehat{Q}_m = Q_m^{+\rho(m)}$. A $Q_m^{+\rho(m)}$ is a '$\rho(m)$-blow-up' of an earlier defined set Q_m. By Q denote the intersection: $\bigcap\limits_{m=1}^{\infty} \widehat{Q}_m$.

Lemma 3.3. *Q is a not empty compact.*

Proof. We have: $Q_{m+1} \subset Q_m^{+\delta_m}$. Using Lemma 3.2 and taking into account that $\delta_m + \rho(m+1) = \delta_m + \rho(m) < \rho(m)$, we derive:

$$\widehat{Q}_{m+1} = Q_{m+1}^{+\rho(m+1)} \subset (Q_m^{+\delta_m})^{+\rho(m+1)} \subset Q_m^{+(\delta_m + \rho(m+1))} \subset Q_m^{\rho(m)} = \widehat{Q}_m.$$

The $\widehat{Q}_m$ form a descending sequence, which makes Q a not empty compact. $\square$

Lemma 3.4. *Q is of measure zero.*

Proof. We show that a $\widehat{Q}_{m+1}$ is $\frac{1}{2^{m-2}}$-half dense in $\widehat{Q}_m$, $m = 1, 2, \ldots$.

In the inequalities $\frac{|Q_{m+1}|}{|Q_m|} < \frac{1}{2} + \frac{1}{2^{m-1}}$ of Lemma 3.1, the term $|Q_{m+1}|$ may for a sufficiently small $\rho(m)$ (requiring a down-adjustment of $[t]$ again) be replaced by $|\widehat{Q}_{m+1}|$ at the price of increasing the $\frac{1}{2^{m-1}}$ on the right side of the inequality to $\frac{1}{2^{m-2}}$. Under consecutive down-adjustments of $[t]$ (and, consequently, of the derived from $[t]$ sequence $\rho(m)$, $m = 1, 2, \ldots$) we come to the inequalities:

$$\frac{|\widehat{Q}_{m+1}|}{|\widehat{Q}_m|} < \frac{1}{2} + \frac{1}{2^{m-2}},\ m = 1, 2, \ldots .$$

This sequence of inequalities implies: $\lim_{m\to\infty} |\widehat{Q}_m| = |Q| = 0$. $\square$

4. In the present section the defined earlier in Section 3. set Q becomes finally related to the given perfect set P.

Again, a little note about the notation: In what follows, at some places we find it convenient to write a segment $[t_1^m, \ldots, t_{2^m}^m]$ of a strongly decreasing $[t]$ rather in the form: $[\tau_1^m, \tau_1^{m,'}, \ldots, \tau_k, \tau_k^{m,'}, \ldots, \tau_{2^{m-2}}, \tau_{2^m-2}^{m,'}]$ in which the letters have been changed to τ_k^m and $\tau_k^{m'}$ and the 2^m terms t_k^m grouped into 2^{m-1} pairs of neighboring terms.

Introduce one more sequence of subsets of $\mathcal{C}$, a *cascade*. It is not a sequence of sets elementary, but that of finite sets. We define it recursively:

Start with a strongly decreasing sequence $[t]$. A based on it cascade is a sequence $Y_m \subset \mathcal{C}$, $m = 1, 2, \ldots$. defined as follows:

Y_1 is a one-point set $\{y\}$, where $y \in \mathcal{C}$ is arbitrary. Assume that the sets of a cascade have been already defined for $1 \le n \le m$ and that Y_m contains 2^{m-1} points. Write the counterclockwise ordered Y_m in the form: $Y_m = \{y_k^m,\ k = 1, \ldots, 2^{m-1}\}$.

Bring in B_m sets based on the same $[t]$. By their definition, we have: $B_m \in \wp(t_1^m, \ldots, t_{2^m}^m)$, that is, B_m contains at least one point out of every couple of points $x, x - t_k^m \in \mathcal{C}$, where $k = 1, \ldots, 2^m$. In particular, for $m > 2$ it contains at least one point out of every of two couples: $\{y_k^m, y_k^m - \tau_k^m\}$ and $\{y_k^m - \tau_k^m - \tau_k^{m'}, y_k^m - \tau_k^m - 2\tau_k^{m'}\}$, both made of points of the Y_m set of the cascade. Select one such point within each of the two couples (should both points from a couple be in B_m, select the one which is closest to y_k^m) and name these two points z_k^m and $z_k^{m'}$.

After this preparation define the Y_{m+1} by the formula:

$$Y_{m+1} = \bigcup_{k=1}^{2^{m-1}} \{z_k^m, z_k^{m'}\}. \tag{Y}$$

Note the implication: $Y_{m+1} \subset B_m$. Also note that in passing from Y_m to Y_{m+1} a point y_k of Y_m is replaced by the generated by that point pair $z_k^m, z_k^{m'}$ of points.

The following lemmas illustrate a few properties of the just introduced concept of a cascade.

Lemma 4.1. *The set* $Y_\infty = \bigcap_{m=1}^{\infty} \overline{\bigcup_{k=m}^{\infty} Y_k}$ *is a not empty perfect set.*

Proof. The intersection of the descending sequence of not empty compacts $\overline{\bigcup_{k=m}^{\infty} Y_k}$ is a not empty compact. As follows from (Y), for m large enough every point of a set $\bigcup_{k=m}^{\infty} Y_k$ has in that set arbitrarily close to it but different from it point. Therefore, Y_∞ cannot have isolated points. □

The following lemma ties a cascade to a corresponding to it Q_m, $m = 1, 2, \dots$, sequence.

Lemma 4.2. *To a cascade* Y_m*,* $m = 1, 2 \dots$*, corresponds a sequence* Q_m*,* $m = 1, 2, \dots$*, such that:*

$$Y_\infty \subset Q_m,\ m = 1, 2, \dots .$$

Proof. First, we shall prove by induction a chain of inclusions: $Y_m \subset Q_m$, $m = 1, 2, \dots$.

The case $m = 1$ is beyond question. Assume $Y_m \subset Q_m$ for a given m.

Since t_1^m is the largest among the terms of the segment $[t_1^m, \dots, t_{2^m}^m$ and since the union of the two couples: $\{y_k^m, y_k^m - \tau_k^m\} \cup \{y_k^m - \tau_k^m - \tau_k^{m'}, y_k^m - \tau_k^m - 2\tau_k^{m'}\}$ is of diameter smaller than $\delta_m = 3t_1^m$ and also since one point of the first couple is in Y_m, both couples are contained within the closed neighborhood $Y_m^{+\delta_m}$ of Y_m, hence $Y_{m+1} \subset Y_m^{+\delta_m}$. Consequently, we have: $Y_{m+1} \subset Q_m^{+\delta_m}$. But according to (Y) also $Y_{m+1} \subset B_m$.

Combinely, the two inclusions yield: $Y_{m+1} \subset Q_m^{+\delta_m} \cap B_m = Q_{m+1}$. The induction is complete. □

(Note that an appropriate adjustment-down of $[t]$ is to be applied to the segment $[t_1^m, \dots, t_{2^m}^m]$ and consecutively to all the following segments, which in limit will create a strongly decreasing limit sequence.)

Corollary 4.3. *To a cascade* Y_m*,* $m = 1, 2, \dots$*, corresponds a sequence* $\widehat{Q}_m$*,* $m = 1, 2, \dots$*, such that* $Y_\infty \subset Q$*.*

Proof. By Lemma 4.2 and the definition of $\widehat{Q}_m$ we have: a sequence $\widehat{Y}_m$, $m = 1, 2, \dots$, with $Y_\infty \subset \widehat{Q}_m$ for $m = 1, 2, \dots$, hence $Y_\infty \subset Q$. □

Lemma 4.4. *For a nonempty perfect set* $P \subset \mathcal{C}$ *there are: a string cascade* Y_m*,* $m = 1, 2 \dots$*, and a sequence* $\widehat{Q}_m$*,* $m = 1, 2, \dots$*, such that* $Y_\infty \subset P \cap Q$*.*

Proof. Starting with an arbitrary strongly decreasing $[t]$, one forms based on it sequences B_m and Y_m, $m = 1, 2, \dots$, and a corresponding by Lemma 4.2 sequence Q_m, $m = 1, 2, \dots$, and that under the restriction that the picked up points to form an Y_m are to be right-hand accumulation points of P. This restriction calls at every stage of construction of a cascade for an appropriate down adjustment of $[t]$ and this results in a sequence of down-adjustments of $[t]$

leading to creation of a strongly decreasing limit sequence. The density-in-itself of P guarantees the possibility of indefinite continuation of such a process.

By the corollary 4.3 one obtains as a result that Y_∞ is contained in both sets P and Q simultaneously. (Note, that except for an at most countable subset all points of P are its both-sided accumulation points. The requirement for right-sidedness comes from our choice of counterclockwise orientation on $\mathcal{C}$.) $\square$

Enter now into the picture rotations of all the sets Q_m from their original positions by a $\theta \in \mathcal{C}$. Since in general the selections of a pair of $z_k^m, z_k^{m'}$ points (out for four choices) which are in $B_m + \theta$ depends on θ, the corresponding sets Y_{m+1} of the cascade depend on θ. Write for them $Y_m(\theta)$.

Corollary 4.5. *For every $\theta \in \mathcal{C}$ the set $P \cap (Q + \theta)$ contains a Cantor-like subset.*

Proof. Lemma 4.4 is equally valid for every position $Q + \theta$ of Q on $\mathcal{C}$. The constructed in Lemma 4.4 corresponding to θ set $Y_\infty(P, \theta)$ is a Cantor-like set included in both P and $Q + \theta$. $\square$

This is the ultimate result we were after.

5. In closing, a few side comments.

What a 'small set' is may be understood in various ways. Questions similar to our may range from difficult to almost trivial. For instance, asking for a Q as in our case but nowhere-dense only (measure zero is not required), is but an elementary exercise.

Another digression: a rather superficial but instantly derivable from Proposition 1 corollary is the following result:

Let $P_k \subset \mathcal{C}$, $k = 1, 2, \dots$, be a countable family of nonempty perfect sets. There is an universal for that family set of first category and measure zero which placed in every position on $\mathcal{C}$ intersects every P_k on a Cantor-like set. Indeed, the union $\cup_k Q^k$ of the corresponding (*by Proposition* 1) *Cantor-like counterparts Q^k to the P_k sets is such a set.*

One may also suggest a generalization of Proposition 1 by replacing the unit circle $\mathcal{C}$ with an arbitrary product of copies of the unit circle, a 'torus-space', $\mathcal{C}^\Xi$ (over an arbitrary set Ξ). Such super-torus is easily made into a linear space under the naturally defined linearity (a shift is the product of rotations in the factor circles). It can be turned into a topological space with weak product-topology and into a measure space carrying product measure of the Lebesgue measures on the factor-circles.

To pursue such generalization an expansion of the concepts of Cantor-like set is be needed: let us say that a subset of $\mathcal{C}^\Xi$ is *Cantor-like in all the coordinates* if all its coordinate projections on the factor-circles are Cantor-like sets. In a similar fashion be defined the congruency of two subsets of $\mathcal{C}^\Xi$ by reducing it to congruency of the projections on all the factor-circles. With that done the following generalization of Proposition 1 would hold:

Proposition 2. *To every nonempty perfect in all the coordinates set $P \subset \mathcal{C}^{\Xi}$ there is a perfect in all the coordinates set $Q \subset \mathcal{C}^{\Xi}$ of product measure zero, such such that every congruent to Q subset intersects P over a nonempty perfect in all the coordinates set.*

We leave it to an interested reader to fill in the needed steps.

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(Received 12.11.2001)

Author's address:
110 S. El Nido
Pasadena, CA 91107
U.S.A.