Georgian Mathematical Journal
Volume 9 (2002), Number 4, 601–605

A DUALITY ON SIMPLICIAL COMPLEXES

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Dedicated to Hvedri Inassaridze on the occasion of his 70th birthday

Abstract. We describe a duality theory for finite simplicial complexes that gives isomorphisms between the (reduced) homology of the complex and the (reduced) cohomology of the dual.

2000 Mathematics Subject Classification: 55M05, 18G30.
Key words and phrases: Abstract simplicial complex, duality.

1. Finite Simplicial Complexes

The usual definition of finite simplicial complex is a set of non-empty subsets of a finite set, closed under non-empty subset formation. For our purposes here, we will omit the non-emptiness and define a finite simplicial complex to be a down-closed subset of the set of subsets of a finite set. We can, and will suppose that the finite set is the integers $0, \dots, N$. We will denote by K the set 2^{N+1} of all subsets of $N+1$ $(= \{0, \dots, N\})$. If $S \subseteq K$ is a finite simplicial complex, then a subset of $n+1$ elements in S is called an n-simplex. We will write an element of S as $[a_0, \dots, a_n]$ with $a_0 < \cdots < a_n$. We also write $[\,]$ for the unique (-1)-simplex. If $\sigma = [a_0, \dots, a_n]$ is an n-simplex, we say that $a_0, \dots, a_n$ are the *vertices* of σ.

We will be dealing with the free abelian group generated by the n-simplexes. We will continue to write $[a_0, \dots, a_n]$, for $a_0 < \cdots < a_n$, but we will also denote by $[a_0, \dots, a_n]$ the element $\operatorname{sgn} p[a_{p0}, \dots, a_{pn}]$ where p is the unique permutation such that $a_{p0} < \cdots < a_{pn}$ and also let $[a_0, \dots, a_n] = 0$ if the vertices are not distinct.

1.1. Homology and cohomology.

Let S be a finite simplicial complex and S_n denote the set of n-simplexes. We let $C_n(S)$ denote the free abelian group generated by S_n, for $n = -1, \dots, N$. We also let $C^n(S)$ denote the abelian group of functions from S_n to $\mathbf{Z}$. For any finite S, it is obviously the case that $C_n(S) \equiv C^n(S)$, but the functors are totally different; for one thing, the first is covariant and the second contravariant.

If $\sigma = [a_0, \dots, a_n]$ is an n-simplex and $0 \le i \le n$, we define $\partial^i = \partial^i_n$ by $\partial^i \sigma = [a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n]$. We call $\partial^i \sigma$ the i-face of σ. All 0-simplexes $[a]$ have the same 0-face, namely $[\,]$.

Then we define the boundary operator $\partial = \partial_n : C_n(S) \to C_{n-1}(S)$ by

$$\partial \sigma = \sum_{i=0}^{n} \partial^i \sigma.$$

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The equation $\partial^i.\partial^j = \partial^{j-1}.\partial^i$ for $i < j$ allows one to prove readily that $\partial_{n-1}.\partial_n = 0$. Thus we can define the homology groups of S by

$$H_n(S) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

For $f : S_n \to \mathbf{Z}$, define $\delta^n f = \delta f : S_{n+1} \to \mathbf{Z}$ by

$$(\delta f)\sigma = f(\partial\sigma).$$

Then $\delta.\delta = 0$ and we define the cohomology groups of S by

$$H^n(S) = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}.$$

Because of the simplex in degree -1, the homology and cohomology groups so defined are equivalent to the usual reduced homology and cohomology groups of these complexes. That is, they are the same except in degree 0 where the groups defined here are free on one less generator. There is also a group in degree -1, which is trivial as soon as there is a single 0-simplex (or a single non-empty simplex), since then the arrow from 0-chains to (-1)-chains is surjective. The coefficient module turns out not as the homology of a point, as in the traditional theory, but as the homology of the complex consisting of the empty set alone. In particular, the homology of a point—or of any simplex—is 0 in all degrees.

2. The Duality

If σ is a simplex, we let σ^c denote the simplex whose vertices are the complement of those of σ. We will say that σ^c is the *complement* of σ. The dimension of σ plus the dimension of σ^c is $N-1$. In particular, $[\,]$ is the complement of K.

A finite simplicial complex is a set of simplexes of K closed under taking of faces. We will think of K as a complex consisting of all the faces. If S is such a complex, define

$$S^* = \{\sigma \in K \mid \sigma^c \notin S\}$$

Proposition 1. *For any simplicial complexes $S \subseteq T \subseteq K$,*

1. *$K^* = \emptyset$ and $\emptyset^* = K$;*
2. *S^* is a simplicial complex;*
3. *$S^{**} = S$*
4. *If $S \subseteq T$, then $T^* \subseteq S^*$;*

Proof. (1) is obvious. If $\sigma \in S^*$, and $\tau \subseteq \sigma$, then $\sigma^c \notin S$. But then $\sigma^c \subseteq \tau^c$, so it must be that $\tau \notin S$ so that $\tau \in S^*$ and so S^* is down closed. For any simplex σ, $\sigma \in S^{**}$ if and only if $\sigma^c \notin S^*$, which is the case if and only if $\sigma^{cc} \in S$ and $\sigma^{cc} = \sigma$. Finally if $S \subseteq T$, then for any $\tau \in T^*$, $\tau^c \notin T$ so that $\tau^c \notin S$ and so $\tau \in S^*$. □

The duality is based on the following.

Proposition 2. *When $m + n = N - 1$, there is a morphism $j : C_n(S^*) \to C^m(K)$ such that the sequence*

$$0 \to C_n(S^*) \to C^m(K) \to C^m(S) \to 0$$

is exact.

Proof. Suppose that $n+m=N-1$. The group $C^m(K)$ has as basis elements σ^*, for σ an m-simplex, defined by

$$\sigma^*(\tau)=\begin{cases}1 & \text{if } \sigma=\tau,\\ 0 & \text{otherwise.}\end{cases}$$

The kernel of the map $C^m(K)\to C^m(S)$ consists of those σ^* for which $\sigma\notin S$. For any n-simplex σ, the map $j(\sigma^{\mathrm{c}})\in C^m(K)$ defined by $j(\sigma^{\mathrm{c}})(\tau)=\sigma^{\mathrm{c}}\cdot\tau$ is $\pm\sigma^*(\tau)$. The reason is that $\sigma^{\mathrm{c}}\cdot\tau=0$ unless σ^{c} is disjoint from τ, that is unless $\sigma=\tau$ and in the latter case, $\sigma^{\mathrm{c}}\cdot\tau=\pm1$. Thus the set of $j(\sigma^{\mathrm{c}})$, for $\sigma\notin S$ is a basis for the kernel. But the set of $j(\sigma^{\mathrm{c}})$ is just S^*. □

Proposition 3. *When $m+n=N-1$, for any n-simplex $\sigma=[a_0,\dots,a_n]$ and m-simplex $\tau=[b_0,\dots,b_m]$,*

$$\partial[a_0,\dots,a_n]\cdot[b_0,\dots,b_m]=(-1)^n[a_0,\dots,a_n]\cdot\delta[b_0,\dots,b_m]$$

Proof. The only way either side can be non-zero is if there is exactly one element in $\{a_0,\dots,a_n\}\cap\{b_0,\dots,b_m\}$ since the number of elements ensures that there is at least one overlap and if there is more than one, there will still be at least one overlap on each face. So suppose that $a_i=b_j$ and there is no other overlap. In that case, using the usual convention whereby a ˆ indicates an omitted term, the left hand side is

$$(-1)^i[a_0,\dots,\hat{a}_i,\dots,a_n]\cdot[b_0,\dots,b_m]$$

and the right hand side is

$$(-1)^j[a_0,\dots,a_n]\cdot[b_0,\dots,\hat{b}_j,\dots,b_m]$$

The permutation required to change one to the other involves $n-i+j$ transpositions and comparing the exponents we see that they differ by exactly $(-1)^n$. □

It follows that for $0\le n\le N$ and $m+n=N-1$, the square

$$\begin{array}{ccc} C_n(S^*) & \xrightarrow{\equiv} & C^m(K)\\ \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\delta}\\ C_{n-1}(S^*) & \xrightarrow[\equiv]{} & C^{m+1}(K)\end{array}$$

either commutes or anticommutes. Since changing the sign of the boundary operator has no effect on the homology, we can replace ∂_n by $(-1)^n\partial$ and make all the squares commute. When this is done, we have,

Corollary 1. *There is an exact sequence of chain complexes*

$$0\to C_\bullet(S^*)\to C^{N-1-\bullet}(K)\to C^{N-1-\bullet}(S)\to 0$$

Theorem 1. *There are canonical isomorphisms $H_n(S^*)\equiv H^{N-2-n}(S)$ for $n=-1,\dots,N$.*

Proof. By taking homology of the sequence above, we get a long exact sequence

$$\begin{aligned}
0 \to H_N(S^*) \to H^{-1}(K) \to H^{-1}(S) \\
\to H_{N-1}(S^*) \to H^0(K) \to H^0(S) \\
\to H_{N-2}(S^*) \to H^1(K) \to H^1(S) \\
\to \quad \cdots\cdots\cdots \\
\to H_0(S^*) \to H^{N-1}(K) \to H^{N-1}(S) \\
\to H_{-1}(S^*) \to H^N(K) \to H^N(S) \to 0
\end{aligned}$$

Since $H_n(K)=0$ for all $-1\le n\le N$, we conclude that $H_n(S^*) \equiv H^{N-2-n}(S)$. □

2.1. **Naturality.** Naturality with respect to simplicial maps does not make sense since the duality depends on the dimension of the simplex that the complex is embedded in. The one case in which it does make sense is that of two subcomplexes embedded in the same simplex, one a subcomplex of the other. It is clear that if $S \subseteq T$, then $K - T \subseteq K - S$ and so the set of complements of $K-T$ is included in the set of complements of $K-S$, that is $T^* \subseteq S^*$. The result is induced maps $H_n(T^*) \to H_n(S^*)$ to go with $H^{N-n-2}(T) \to H^{N-n-2}(S)$.

Proposition 4. *Suppose that $S \subseteq T$ are subcomplexes of K. Then for $n+m = N-1$, the square*

$$\begin{tikzcd}
C_n(T^*) \arrow[r] \arrow[d] & C^m(K) \arrow[d] \\
C_n(S^*) \arrow[r] & C^m(K)
\end{tikzcd}$$

in which the left hand map is induced by the inclusion $T^* \subseteq S^*$ is commutative.

Proof. A simplex in T^* induces the same linear functional on $C_m(K)$ as it does in S^*. □

Proposition 5. *Suppose S and T are subcomplexes of the same simplex K with $S \subseteq T$. Then the square*

$$\begin{tikzcd}
H_n(T^*) \arrow[r, "\equiv"] \arrow[d] & H^{N-n-2}(T) \arrow[d] \\
H_n(S^*) \arrow[r, "\equiv"'] & H^{N-n-2}(S)
\end{tikzcd}$$

commutes.

Proof. From the preceding proposition we see that the top left and bottom left squares of the diagram

$$
\begin{tikzcd}
0 \arrow[r] & C_n(T^*) \arrow[r] \arrow[dr] \arrow[dd] & C^m(K) \arrow[r] \arrow[dr] \arrow[dd] & C^m(T) \arrow[r] \arrow[dr] \arrow[dd] & 0 & \\
& 0 \arrow[r] & C_n(S^*) \arrow[r] \arrow[dd] & C^m(K) \arrow[r] \arrow[dd] & C^m(S) \arrow[r] \arrow[dd] & 0 \\
0 \arrow[r] & C_{n-1}(T^*) \arrow[r, dashed] \arrow[dr] & C^{m+1}(K) \arrow[r, dashed] \arrow[dr] & C^{m+1}(T) \arrow[r, dashed] \arrow[dr] & 0 & \\
& 0 \arrow[r] & C_{n-1}(S^*) \arrow[r] & C^{m+1}(K) \arrow[r] & C^{m+1}(S) \arrow[r] & 0
\end{tikzcd}
$$

commute, while the commutation of the left front and left rear squares was proved in 2. All the squares on the right obviously commute and hence the whole diagram does. Since the isomorhism in question is just the connecting homomorphism in the homologies of the front and back complexes, the conclusion is standard. □

3. Spanier–Whitehead Duality?

Several algebraic topologists that I have discussed this with believe that it is the finite simplicial version of the Spanier–Whitehead duality. One, in fact, suggested that one way that the SW duality might have been discovered would be by beginning with the duality described here and extending it to CW-complexes. I have not found a complete description of the SW duality; it does seem to be mentioned in standard texts on algebraic topology. It has been described to me as follows. Let C be a CW-complex and let S^N be a sufficiently high dimensional sphere that there is an embedding $C \subseteq S^N$. The complementary space $S^N - C$ is not a CW-complex, but can be contracted in some way to one. Call the resultant space C'. C' is the dual space of C and its stable homotopy is equivalent to the stable cohomotopy of C, suitably reindexed.

There are obvious differences, but also intriguing similarities between the duality here and the SW duality. Although we embed in a simplex, we could have just as well embedded into a sphere and stuck to non-empty simplexes. In that case, the equivalence between the homology and cohomology would be off by one in degree and codegree 0. An obvious point is that both constructions depend on the dimension of the sphere or simplex.

(Received 18.03.2002)

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