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DIFFERENTIAL EQUATIONS, SPENCER COHOMOLOGY, AND COMPUTING RESOLUTIONS

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To Hvedri Inassaridze, on his seventieth birthday

Abstract. We propose a new point of view of the Spencer cohomology appearing in the formal theory of differential equations based on a dual approach via comodules. It allows us to relate the Spencer cohomology with standard constructions in homological algebra and, in particular, to express it as a Cotor. We discuss concrete methods for its construction based on homological perturbation theory.

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1. Introduction

This paper ties three fairly different areas of research, viz. differential equations, (co)homological algebra, and symbolic computation, together. While many people working in these individual areas will find the sections concerning their particular area quite familiar (and perhaps a bit too elementary), we expect that they will find the other sections much less familiar (and perhaps a bit too dense). We have made an attempt to provide a basic background for all areas upon which we touch and hope that this will encourage more multidisciplinary research in these areas.

The definition of involutivity for a system of partial differential equations has had a very long and convoluted history. First works on overdetermined systems go back at least to Clebsch and Jacobi. In the middle of the 19th century the analysis of homogeneous linear first order systems in one unknown function was a very popular subject (nowadays this has been superseded by the geometric view of the Frobenius theorem). Most older textbooks like [13, 26] contain a chapter on this theory (with references to the original works).

The late 19th and early 20th century saw a flurry of activities in this field extending the theory to more and more general systems. One line of research lead to the Janet-Riquier theory [46, 70, 86, 87] with its central notion of a *passive* system. We will later meet some ideas from this theory in the combinatorial approach to involution. Another line of research culminated in the Cartan-Kähler theory of exterior differential systems [11, 14, 50] (a dual version based on vector fields instead of differential forms was developed by Vessiot [88]). In this

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approach one usually speaks of *involutive* systems, although Weber attributed this terminology to Lie in his encyclopedia article [89].

Roughly between 1950 and 1970 a more sophisticated point of view emerged combining many elements of the old approaches with new and more abstract techniques. Ehresmann's theory of jet spaces [20, 21, 22] allows for an intrinsic geometric treatment of differential equations without resorting to differential forms. *Formal integrability*, i. e. the existence of formal power series solutions, may then be interpreted as a straightforward geometric concept.

Somewhat surprisingly, it has turned out that for many purposes such a purely geometric approach does *not* suffice. This begins with the simple fact that the geometric definition of formal integrability is not effectively verifiable, as it consists of infinitely many conditions. Thus geometry had to be complemented by some algebraic (mainly homological) tools, in particular with *Spencer cohomology* which is one of the topics of this article. In the Cartan-Kähler theory the famous Cartan test for involution requires checking a relation between certain integers (encoding dimensions) as will be explained in more detail below. Serre, Guillemin and Sternberg [39] showed that this test represents in fact a homological condition, namely the vanishing of certain Tor groups, and is related to a complex introduced earlier by Spencer [82] (a dual approach to an algebraic definition of involution was already earlier given by Matsushima [64]). Later, the theory was thoroughly studied by Spencer [83], Quillen [66] and Goldschmidt [31, 32].

The arising theory is often called the *formal theory of differential equations*. Textbook presentations may be found e. g. in [11, 17, 52, 65]. One interpretation of the name is that it concerns itself with formal power series solutions for arbitrary systems, i. e. also for under- and overdetermined ones, and indeed we will use this approach for the brief introduction to the theory given in Section 3. However, restricting the formal theory to this one aspect would give a much too narrow picture.

It is probably fair to say that the full meaning and importance of the ideas surrounding involution are still not properly understood. While many facets have emerged, the complete picture has remained elusive. Recently, it has been shown that in an algebraic context it is related to the Castelnuovo-Mumford regularity [79] and that in numerical analysis obstruction to involution may become integrability conditions in a semi-discretization [80].

In more detail, we associate with every system of differential equations of order q in m unknown functions of n variables a homogeneous degree q subspace $N_q \subseteq C_q^m$ where C is the set of all polynomials in $x_1, \dots, x_n$ with coefficients in $\Bbbk$. The space N_q is called the *geometric symbol*[1] of the differential system (see Sects. 3.3 and 3.4).

We set $N_q^{(s)} = \{f \in N_q \mid \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1, \dots, s\}$ where $f = (f_1, \dots, f_m)$ and the derivatives are taken coordinate-wise. One further defines the *first prolongation* of the symbol $N_{q+1} = \{f \in C_{q+1}^m \mid \frac{\partial f}{\partial x_i} \in N_q \text{ for } i = 1, \dots, n\}$

[1]In [11, p. 116] the space N_q is called a *tableau* and its annihilator $N_q^\perp$ a symbol.

consisting of all integrals of degree $q+1$ of elements in N_q. It is well-known [11, 39] that

$$\dim N_{q+1} \leq \dim N_q + \dim N_q^{(1)} + \cdots + \dim N_q^{(n-1)} . \tag{1.1}$$

The symbol N_q is said to be *involutive*, if coordinates $x_1, \ldots, x_n$ exist such that (1.1) is actually an equality.

The theory proceeds by recursively defining higher prolongations $N_{q+k+1} = \{f \in C^m_{q+k+1} \mid \frac{\partial f}{\partial x_i} \in N_{q+k}, \text{for } i = 1, \ldots, n\}$. The vector space $N = \bigcup_{k=1}^{\infty} N_{q+k}$ is called the infinite prolongation; we will see in Section 3.4 that it has the structure of a polynomial *comodule*. Since each N_{q+k} is again a symbol (of a prolonged differential system), what we have said above about the involutivity of N also applies to each N_{q+k} and a classic theorem gives that N_{q+r} becomes involutive for some $r \geq 0$ (see the remark after Theorem 5.2).

It was noted in [39] that N is dual to a graded A-module M where $A = \Bbbk[x_1, \ldots, x_n]$ is the polynomial algebra over $\Bbbk$. It was also noted in [39] that the involutivity condition above is equivalent to the maps

$$M_{q+r+1}/(x_1, \ldots, x_k)M_{q+r} \longrightarrow M_{q+r+2}/(x_1, \ldots, x_k)M_{q+r+1} \tag{1.2}$$

given by multiplication by x_{k+1} being one-one for all $k = 0, \ldots, n-1$ and all $r \geq 0$. It is, of course, now common to say that if the above condition holds, the sequence $x_1, \ldots, x_n$ is *quasiregular* for M. In an appendix to [39], Serre showed that the quasiregular condition for a module M is *generically* equivalent to the vanishing of $\operatorname{Tor}^A(M, \Bbbk)$ (see Section 5.1) in positive degrees. The authors observe that the Koszul complex for computing Tor is dual to the Spencer complex referred to above. Thus, by duality, Serre's result can be translated to a statement about the vanishing of Spencer cohomology.

Various authors have discussed the duality between the Spencer complex and the Koszul complex [11, 31, 63, 72]. First we will show that this duality is not as ad hoc as it seems to be from the literature. In fact, by introducing coalgebras and comodules, we show that the duality is a special case of one that exists between comodules over a coalgebra C and modules over the dual algebra $A = C^*$ (Section 2.12). This leads to the identification of Spencer cohomology as a Cotor (Section 6.7).

Thus, the knowledge of the vanishing of a certain Tor (or Cotor) gives some computational insight in the completion of general systems of partial differential equations and is also useful for the concrete determination of formal power series solutions. The actual values of the ranks of the Tor groups provide us with a coordinate independent criterion for involutive symbols. Beyond the order at which the system becomes involutive, the unique determination of the coefficients of power series solutions is straightforward.

For this and other reasons, we are interested in computing $\operatorname{Tor}^A(M, \Bbbk)$ effectively for a module M over the polynomial algebra A. Fortunately, there are many ways to do this explicitly. Perhaps the most familiar to those readers who are acquainted with *computer algebra* is the program Macaulay 2 [25]. That program uses what we call Schreyer methods (Section 4.1). We will look at some novel methods for deriving other classes of "small" resolutions that can be used for computing Tor as well.

In this article, we restrict to the case where the module M is generated by elements of homogeneous degree q. But we note that there are generalizations to cases where M contains elements of mixed degrees but still homogeneous. These generalizations are also useful in the context of formal theory, as they appear in more efficient versions of the basic completion algorithm [40]. However, as this is a very technical subject, we will study it elsewhere.

We finally introduce an interesting algorithm for computing minimal resolutions of monomial modules (Section 7) over the polynomial ring, and give examples in Section 8.

2. (Co)Algebra

Throughout this paper, $\mathbb{k}$ will denote a field of characteristic zero.

It is assumed that the reader is familiar with basic notions such as the tensor product of vector spaces, algebras over a field, modules over an algebra, etc. When tensor products are taken over $\mathbb{k}$, we will omit the subscript in $\otimes_{\mathbb{k}}$. We will quickly review coalgebras and comodules. More details can be found in [59].

2.1. Coalgebras and Comodules.

Recall that a coalgebra over $\mathbb{k}$ is a vector space equipped with a linear map $C \xrightarrow{\Delta} C \otimes C$ (called the *comultiplication*) and $\epsilon : C \longrightarrow \mathbb{k}$ (called the *counit*) such that the coassociativity and counit axioms hold (see below). It has become customary to write the result of comultiplication in the following form which is called the Heyneman-Sweedler (H-S) notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation over the indices which run in parallel is assumed. See [59, Sec. A.3.2]).

Using H-S notation, coassociativity is expressed as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$$

and the counit axioms are

$$c_{(1)}\epsilon(c_{(2)}) = c = \epsilon(c_{(1)})c_{(2)}$$

A vector space N over $\mathbb{k}$ is said to be a (right) *comodule* over the coalgebra C if there is a $\mathbb{k}$-linear map $\rho : M \longrightarrow M \otimes C$ such that

$$(1_M \otimes \epsilon)\rho = 1_M$$

and

$$(\rho \otimes 1_C)\rho = (1_M \otimes \Delta)\rho.$$

We use an extended H-S notation ([59]) for the action of the structure map ρ in a right comodule:

$$\rho(m) = m^{\langle 1 \rangle} \otimes m^{(2)} \in M \otimes C.$$

Similar remarks apply to left comodules.

2.1.1. *The Dual Algebra.* Recall that the algebra dual to C is given by $A = C^*$ where C^* is the linear dual, $C^* = hom_{\Bbbk}(C, \Bbbk)$ and the product is given by

$$\langle \alpha\beta, c\rangle = \alpha(c_{(1)})\beta(c_{(2)}) \tag{2.1}$$

for $\alpha, \beta \in A$ and $c \in C$ and where $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing $\langle \gamma, c\rangle = \gamma(c)$ [59].

2.2. **Graded Modules.** Let A be an algebra over $\Bbbk$. A left A-module M is said to be a (non-negatively) *graded module* over A if $M = \oplus_{n=0}^{\infty} M_n$ as an Abelian group. The subgroup M_n is called the set of elements of homogeneous degree n. If $x \in M_n$, we write $|x| = n$ for its degree. It is convenient, at times, to think of a graded module as the sequence $(M_0, M_1, \dots, M_n, \dots)$ and work in M degree-wise.

If M and N are two graded modules over A, a module map $f : M \longrightarrow N$ is said to be graded of degree r if $f|_{M_n} : M_n \longrightarrow N_{n+r}$ for some fixed integer r. If $r = 0$, we simply say that f is a (graded) module map.

A submodule N of a graded module M is said to be a graded submodule if N is a graded module and for all $n \geq 0$, $N_n \subseteq M_n$. It is clear that the kernel and image of a graded module map are both graded modules.

A graded module is said to be of *finite type* if for all n, M_n is finitely generated over A.

2.2.1. *Graded Dual.* If M is a graded module over $\Bbbk$, its graded dual is the module M^* where $M_n^* = hom_{\Bbbk}(M_n, \Bbbk)$. Note that if M is of finite type, then so is M^*.

2.2.2. *Sign Convention.* We adopt the usual sign convention which states that if one element x "passes by" another y, the result must be multiplied by $(-1)^{|x||y|}$. Thus, for example, if f and g are two maps, their tensor product is given by

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

2.3. **Graded Algebras.** A graded algebra A over $\Bbbk$ is a graded module over $\Bbbk$ such that the multiplication $m : A \otimes_{\Bbbk} A \longrightarrow A$ satisfies

$$A_i \otimes A_j \xrightarrow{m} A_{i+j}.$$

When we talk about a graded module over a graded algebra, it is assumed that the structure map $\mu : A \otimes M \longrightarrow M$ satisfies $\mu|_{A_i \otimes M_j} : A_i \otimes M_j \longrightarrow M_{i+j}$.

2.4. **Graded Coalgebras.** A graded coalgebra C over $\Bbbk$ is a graded module over $\Bbbk$ such that the comultiplication $\Delta : C \longrightarrow C \otimes C$ satisfies

$$C_i \xrightarrow{\Delta} \sum_{r+s=i} C_r \otimes C_s.$$

When we talk about a graded comodule over a graded coalgebra, it is assumed that the structure map $\rho : M \longrightarrow M \otimes C$ satisfies $\rho|_{M_i} : M_i \longrightarrow \sum_{r+s=i} M_r \otimes C_s$.

2.5. **Bialgebras.** If a given algebra B is also a coalgebra, we say that it is a bialgebra if $\Delta : B \longrightarrow B \otimes B$ is a homomorphism of algebras where $B \otimes B$ has the tensor product structure $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$. See [59, Section 1.5] for more details.

2.6. **Chain/Cochain Complexes.** A *chain complex* over a $\Bbbk$-algebra A is a module X equipped with a module map $d : X \longrightarrow X$ of degree -1 such that $d^2 = 0$. We call the map d_n the n^{th} *differential.* The n^{th} homology module of X, denoted by $H_n(X)$ is, by definition, the quotient module $\ker(d_n)/\mathrm{im}\,(d_{n+1})$. Elements of $\ker(d)$ are called *cycles* and elements in $\mathrm{im}\,(d)$ are called *boundries.*

A *cochain complex* over A is a module Y equipped with a module map $\delta : Y \longrightarrow Y$ of degree $+1$ such that $\delta^2 = 0$. The n^{th} cohomology module of Y, denoted by $H^n(Y)$ is, by definition, the quotient module $\ker(d_n)/\mathrm{im}\,(d_{n-1})$. Elements of $\ker(d)$ are called *cocycles* and elements in $\mathrm{im}\,(d)$ are called *coboundries.* Note that if X is a chain complex, then the linear dual $X^* = \mathrm{Hom}_A(X, A)$ is a cochain complex in the obvious way.

2.6.1. *Chain Maps and Homotopies.* A *chain map* $f : X \longrightarrow Y$ is a module map that makes the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "d_n"'] & Y_n \arrow[d, "d_n"] \\ X_{n-1} \arrow[r, "f_{n-1}"] & Y_{n-1} \end{tikzcd}$$

commute. It is easy to see that this condition causes any chain map to induce an A-linear map on homology $H_*(f) : H_*(X) \longrightarrow H_*(Y)$ in the obvious way. Note that the identity map on X is a chain map.

Two chain maps $X \xrightarrow{f,g} Y$ are said to be *chain homotopic* is there is a degree one map $X \xrightarrow{\phi} Y$ such that

$$d\phi + \phi d = f - g. \tag{2.2}$$

2.7. **Differential Graded (Co)Algebras.** A differential graded algebra over $\Bbbk$ is a graded algebra which is also a chain complex. It is furthermore assumed that the differential d is a derivation, i.e.

$$d(ab) = d(a)b + (-1)^{|a|} ad(b).$$

A differential graded coalgebra over $\Bbbk$ is a graded coalgebra which is also a chain complex and for which the differential ∂ is a coderivation. The notion of coderivation is completely dual to that of derivation, i.e. the differential ∂ must satisfy

$$\Delta\partial = (\partial \otimes 1 + 1 \otimes \partial)\Delta.$$

The reader is invited to work this out using H-S notation from Section 2.1.

2.8. **Resolutions.** Let M be a (left) A-module. A free resolution [15, 62] of M over A is a sequence of free A-modules X_i and A-linear maps

$$\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

such that sequence is exact, i.e. $\ker(d_{n-1}) = \mathrm{im}\,(d_n)$ for all $n \geq 1$ and $\ker(\epsilon) = \mathrm{im}\,(d_0)$. We always associate the chain complex X given by

$$\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_0 \longrightarrow 0$$

to a given free resolution. Note that $H_n(X) = 0$ for $n \geq 1$ and $H_0(X) = X_0/\mathrm{im}\,(d_0) \cong X_0/\ker(\epsilon) \cong \mathrm{im}\,(\epsilon) = M$.

If we give M the trivial differential, we extend ϵ to all of X by setting it to be zero on elements of degree greater than zero and we obtain a chain map

$$X \xrightarrow{\epsilon} M \longrightarrow 0 \tag{2.3}$$

such that ϵ is an isomorphism in homology.

A *contracting homotopy* for X is a degree one map ψ linear over $\Bbbk$ (but not generally over A) such that $d\psi + \psi d = 1$, i.e. a chain homotopy between the identity map and the zero map.

Note that more generally, one talks about *projective* resolutions, i.e. resolutions in which each X_i is projective over A. An A-module is projective if and only if it is a direct summand of a free A-module. All the resolutions in this paper however will be free. In fact, we will only consider resolutions over the polynomial algebra and it is well-known [67, 84] that any projective module is free in that case.

Similar remarks apply to right modules.

2.9. **The Polynomial Bialgebra.** Let $A = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\Bbbk$. Note that A is naturally graded by setting A_i equal to the subspace of polynomials of homogeneous degree i for $i = 0, 1, \ldots$. Note also that it is of finite type. Define a map $A \xrightarrow{\epsilon} \Bbbk$ where $\epsilon(p) = p(0)$ (i.e. the constant term of p).

We will write monomials as $\mathbf{x}^\alpha$ where

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

and the components of $\alpha = (\alpha_1, \ldots, \alpha_n)$ are non negative integers.

The following relation on monomials

$$\mathbf{x}^\alpha \preceq \mathbf{x}^\beta \text{ iff } \alpha_i \leq \beta_i \text{ for all } i = 1, \ldots, n. \tag{2.4}$$

will be used later.

Remark 2.1. A total order on monomials that satisfies $u < v \implies mu < mv$ and $1 < m$, for all monomials $m, u, v \neq 1$ in A, is called a *monomial order* on A.

The algebra A also possesses a coproduct given as follows.

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1 \tag{2.5}$$

$$\Delta(fg) = \Delta(f)\Delta(g), \text{ for } f, g \in A. \tag{2.6}$$

The counit is given by ϵ above.

It follows by an easy computation that

$$\Delta(\mathbf{x}^\alpha) = \sum_{r_1,\ldots,r_n=0}^{\alpha_1,\ldots,\alpha_n} \binom{\alpha_1}{r_1} \cdots \binom{\alpha_n}{r_n} \mathbf{x}^{\mathbf{r}} \otimes \mathbf{x}^{\alpha-\mathbf{r}} \tag{2.7}$$

where $r = (r_1, \ldots, r_n)$ and $\alpha - r = (\alpha_1 - r_1, \ldots, \alpha_n - r_n)$.

Note that combinatorially, the above formula for the coproduct is completely determined by the binomial expansion of $(\mathbf{x}+\mathbf{y})^\alpha$. This means that one can view the coproduct in the following way. For any polynomial $f(\mathbf{x})$ consider $f(\mathbf{x}+\mathbf{y})$ as an element of $\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n] = \Bbbk[x_1, \ldots, x_n] \otimes \Bbbk[y_1, \ldots, y_n]$. If we expand $f(\mathbf{x}+\mathbf{y})$ and replace y_i by x_i for $i = 1, \ldots, n$, we get exactly the formula determined by (2.7) above. Thus we have the following proposition.

Proposition 2.2. *The coproduct in the polynomial bialgebra A may be written as*

$$\Delta(f) = \sum \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \otimes x_1^{i_1} \cdots x_n^{i_n}. \tag{2.8}$$

Proof. For any polynomial f, the Taylor series expansion of $f(\mathbf{x}+\mathbf{y})$ is given exactly by the formula above. □

2.9.1. *The Dual Bialgebra.* The following is well-known (see, e.g., [16]), but we include it for completeness.

Note that $\Bbbk[x_1, ..., x_n] \cong \otimes^n \Bbbk[x]$. Consider the bialgebra dual to $\Bbbk[x]$. Letting $\gamma_i(x)$ be the linear dual to x^i, we have

$$\langle \Delta\gamma_i(x), x^r \otimes x^s \rangle = \delta_i^{r+s}$$

so that

$$\Delta\big(\gamma_i(x)\big) = \sum_{r+s=i} \gamma_r(x) \otimes \gamma_s(x). \tag{2.9}$$

Also,

$$\langle \gamma_i(x)\gamma_j(x), x^r \rangle = \langle \gamma_i(x) \otimes \gamma_j(x), \Delta(x^r) \rangle$$

But recall from above that

$$\Delta(x^r) = \sum_{n+m=r} \binom{r}{n} x^n \otimes x^{n-r}.$$

Thus,

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x). \tag{2.10}$$

The algebra $\Bbbk[x]^*$ is called the *divided power* algebra and is usually denoted by $\Gamma[x]$. Now recall that $\Bbbk$ has characteristic zero and so if we write $y = \gamma_1(x)$, we have

$$\gamma_i(x) = \frac{y^i}{i!}. \tag{2.11}$$

This follows immediately from $\gamma_1(x)^i = i!\gamma_i(x)$. Thus setting $z_i = \frac{y^i}{i!}$ we have $z_i z_j = z_{i+j}$ and this is just the polynomial algebra. The analogous results hold for the tensor product $\otimes^n \Bbbk[x]$ and therefore for $A = \Bbbk[x_1, \ldots, x_n]$.

It is left to the interested reader to see that the graded dual A^* also has a coproduct dual to the product in A and that $A^* \cong A$ as bialgebras.

2.10. **Coordinate Free Versions of the Polynomial (Co)Algebra.** Let V be a finite-dimensional vector space over $\mathbb{k}$. The tensor algebra $\mathscr{F}(V)$ is given by

$$\mathscr{F}(V) = \sum_{i=0}^{\infty} \otimes^i V$$

where $\otimes^0 V = \mathbb{k}$ and $\otimes^1 V = V$. The product is given by

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_l) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l.$$

If "coordinates" are chosen, i.e. a basis $\{v_1, \dots, v_n\}$ is chosen for V, one identifies this algebra as the algebra $\mathbb{k}\langle v_1, \dots, v_n\rangle$ of non-commuting polynomials in the variables $\{v_1, \dots, v_n\}$. It is the free non-commutative algebra generated by V. We assume that the elements of V are all of degree zero (or of even degree). Let I be the ideal of $\mathscr{F}(V)$ generated by the set $\{xy - yx \mid x, y \in V\}$. The symmetric algebra on V is the quotient algebra

$$\mathscr{S}(V) = \mathscr{F}(V)/I.$$

If one chooses coordinates as above, it is clear that $\mathscr{S}(V)$ is isomorphic to the polynomial algebra $\mathbb{k}[v_1, \dots, v_n]$.

The tensor coalgebra on V is given by

$$\mathfrak{C}(V) = \sum_{i=0}^{\infty} \otimes^i V$$

with coproduct given by

$$\Delta(v_1 \otimes \cdots \otimes v_k) = \sum_{i=0}^{k} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k)$$

where the terms for $i = 0$ and $i = k$ are $1 \otimes (v_1 \otimes \cdots \otimes v_k)$ and $(v_1 \otimes \cdots \otimes v_k) \otimes 1$ respectively. In fact, $\mathfrak{C}(V)$ is the cofree coalgebra on V (also see [36] for a more general construction).

Note that the symmetric group S_n acts on $\mathfrak{C}(V)_n = \otimes^n V$ in the obvious way, i.e. by permuting the tensorands. It is clear that the subspace $\mathfrak{S}(V)$ of symmetric tensors (i.e. those left invariant under the action of the symmetric group) is invariant under the coproduct Δ. Thus, $\mathfrak{S}(V) \subseteq \mathfrak{C}(V)$ is a subcoalgebra. If coordinates are chosen, it is not difficult to see that $\mathfrak{S}(V)$ is isomorphic to the polynomial coalgebra $\mathbb{k}[v_1, \dots, v_n]$. The interested reader should see [7, 8] for more details.

2.11. **The Exterior Bialgebra.** The exterior algebra $E = E[u_1, \dots, u_n]$ over $\mathbb{k}$ is generated as an algebra by $\{u_1, \dots, u_n\}$ and is subject to the relations $u_i u_j = -u_j u_i$. As such, it has a basis given by $\{u_I \mid I = (i_1, \dots, i_k), 1 \le i_1 < \cdots < i_k \le n, k = 0, \dots, n\}$ where if $I = (i_1, \dots, i_k)$, then $u_I = u_{i_1} \dots u_{i_k}$. Generally, we will eliminate the tensor sign in dealing with elements of E. Note that E is a graded algebra over $\Bbbk$ where $|u_I| = |I|$ and $|I|$ denotes the cardinality of I.

The coproduct in E is determined by

$$\begin{aligned} \Delta(u_i) &= u_i \otimes 1 + 1 \otimes u_i, && (2.12)\\ \Delta(uv) &= \Delta(u)\Delta(v), \quad \text{for } u, v \in E. && (2.13) \end{aligned}$$

Thus, for example, note that

$$\begin{aligned} \Delta(u_i u_j) &= (u_i \otimes 1 + 1 \otimes u_i)(u_j \otimes 1 + 1 \otimes u_j)\\ &= 1 \otimes u_i u_j - u_j \otimes u_i + u_i \otimes u_j + u_i u_j \otimes 1 \end{aligned}$$

and so on.

The counit is given by $\epsilon(r) = r$ when r is a scalar, while $\epsilon(u_I) = 0$ for $|I| > 0$.

2.11.1. *The Dual Bialgebra.* Again, it is well-known that $E = E[u_1, \ldots, u_n]$ is self-dual, i.e. the dual E^* is a bialgebra and $E^* \cong E$ as bialgebras.

2.12. **More Duality.** We need to recall some basic results concerning a correspondence between subcomodules and submodules. All of the background material can be found in [59, §A.4.2, pp. 273]. If V is a vector space over $\Bbbk$ and $W \subseteq V$ is a subspace, the inclusion map $W \xrightarrow{\iota} V$ gives rise to the onto dual map $V^* \xrightarrow{\iota^*} W^*$. The kernel of this map is usually denoted by $W^\perp$. Thus,

$$W^\perp = \{\nu \in V^* \mid \nu(w) = 0 \text{ for all } w \in W\} \text{ and } V^*/W^\perp \cong W^*. \tag{2.14}$$

If $U \subseteq V^*$ is a subspace, we similarly define

$$U^\perp = \{v \in V \mid \mu(v) = 0 \text{ for all } \mu \in U\}. \tag{2.15}$$

A subspace Z of either V or V^* is said to be closed, if and only if $Z = Z^{\perp\perp}$. By [59, §A.4], there is a one-one inclusion reversing the correspondence $W \mapsto W^\perp$ between subspaces of V and closed subspaces of V^*.

Now suppose that C is a coalgebra and $N \xrightarrow{\rho} N \otimes C$ is a comodule. Recall that $A = C^*$ is an algebra (Section 2.1.1). The dual vector space N^* is a (right) module over C^* with action determined by

$$\langle \nu\alpha, c\rangle = \langle \nu \otimes \alpha, \rho(c)\rangle = \langle \nu c^{\langle 1\rangle}\rangle\langle \alpha, c^{(2)}\rangle. \tag{2.16}$$

For the Proposition that follows, we need the following.

Lemma 2.3. *If M is locally finite over $\Bbbk$, any subspace (including itself) is closed.*

Proof. This follows by applying Proposition A.4.2 in [59] degree-wise. □

Recall that if N is a comodule over C and $M \subseteq N$ is a subspace, M is a subcomodule if and only if $\rho(M) \subseteq M \otimes C$. We have

Proposition 2.4. *Let C be a coalgebra and N a right comodule over C which is locally finite over $\Bbbk$. If $M \subseteq N$ is a subspace, then $M^\perp \subseteq N^*$ is a submodule, if and only if M is a subcomodule.*

Proof. Note that M is a subcomodule, if and only if for all $m \in M$, $m^{\langle 1 \rangle} \in M$. Thus if $\mu \in M^{\perp}$ and $\alpha \in C^*$, we have that for all $m \in M$,

$$\langle \mu\alpha, m \rangle = \langle \mu, m^{\langle 1 \rangle} \rangle \langle \alpha, m^{(2)} \rangle = 0$$

and so $\mu\alpha \in M^{\perp}$. For the converse, suppose that $M^{\perp}$ is a submodule. Let $m \in M$, we need to show that $m^{\langle 1 \rangle} \in N$. We know that for all $\alpha \in C^*$ and all $\nu \in M^{\perp}$, $\langle \nu, m^{\langle 1 \rangle} \rangle \langle \alpha, m^{(2)} \rangle = 0$. It follows that for all $\nu \in M^{\perp}$, $\langle \nu, m^{\langle 1 \rangle} \rangle = 0$ (take $\alpha = (m^{\langle 1 \rangle})^*$, the element dual to $m^{\langle 1 \rangle}$ in the last equation). Thus, $m^{\langle 1 \rangle} \in M^{\perp\perp} = M$ since M is closed by the last lemma. □

This is analogous to [59, Proposition 1.2.4].

2.13. **Polynomial Comodules and Modules.** Let $C = k[x_1, ..., x_n]$ be the polynomial bialgebra (2.9) with the coproduct given by Proposition 2.2. Note that $N \subseteq C$ is a subcomodule if and only if $\Delta(N) \subseteq N \otimes C$, if and only if for all $p \in N$,

$$\frac{\partial^k p}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \in N \tag{2.17}$$

for all $k \in \mathbb{N}$ and all $i_j \in \mathbb{N}$ such that $i_1 + \cdots + i_n = k$.

In general, if D is a coalgebra, D^n is a comodule over D with the following structure map

$$\rho(d_1, \ldots, d_n) = ((d_1)_{(1)}, \ldots, (d_n)_{(1)}) \otimes ((d_1)_{(2)} + \cdots + (d_n)_{(2)}). \tag{2.18}$$

We will call any comodule N over D which is isomorphic to D^n a *free comodule* over D.

Note that for the polynomial coalgebra C, $N \subseteq C^n$ is a subcomodule, if and only if (2.17) holds in each coordinate.

Remark 2.5. A module over an algebra A is finitely generated if and only if it is a quotient of A^n for some $n \in \mathbb{N}$. It is natural to say that a comodule M over a coalgebra D is finitely cogenerated if it is a submodule of D^n where D^n has the above comodule structure. In fact, such a definition was given in [68] where more information can be found.

Using Proposition 2.4 we have the following.

Proposition 2.6. *Let C be a coalgebra and $N \subseteq C^m$ a subspace that is locally finite. Then N is a subcomodule, if and only if $N^{\perp} \subseteq A^m$ is a submodule where A is the graded dual algebra C^*. Furthermore, we have an isomorphism $A^m/N^{\perp} \cong N^*$.*

Proof. The first assertion is a special case of Proposition 2.4. The second part follows by applying the second equation of (2.14) coordinate-wise. □

2.14. **Cogeneration.** Let $Y \subset C$ be a set of homogeneous polynomials of degree r. We are interested in the subcomodule $N \subset C$ cogenerated by Y. Let

N_r be the $\Bbbk$-linear span of Y. For the components of lower and higher degree, respectively, set

$$N_{r-j} = \left\{ \frac{\partial^j p}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \mid p \in N_r,\ i_1 + \cdots + i_n = j \right\}, \tag{2.19}$$

and

$$N_{r+j} = \left\{ p \in C \mid \frac{\partial^j p}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \in N_r,\ \forall\, i_1 + \cdots + i_n = j \right\} \tag{2.20}$$

(working coordinate-wise in C^m). Clearly, $N = \sum_{i=0}^{\infty} N_i$ is the comodule cogenerated by Y.

Remark 2.7. Since the notation $(m_1, \ldots, m_k)$ is in wide use for the submodule generated by a subset $\{m_1, \ldots, m_k\}$ of a module M, we will use the notation

$$N =)y_1, \ldots, y_k($$

for the comodule cogenerated by $Y = \{y_1, \ldots, y_k\} \subseteq C^m$.

3. Differential Equations

We will briefly outline some of the basic ideas of the formal theory of differential equations. For more details and proofs we must refer to the literature (see, e.g., [17, 18, 52, 65, 77] and references therein); our exposition follows mainly [77]. It should be noted that a number of alternative approaches to general systems of differential equations exist. This includes in particular the more algebraic Janet-Riquier theory (some elements of which have been incorporated into the formal theory) [46, 70], differential ideal theory [51, 71] for equations with at most polynomial nonlinearities, or the theory of exterior differential systems [11]. The latter one goes mainly back to Cartan and Kähler and is based on representing partial differential equations with differential forms. While it is equivalent to the theory we will describe, it is highly non-trivial to exhibit this equivalence.

For notational simplicity we will mainly use a global language in the sequel, although it must be stressed that most constructions are purely local, in fact often even pointwise.

3.1. **Formal Geometry.** Geometric approaches to differential equations are based on *jet bundles* [73]. The independent and dependent variables are modeled by a fibered manifold $\pi : \mathscr{E} \to \mathscr{B}$ (this means that π is a surjective submersion from the total space $\mathscr{E}$ onto the base space $\mathscr{B}$). The simplest example of such a fibered manifold is a trivial bundle where $\mathscr{E} = \mathscr{B} \times \mathscr{U}$ and π is simply the projection on the first factor. In fact, locally, in the neighborhood of a point $e \in \mathscr{E}$ any fibered manifold looks like a trivial bundle. Readers unfamiliar with manifolds may simply think of the example $\mathscr{B} = \mathbb{R}^n$ and $\mathscr{U} = \mathbb{R}^m$.

A section is a map $\sigma : \mathscr{B} \to \mathscr{E}$ such that $\sigma \circ \pi = 1_{\mathscr{B}}$. This generalizes the notion of (the graph of) a function, as in a trivial bundle a section is always of the form $\sigma(b) = \big(b, \mathbf{s}(b)\big)$ with a function $\mathbf{s} : \mathscr{B} \to \mathscr{U}$. A point in the qth order jet bundle $J_q\mathscr{E}$ corresponds to an equivalence class of smooth sections of

$\mathscr{E}$ which have at point $b \in \mathscr{B}$ a contact of order q, i. e. in local coordinates, their Taylor expansions at b coincide up to order q.

If $(x_1, \dots, x_n)$ are local coordinates on $\mathscr{B}$ and $(u_1, \dots, u_m)$ are fiber coordinates on $\mathscr{E}$, then we can extend them to coordinates on $J_q\mathscr{E}$ by the jet variables $p_{\alpha,\mu}$ where the multi index $\mu = [\mu_1, \dots, \mu_n]$ has a length $|\mu| = \mu_1 + \cdots + \mu_n \leq q$. For notational simplicity, we will often identify u_α with $p_{\alpha,[0,\dots,0]}$. Locally, a section $\sigma : \mathscr{B} \to \mathscr{E}$ may be written as $\sigma(\mathbf{x}) = \bigl(\mathbf{x}, \mathbf{s}(\mathbf{x})\bigr)$. Such a section induces a *prolonged section* $j_q\sigma(\mathbf{x}) = \bigl(\mathbf{x}, \mathbf{s}(\mathbf{x}), \partial_{\mathbf{x}}\mathbf{s}(\mathbf{x})\bigr)$ of the fibered manifold $\pi^q : J_q\mathscr{E} \to \mathscr{B}$. Here $\partial_{\mathbf{x}}\mathbf{s}$ represents all derivatives of the function $\mathbf{s}$ up to order q, i. e. the variable $p_{\alpha,\mu}$ gets assigned the value $\partial^{|\mu|}s_\alpha(\mathbf{x})/\partial\mathbf{x}^\mu$. This clearly demonstrates that locally we may interpret the coordinates of a point of $J_q\mathscr{E}$ as the coefficients of truncated Taylor series of functions $\mathbf{s}(\mathbf{x})$ and thus the jet variables as derivatives of such functions.

The jet bundles form a natural hierarchy with projections $\pi^q_r : J_r\mathscr{E} \to J_q\mathscr{E}$ for $r < q$. Of particular importance are the projections π^q_{q-1}. Let us denote by $V\mathscr{E} \subset T\mathscr{E}$ the vertical bundle, i. e. the kernel of the map $T\pi$. It is not difficult to prove the following proposition, for example by studying the transformation law of the jet coordinates under a change of variables in $\mathscr{E}$ (see below).

Proposition 3.1. *The jet bundle $J_q\mathscr{E}$ of order q is an affine bundle over the jet bundle $J_{q-1}\mathscr{E}$ of order $q-1$ modeled on the vector bundle $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$* (*see Section* 2.10 *for notation*).

This observation is the key for the introduction of algebraic techniques into the geometric theory (and sometimes even used for the intrinsic definition of jet bundles). In local coordinates, this affinity has the following meaning. Assume that we perform changes of coordinates $\mathbf{x} \to \mathbf{y}$ and $\mathbf{u} \to \mathbf{v}$. They induce via the chain rule changes of the jet coordinates $u_{\alpha,\mu} \to v_{\alpha,\mu}$. For the coordinates of order q, these are of the form $v_{\alpha,\mu} = \sum_{\beta=1}^{m} \sum_{|\nu|=q} A_{\beta,\nu} u_{\beta,\nu} + B$ where $A_{\beta,\nu}$ and B are (polynomial) functions of the jet coordinates $u_{\gamma,\lambda}$ with $|\lambda| < q$.

Remark 3.2. In the literature one usually considers instead of $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$ the isomorphic vector bundle $\mathscr{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$ (where $\mathscr{S}_q$ denotes the q-fold symmetric product). However, in view of Prop. 2.2 our choice appears much more natural and more consistent with the interpretation of the jet variables as derivatives of functions. At this point here, there is no real difference between using $\mathscr{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$ or $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$, as only the vector space structure matters. This will change in Section 3.4 where the natural comodule structure of $\mathfrak{S}(T^*\mathscr{B}) \otimes V\mathscr{E}$ is needed.

A *differential equation* of order q is now defined as a fibered submanifold $\mathscr{R}_q \subseteq J_q\mathscr{E}$. Note that this definition does not distinguish between a scalar equation and a system, as nothing is said about the codimension of $\mathscr{R}_q$. A *solution* is a section $\sigma : \mathscr{B} \to \mathscr{E}$ such that the image of the prolonged section $j_q\sigma : \mathscr{B} \to J_q\mathscr{E}$ is a subset of $\mathscr{R}_q$. Locally, such a submanifold is described by some equations $\Phi_\tau(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0$ with $\tau = 1, \dots, t$ and a section is a solution, if $\Phi_\tau\bigl(\mathbf{x}, \mathbf{s}(\mathbf{x}), \partial_{\mathbf{x}}\mathbf{s}(\mathbf{x})\bigr) \equiv 0$. Thus we recover the familiar form of a system of partial differential equations and its solutions.

Two natural operations with differential equations are projection and prolongation. The first one is easy to describe intrinsically but difficult to perform effectively. For the second one the local description is much easier than the intrinsic one. The *projection* (to order $r < q$) of the differential equation $\mathscr{R}_q$ is simply defined as $\mathscr{R}_r^{(q-r)} = \pi_r^q(\mathscr{R}_q) \subseteq J_r\mathscr{E}$. In local coordinates, the projection requires the elimination (by purely algebraic operations) of the jet variables of order greater than r in as many local equations $\Phi_\tau = 0$ as possible. Those equations that do not depend on any of these variables "survive" the projection and define the submanifold $\mathscr{R}_r^{(q-r)}$. Obviously, for nonlinear equations the elimination might be impossible to do effectively.

As both $\mathscr{R}_q$ and $J_q\mathscr{E}$ are again fibered manifolds over $\mathscr{B}$, we may form jet bundle over them. Note that $J_r(J_q\mathscr{E})$ is not the same as $J_{q+r}\mathscr{E}$; in fact the latter one may be identified with a submanifold of the former one. Now we may define the (r-fold) *prolongation* of $\mathscr{R}_q$ as the differential equation $\mathscr{R}_{q+r} = J_r(\mathscr{R}_q) \cap J_{q+r}\mathscr{E} \subseteq J_{q+r}\mathscr{E}$ where the intersection is understood to take place in $J_r(J_q\mathscr{E})$. Thus we obtain an equation of order $q + r$. Note that this simple formula is only obtained because we make a number of implicit identifications. A rigorous expression would require a number of inclusion maps.

In local coordinates, prolongation is performed with the help of the formal derivative. Let $\Phi : J_q\mathscr{E} \to \mathbb{R}$ be a smooth function. Then its *formal derivative* with respect to x_k, denoted by $D_k\Phi$, is a real-valued function on $J_{q+1}\mathscr{E}$ locally defined by

$$D_k\Phi = \frac{\partial \Phi}{\partial x_k} + \sum_{\alpha=1}^{m} \sum_{0\leq|\mu|\leq q} \frac{\partial \Phi}{\partial u_{\alpha,\mu}} u_{\alpha,\mu+1_k}\,. \tag{3.1}$$

Here 1_k denotes the multi index where all entries are zero except the kth which is one and the addition $\mu+1_k$ is defined componentwise (thus effectively, the kth entry of μ is increased by one). Note that the formal derivative $D_k\Phi$ is always a *quasi-linear* function, i. e. it is linear in the derivatives of order $q + 1$. The prolonged equation $\mathscr{R}_{q+1}$ is now locally described by all the equations $\Phi_\tau = 0$ describing $\mathscr{R}_q$ and in addition all the formal derivatives $D_k\Phi_\tau = 0$. More generally, we need for the local description of the r-fold prolongation $\mathscr{R}_{q+r}$ all equations $D_\nu\Phi_\tau = 0$ where ν runs over all multi indices with $0 \leq |\nu| \leq r$.

3.2. **Formal Integrability.** One could be tempted to think that prolongation and projection are a kind of inverse operations, i. e. if we first prolong and then project back that we obtain again the original differential equation. However, this is not true. In general, we only find that $\mathscr{R}_q^{(1)} = \pi_q^{q+1}(\mathscr{R}_{q+1}) \subseteq \mathscr{R}_q$. If it is a proper submanifold, this signals the appearance of integrability conditions in the classical language. In fact, in many cases the combination of a prolongation with a projection corresponds to taking cross-derivatives (as the differential analogue of S-polynomials in the theory of Gröbner bases). Differential equations where at no order of prolongation integrability conditions appear are particularly important and are therefore given a special name.

Definition 3.3. The differential equation $\mathscr{R}_q \subseteq J_q\mathscr{E}$ is called *formally integrable*, if the equality $\mathscr{R}^{(1)}_{q+r} = \pi^{q+r+1}_{q+r}(\mathscr{R}_{q+r+1}) = \mathscr{R}_{q+r}$ holds for all integers $r \geq 0$.

In order to explain this terminology, we consider the order by order construction of formal power series solutions for an equation $\mathscr{R}_q$. For this purpose, we expand the solution around some point $b \in \mathscr{B}$ with local coordinates $\bar{\mathbf{x}}$ into a formal power series, i. e. we make the ansatz

$$u_\alpha(\mathbf{x}) = \sum_{|\mu|\geq 0} \frac{c_{\alpha,\mu}}{\mu!}(\mathbf{x} - \bar{\mathbf{x}})^\mu \ .$$

Entering this series ansatz into a local representation $\Phi_\tau(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0$ with $1 \leq \tau \leq t$ of our differential equation $\mathscr{R}_q$ and evaluating at the chosen point b yields for the coefficients $c_{\alpha,\mu}$ the algebraic equations $\Phi_\tau(\bar{\mathbf{x}}, \mathbf{c}) = 0$ where the vector $\mathbf{c}$ represents all coefficients $c_{\alpha,\mu}$ with $0 \leq |\mu| \leq q$. Thus we only have to substitute x_i by $\bar{x}_i$ and $p_{\alpha,\mu}$ by $c_{\alpha,\mu}$. In general, these are nonlinear equations and the solution space may have a very complicated structure consisting of several components with differing dimensions etc.

If we apply the same procedure to the prolonged equation $\mathscr{R}_{q+1}$, we obtain further algebraic equations for the coefficients $c_{\alpha,\mu}$, namely $D_k\Phi_\tau(\bar{\mathbf{x}}, \mathbf{c}) = 0$ where now the vector $\mathbf{c}$ represents all coefficients $c_{\alpha,\mu}$ with $0 \leq |\mu| \leq q + 1$. Note that due to the quasi-linearity of the formal derivative (3.1), we may consider these additional equations as an inhomogeneous *linear* system for those coefficients $c_{\alpha,\mu}$ with $|\mu| = q + 1$ where both the matrix and the right hand side depend on the coefficients of lower order.

We may iterate this construction: entering our ansatz into $\mathscr{R}_{q+r}$ and evaluating at b yields the additional algebraic equations $D_\nu\Phi_\tau(\bar{\mathbf{x}}, \mathbf{c}) = 0$ where ν runs over all multi indices of length r. These equations contain all coefficients $c_{\alpha,\mu}$ with $0 \leq |\mu| \leq q + r$ and may be interpreted as an inhomogeneous linear system for the coefficients $c_{\alpha,\mu}$ with $|\mu| = q+r$ depending parametrically on the coefficients of lower order.

Note that this construction only makes sense for a formally integrable equation. Obviously, we can do the computation only up to some finite order $\hat{q} \geq q$. If the differential equation is not formally integrable, we cannot be sure that at some higher order integrability conditions of order less than or equal to $\hat{q}$ are hidden. These conditions would impose further restrictions on the coefficients $c_{\alpha,\mu}$ with $|\mu| \leq \hat{q}$ which we have not taken into account.

Furthermore, the validity of the interpretation of the equations obtained from the prolongation $\mathscr{R}_{q+r}$ as a linear system for the coefficients $c_{\alpha,\mu}$ with $|\mu| = q+r$ relies on the following observation which only holds for a formally integrable differential equation. In general, the matrix of this system does not have full rank. Thus it may be possible to generate zero rows by elementary row operations. The corresponding right hand side is guaranteed to vanish only for a formally integrable equation. Otherwise we would obtain an additional equation for the lower order coefficients. Indeed, this is just the effect of an integrability condition!

We may summarize these considerations in form of a local existence theorem for formal power series solutions which also explains the terminology "formal integrability".

Proposition 3.4. *Let the differential equation $\mathscr{R}_q$ be formally integrable. Then it possesses formal power series solutions.*

It should be mentioned that a serious problem with the concept of formal integrability is an effective criterion for verifying it. Obviously, the definition above contains infinitely many conditions. This is one of the reason why pure geometry does not suffice for the analysis of overdetermined systems; it must be complemented by algebraic, mainly homological, tools. In the sequel, we will not bother with deriving a criterion for formal integrability but instead introduce at once the stronger notion of an *involutive* system.

3.3. **The Geometric Symbol.** As $\pi^q_{q-1} : J_q\mathscr{E} \to J_{q-1}\mathscr{E}$ is an affine bundle modeled on the vector bundle $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$, we may consider for any differential equation $\mathscr{R}_q \subseteq J_q\mathscr{E}$ the vector bundle $\mathscr{N}_q = V^{(q)}\mathscr{R}_q \subseteq V^{(q)}J_q\mathscr{E}$ (where $V^{(q)}J_q\mathscr{E}$ denotes the vertical bundle with respect to the projection π^q_{q-1}) as a subbundle of $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$. It is called the *(geometric) symbol* of the differential equation $\mathscr{R}_q$. Note that while the geometric symbol is indeed closely related to the classical (principal) symbol introduced in many textbooks on partial differential equations (see, e.g., [69]), it is not the same.

In local coordinates, the symbol may be described as the solution space of a linear system of equations. Let as usual the equations $\Phi_\tau = 0$ with $1 \leq \tau \leq t$ form a local representation of the differential equation $\mathscr{R}_q$. We denote local coordinates on the vector space $\mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$ by $v_{\alpha,\mu}$ where $1 \leq \alpha \leq m$ and $|\mu| = q$ (the coefficients with respect to the basis $d\mathbf{x}^\mu \otimes \partial_{u_\alpha}$). Then the symbol $\mathscr{N}_q$ consists of those points $\mathbf{v}$ for which

$$\sum_{\alpha=1}^{m} \sum_{|\mu|=q} \frac{\partial \Phi_\tau}{\partial u_{\alpha,\mu}} v_{\alpha,\mu} = 0\,, \qquad 1 \leq \tau \leq t\,. \tag{3.2}$$

Note that this is a pointwise construction. Strictly speaking, we choose a point $\rho \in \mathscr{R}_q$ and evaluate the coefficient matrix of the linear system (3.2) at this point so that we obtain a real matrix. The rank of this matrix could vary with ρ, but we will always assume that this is not the case, so that $\mathscr{N}_q$ indeed forms a vector bundle over $\mathscr{R}_q$.

The symbol is most easily understood for linear systems. There it is essentially just the principal part of the system (i. e. the terms of maximal order), however, considered no longer as differential but as algebraic equations. For a nonlinear system, the local equations (3.2) describing the symbol at a point $\rho \in \mathscr{R}_q$ are obtained by first linearizing the local representation $\Phi_\tau = 0$ at ρ and then taking the principal part.

We may also introduce the prolonged symbols $\mathscr{N}_{q+r} = V^{(q+r)}\mathscr{R}_{q+r} \subseteq \mathfrak{S}_{q+r}(T^*\mathscr{B}) \otimes V\mathscr{E}$. It should be noted that their construction does not require one to actually compute the prolonged differential equations $\mathscr{R}_{q+r}$; they are already completely determined by $\mathscr{N}_q$. Indeed, we may compute them as the intersection $\mathscr{N}_{q+r} = \big(\mathfrak{S}_r(T^*\mathscr{B}) \otimes \mathscr{N}_q\big) \cap \big(\mathfrak{S}_{q+r}(T^*\mathscr{B}) \otimes V\mathscr{E}\big)$ (which is understood to take place in $\bigotimes^{q+r} T^*\mathscr{B} \otimes V\mathscr{E}$). Locally, $\mathscr{N}_{q+r}$ is the solution space of the following linear system of equations:

$$\sum_{\alpha=1}^{m} \sum_{|\mu|=q} \frac{\partial \Phi_\tau}{\partial u_{\alpha,\mu}} v_{\alpha,\mu+\nu} = 0\,, \qquad 1 \leq \tau \leq t\,, \quad |\nu| = r\,. \tag{3.3}$$

Again, this is a consequence of the quasi-linearity of the formal derivative (3.1). We have met these linear systems already in our discussion of formal integrability: (3.3) is the homogeneous part of the linear system determining the coefficients $c_{\alpha,\mu}$ with $|\mu| = q + r$.

3.4. **The Symbol Comodule.** The symbol $\mathscr{N}_q$ and its prolongations $\mathscr{N}_{q+r}$ are defined as subspaces of $\mathfrak{S}_{q+r}(T^*\mathscr{B}) \otimes V\mathscr{E}$ for $r \geq 0$. Recall (Section 2.10) that the graded vector space $\mathfrak{S}(T^*\mathscr{B})$ possesses a natural *co*algebra structure; more precisely it is isomorphic to the polynomial coalgebra $C = \Bbbk[x_1, \dots, x_n]$ with the coproduct Δ defined by (2.7). Note that no canonical isomorphism between $\mathfrak{S}(T^*\mathscr{B})$ and C exists. We must first choose a basis $\{\omega_1, \dots, \omega_n\}$ of the cotangent bundle $T^*\mathscr{B}$; then we have the trivial isomorphism defined by $\omega_i \leftrightarrow x_i$. Given some local coordinates $\mathbf{x}$ on $\mathscr{B}$ a simple choice is of course $\omega_i = dx_i$.

Given such an isomorphism between $\mathfrak{S}(T^*\mathscr{B})$ and the polynomial coalgebra C, we may consider $\mathfrak{S}(T^*\mathscr{B}) \otimes V\mathscr{E}$ as a free polynomial comodule of rank m over C. For the proof of the following proposition, it turns out to be convenient to use as a vector space basis of C not the usual monomials $\mathbf{x}^\nu$ but the multivariate divided powers $\gamma_\nu(\mathbf{x}) = \mathbf{x}^\nu/\nu!$ (cf. Section 2.9.1).

Proposition 3.5. *Let* $N =)\mathscr{N}_q(\subseteq \mathfrak{S}(T^*\mathscr{B}) \otimes V\mathscr{E}$ *be the polynomial subcomodule cogenerated by the symbol* $\mathscr{N}_q \subseteq \mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$. *Then* $N_{q+r} = \mathscr{N}_{q+r}$ *for all* $r \geq 0$.

Proof. This follows immediately from a comparison of (3.3) and (2.20). Recall that an element of the prolonged symbol $\mathscr{N}_{q+r}$ is of the form

$$f = \sum_{\alpha=1}^{m} \sum_{|\nu|=q+r} v_{\alpha,\nu} \partial_{u_\alpha} \otimes d\mathbf{x}^\mu$$

where the coefficients $v_{\alpha,\nu}$ form a solution of the linear system (3.3). We identify this element with the following vector in C^m:

$$f = \Big(\sum_{|\nu|=q+r} v_{1,\nu}\gamma_\nu(\mathbf{x}), \dots, \sum_{|\nu|=q+r} v_{m,\nu}\gamma_\nu(\mathbf{x}) \Big)\,.$$

It is now a straightforward exercise to verify that f corresponds to an element of $\mathscr{N}_{q+r}$, if and only if $\partial^r f/\partial \mathbf{x}^\mu$ corresponds for all multi indices μ with $|\mu| = r$ to an element of $\mathscr{N}_q$. $\square$

Note that this "dual interpretation" of the symbol leads also to a "reversal" of the direction of differentiation. While in the jet bundles differentiation yields equations of higher order, here in the comodule N differentiation leads to the components of lower degree. But, of course, this has to be expected in a dualization and explains why in the proof above we must use the divided powers as basis: the product $x_i\gamma_\nu(\mathbf{x})$ is just the integral $\int \gamma_\nu(\mathbf{x})dx_i$.

Dually, we may consider the left hand sides of the equations in (3.2) as elements of the dual space $\bigl(\mathfrak{S}_q(T^*\mathscr{B})\otimes V\mathscr{E}\bigr)^* \cong \mathfrak{S}_q(T\mathscr{B})\otimes V^*\mathscr{E}$ and similarly for the prolonged symbols. This leads naturally to considering the graded vector space $\mathfrak{S}(T\mathscr{B})\otimes V^*\mathscr{E}$ as a free module over the polynomial *algebra* $A = C^*$. Again an isomorphism between $\mathfrak{S}(T\mathscr{B})$ and A can be given only after a basis $\{w_1,\dots,w_n\}$ of $T\mathscr{B}$ has been chosen. Local coordinates $\mathbf{x}$ on $\mathscr{B}$ induce as a simple choice the basis $w_i = \partial_{x_i}$.

Now we may take the equations defining $\mathscr{N}_q$ and consider the submodule generated by their left hand sides. Again it is trivial to see that the left hand sides of the equations defining $\mathscr{N}_{q+r}$, i. e. (3.3), form the component of degree $q + r$. In fact, we get by Proposition 2.4 that these higher components are just $(N^\perp)_{q+r}$. This simple relation does not hold in the lower degrees, as there our submodule vanishes in contrast to $N^\perp$. As we will see, this is not of any consequence in terms of the connections with the formal theory.

3.5. **Involution.** As by definition $\mathscr{N}_q \subseteq \mathfrak{S}_q(T^*\mathscr{B})\otimes V\mathscr{E}$, we may interpret an element $\sigma\in\mathscr{N}_q$ as a ($V\mathscr{E}$ valued) multilinear form. A coordinate system $\mathbf{x}$ on the base manifold $\mathscr{B}$ induces a basis $\{\partial_{x_1},\dots,\partial_{x_n}\}$ of the tangent bundle $T\mathscr{B}$. We introduce for $1\le k\le n$ the subspaces

$$\mathscr{N}_{q,k}=\bigl\{\sigma\in\mathscr{N}_q \mid \sigma(\partial_{x_i},v_1,\dots,v_{q-1})=0,\ \forall 1\le i\le k,\ v_1,\dots,v_{q-1}\in T\mathscr{B}\bigr\} \tag{3.4}$$

and set $\mathscr{N}_{q,0}=\mathscr{N}_q$. They define a filtration of $\mathscr{N}_q$:

$$0=\mathscr{N}_{q,n}\subset\mathscr{N}_{q,n-1}\subset\cdots\subset\mathscr{N}_{q,1}\subset\mathscr{N}_{q,0}=\mathscr{N}_q\,. \tag{3.5}$$

We introduce the *Cartan characters* of the symbol $\mathscr{N}_q$ as the integers

$$\alpha_q^{(k)}=\dim\mathscr{N}_{q,k-1}-\dim\mathscr{N}_{q,k}\,,\quad 1\le k\le n\,. \tag{3.6}$$

Although it is not obvious from their definition, one can show that they always form a descending sequence: $\alpha_q^{(1)}\ge\cdots\ge\alpha_q^{(n)}\ge 0$ [11, 77].

Using the identification of $\mathfrak{S}_q(T^*\mathscr{B})\otimes V\mathscr{E}$ with C_q^m introduced in the previous section we may give an alternative description of these spaces

$$\mathscr{N}_{q,k}=\bigl\{\sigma\in\mathscr{N}_q \mid \frac{\partial\sigma}{\partial x_i}=0,\ \forall 1\le i\le k\bigr\} \tag{3.7}$$

where the differentiations are understood coordinate-wise. This identification also allows us to introduce the maps $\partial_k:\mathscr{N}_{q+1,k-1}\longrightarrow\mathscr{N}_{q,k-1}$ and we obtain the following algebraic version of the famous Cartan test [11].

Proposition 3.6. *We have the inequality*

$$\dim \mathscr{N}_{q+1} \leq \dim \mathscr{N}_{q,0} + \dim \mathscr{N}_{q,1} + \cdots + \dim \mathscr{N}_{q,n-1} = \sum_{k=1}^{n} k\alpha_q^{(k)} . \tag{3.8}$$

Equality holds, if and only if all the maps ∂_k are surjective.

Proof. For each $1 \leq k \leq n$ the exact sequence

$$0 \longrightarrow \mathscr{N}_{q+1,k} \longrightarrow \mathscr{N}_{q+1,k-1} \xrightarrow{\partial_k} \mathscr{N}_{q,k-1}$$

implies the inequality $\dim \mathscr{N}_{q+1,k-1} - \dim \mathscr{N}_{q+1,k} \leq \dim \mathscr{N}_{q,k-1}$. Summing over k and using the definition (3.6) of the Cartan characters yields (3.8). Equality in (3.8) requires equality in all these dimension relations, but this is equivalent to the surjectivity of all the maps ∂_k. □

Definition 3.7. The symbol $\mathscr{N}_q$ is *involutive*, if there exist local coordinates $\mathbf{x}$ on the base manifold $\mathscr{B}$ such that we have equality in (3.8).

Here we encounter for the first time the problem of *quasi-* or *δ-regularity.* If we obtain in a given coordinate system a strict inequality in (3.8), we cannot conclude that the symbol $\mathscr{N}_q$ is not involutive. It could be that we have simply taken a "bad" coordinate system. However, this problem is less severe than it might seem at first glance, as one can show that for an involutive symbol one finds *generically* equality in (3.8).

3.6. **A Computational Criterion for Involution.** Computationally, it is easier to work with equations instead of their solutions. Therefore we present now an effective realization of Def. 3.7 of an involutive symbol $\mathscr{N}_q$ in terms of the linear system (3.2) describing it locally. This combinatorial approach is based on ideas from the Janet-Riquier theory,[2] in particular the idea of multiplicative variables, and forms the basis of a computer algebra implementation of the formal theory [40].

Let us denote the matrix of the linear system (3.2) by $\mathbf{N}_q$. In order to simplify the analysis we transform it into row echelon form. However, before we order the columns in a certain manner. We define the *class* of a multi index $\mu = [\mu_1, \ldots, \mu_n]$ as $\operatorname{cls}\mu = \min\{i \mid \mu_i \neq 0\}$. Then we require that the column corresponding to the unknown $v_{\alpha,\mu}$ is always to the left of the column corresponding to $v_{\beta,\nu}$, if $\operatorname{cls}\mu > \operatorname{cls}\nu$. If the multi indices μ and ν possess the same class, it does not matter how the two columns are ordered. A simple way to achieve this ordering is to use the reverse lexicographic order defined by $v_{\alpha,\mu} \prec v_{\beta,\nu}$, if either the first non-vanishing entry of $\mu - \nu$ is positive or $\mu = \nu$ and $\alpha < \beta$

After having computed the row echelon form (without column permutations!), we analyze the location of the pivots and define $\beta_q^{(k)}$ as the number of pivots that

[2]Note however that within the classical Janet-Riquier theory neither the notion of a symbol nor the concept of involution appears. Using the more modern theory of involutive bases [29, 30] which may be considered as a combination of (a generalization of) the Janet-Riquier theory with Gröbner bases one may make the relation to involutive symbols more precise [77].

lie in a column corresponding to an unknown $v_{\alpha,\mu}$ with $\operatorname{cls}\mu = k$. The numbers $\beta_q^{(k)}$ with $1 \leq k \leq n$ are sometimes called the *indices* of $\mathscr{N}_q$. Obviously, their definition is coordinate dependent: in different coordinate systems $\mathbf{x}$ on the base manifold $\mathscr{B}$ different values may be obtained. However, it is not difficult to see that only a few coordinate systems yield different values. Generic coordinates lead to such values of the indices that the sum $\sum_{k=1}^{n} k\beta_q^{(k)}$ becomes maximal; such coordinates are called *δ-regular*. More precisely, any coordinate system can be transformed into a δ-regular one with a linear transformation defined by a matrix coming from a Zariski open subset of $\mathbb{R}^{n\times n}$.

Proposition 3.8. *The symbol $\mathscr{N}_q$ is involutive, if and only if local coordinates $\mathbf{x}$ on the base manifold $\mathscr{B}$ exist such that the matrix $\mathbf{N}_{q+1}$ of the prolonged symbol $\mathscr{N}_{q+1}$ satisfies*

$$\operatorname{rank}\mathbf{N}_{q+1} = \sum_{k=1}^{n} k\beta_q^{(k)}\,. \tag{3.9}$$

Proof. The Cartan criterion (3.8) is formulated in terms of dimensions of linear spaces; (3.9) is essentially an equivalent reformulation in terms of the ranks of the associated linear systems. It is a consequence of the following simple relation between the indices $\beta_q^{(k)}$ and the above introduced Cartan characters $\alpha_q^{(k)}$

$$\alpha_q^{(k)} = m\binom{q+n-k-1}{q-1} - \beta_q^{(k)}\,, \qquad 1 \leq k \leq n\,. \tag{3.10}$$

This relation stems from a straightforward combinatorial argument. We consider again the elements of $\mathscr{N}_q$ as polynomials ordered according to the TOP lift of the degree reverse lexicographic term order [2]. Then the Cartan character $\alpha_q^{(k)}$ gives us the dimension of the subspace of elements in $\mathscr{N}_q$ which have a leader of class k, as these elements make the difference between $\mathscr{N}_{q,k}$ and $\mathscr{N}_{q,k-1}$. In C_q^m there are $m\binom{q+n-k-1}{q-1}$ monomials of class k and by our above described preparation of the matrix $\mathbf{N}_q$ we have $\beta_q^{(k)}$ equations for them. This yields (3.10). □

The criterion (3.9) has a simple interpretation. By (3.3), we obtain the equations describing the prolonged symbol $\mathscr{N}_{q+1}$ by formally differentiating each equation in (3.2) with respect to all independent variables. Now assume that we have transformed the matrix $\mathbf{N}_q$ to row echelon form. We assign to each row the *multiplicative variables* $x_1, \ldots, x_k$, if the pivot of the row corresponds to an unknown $v_{\alpha,\mu}$ with $\operatorname{cls}\mu = k$.

It is now easy to see that all rows in $\mathbf{N}_{q+1}$ that stem from a formal differentiation with respect to a multiplicative variable are linearly independent (each has its pivot in a different column). As we have $\beta_q^{(k)}$ rows of class k, this observation implies that for *any* symbol the inequality $\operatorname{rank}\mathbf{N}_{q+1} \geq \sum_{k=1}^{n} k\beta_q^{(k)}$ holds (which is just a dual formulation of Prop. 3.6). Involutive symbols are precisely those that realize this lower bound so that this important idea of multiplicative variables (which goes back to Janet [46]) provides us with a *unique* way to generate all relevant equations in the prolongations of an involutive symbol.

These ideas concerning involutive symbols can be considerably generalized to a complete theory of so-called involutive bases which are a special kind of Gröbner bases. These bases contain much structural information. For more details see [78, 79]. A deeper study of the problem of δ-regularity (and an algorithmic solution of it) is contained in [40].

3.7. **Involutive Differential Equations.** Given the above definition of an involutive symbol, we may now finally introduce the notion of an involutive differential equation.

Definition 3.9. The differential equation $\mathscr{R}_q \subseteq J_q\mathscr{E}$ is called *involutive*, if it is formally integrable and if its symbol $\mathscr{N}_q$ is involutive.

Involutive differential equations have many pleasant properties. For example, one obtains a much better existence and uniqueness theory than for merely formally integrable systems. Somewhat surprisingly, involution is also of importance for numerical analysis, as obstructions to involution may become integrability conditions upon semi-discretization [80].

Recall that our existence result Prop. 3.4 for formally integrable systems does not speak about *unique* solutions. In general, the algebraic systems determining the Taylor coefficients at each order are underdetermined which simply reflects that differential equations usually have infinitely many solutions. It is the task of initial and/or boundary conditions to remove this arbitrariness. For involutive systems one may (algorithmically) derive the right form of initial conditions to ensure the existence of a unique formal solution with the help of some algebraic theory [77].

The extension of Prop. 3.4 to a strong existence (and uniqueness) theorem in some functions space is highly non-trivial. A general result is known only for analytic systems. Here the so-called Cartan normal form of an involutive system allows us via repeated application of the well-known Cauchy-Kovalevskaya theorem [69] to prove the convergence of our formal power series solutions.

Theorem 3.10 (Cartan-Kähler)**.** *Let $\mathscr{R}_q \subseteq J_q\mathscr{E}$ be an involutive differential equation. Assume that $\mathscr{R}_q$ is an analytic submanifold, i. e. it has local representations $\Phi_\tau = 0$ with $1 \leq \tau \leq t$ where the functions Φ_τ are real analytic. Then the Cauchy problem for $\mathscr{R}_q$ possesses for real analytic initial conditions a unique real analytic solution.*

It is important to note that the uniqueness is only within the space of real analytic functions. Thus the Cartan-Kähler theorem does not exclude the existence of further non-analytic solutions. However, for linear systems it is straightforward to extend the classical Holmgren theorem on the uniqueness of C^1 solutions [77].

Obviously, Def. 3.3 of a formally integrable system requires to check infinitely many conditions and thus is not constructive. At first sight, one could think that involution does not help us here, as the definition of an involutive equation includes formal integrability. However, we have the following proposition showing the power of the concept of involution. It requires a simple lemma which will be an easy consequence of looking at things homologically in Section 5.3.

Lemma 3.11. *If the symbol $\mathscr{N}_q$ is involutive, then the prolonged symbols $\mathscr{N}_{q+r}$ are also involutive for all integers $r > 0$.*

Proposition 3.12. *Let the symbol $\mathscr{N}_q$ of the differential equation $\mathscr{R}_q$ be involutive. Then $\mathscr{R}_q$ is involutive, if and only if $\mathscr{R}_q^{(1)} = \mathscr{R}_q$.*

Proof. Essentially, this is a corollary to Lemma 3.11 and a rather technical property of differential equations with involutive symbol, namely that for such equations $(\mathscr{R}_q^{(1)})_{+1} = \mathscr{R}_{q+1}^{(1)}$ (if we denote a prolongation by ρ and a projection by π, this means that if $\mathscr{N}_q$ is involutive, then $\rho \circ \pi \circ \rho = \pi \circ \rho^2$). This property is non-trivial, as it states that for such differential equations any integrability condition arising after two prolongations is also obtainable by differentiating an integrability condition arising after only one prolongation. For general differential equations this is surely not the case.

Now, given this result, we may conclude that $\mathscr{R}_{q+1}^{(1)} = (\mathscr{R}_q^{(1)})_{+1} = \mathscr{R}_{q+1}$ by assumption. As the prolonged symbols $\mathscr{N}_{q+r}$ are again involutive by Lemma 3.11, we may iterate this argument and find $\mathscr{R}_{q+r}^{(1)} = \mathscr{R}_{q+r}$ for all $r \geq 0$. Thus $\mathscr{R}_q$ is formally integrable. □

Hence for a differential equation $\mathscr{R}_q$ with involutive symbol it is no longer necessary to check an infinite number of prolongations: if no integrability conditions appear in the next prolongation, none will show up at higher order. This result is one of the two key ingredients for the *Cartan-Kuranishi theorem* below. The second one is the following result which will also be an easy consequence of looking at things homologically (the proof is given after Theorem 6.5 in Section 4).

Proposition 3.13. *Every symbol $\mathscr{N}_q \subseteq \mathfrak{S}_q(T^*\mathscr{B}) \otimes V\mathscr{E}$ becomes involutive after a finite number of prolongations.*

The question naturally arises what we should do, if we encounter an equation which is not involutive. The answer is simple: we complete it to an involutive one. The next theorem ensures that we may always do this without altering the (formal) solution space. The completion should be considered as a differential analogue to the construction of a Gröbner basis for a polynomial ideal: the addition of further generators to the original basis does not change the considered ideal, but many of its properties become more transparent, if we know a Gröbner basis.

Theorem 3.14 (Cartan-Kuranishi). *For every (sufficiently regular) differential equation $\mathscr{R}_q$ there exist two integers $r, s \geq 0$ such that the differential equation $\mathscr{R}_{q+r}^{(s)}$ is involutive.*

Proof. For lack of space we only sketch a constructive proof of this important theorem. As the name indicates, it stems originally from the theory of exterior systems and was first proven in full generality by Kuranishi [53]. Our proof follows the one presented in [65].

A simple algorithm for completing the differential equation $\mathscr{R}_q$ to an involutive equation consists of two nested loops. In the inner one, the equation is prolonged until its symbol becomes involutive (i. e. we increase the counter r). The termination of this loop follows immediately from Proposition 3.13. Once we have reached an involutive symbol, we check whether in the next prolongation integrability conditions appear. If not, we have reached an involutive equation by Proposition 3.12. Otherwise, we add these conditions (i. e. increase the counter s by one) and start anew. The termination of the outer loop can be shown with the help of a Noetherian argument. □

Note that the two equations $\mathscr{R}_q$ and $\mathscr{R}^{(s)}_{q+r}$ are in so far equivalent as they possess the same (formal) solution space. Indeed, neither prolongations nor the addition of integrability conditions can affect the solution space, as any of these additional equations is automatically satisfied by any solution of $\mathscr{R}_q$.

While the above sketched proof is surely constructive, it does not immediately yield an algorithm in the strict sense of computer science. For a number of steps in the "algorithm" it is not obvious whether they may be performed effectively. One of the greatest obstacles consists of effectively checking whether $\mathscr{R}^{(1)}_q = \mathscr{R}_q$. In local coordinates, this amounts to checking the functional independence of equations. Thus we must compute the rank of a Jacobian *on the submanifold* $\mathscr{R}_q$. For linear equations this is easily done via Gaussian elimination. For polynomial equations it can be done with Gröbner bases, although it may become rather expensive. For arbitrary equations no algorithm is known. A rather direct translation of the outlined completion procedure into a computer algebra package was presented in [75, 76]. A much more efficient version for linear equations enhancing the procedure with algebraic ideas from the theory of involutive bases was presented in [40].

Thanks to the Cartan-Kuranishi theorem, we may always assume in the analysis of a general differential equation (i. e. an equation not in Cauchy-Kovalevskaya form) that we are actually dealing with an involutive equation. This is a considerable simplification, as involutive equations possess local representations in a normal form (corresponding to the above described row echelon form of the symbol) making many of its properties much more transparent. The completion to an equivalent involutive equation is therefore a central algorithm for general equations.

4. More (Co)homological Algebra

There is quite a bit of literature on resolutions over the polynomial algebra $A = \Bbbk[x_1, ..., x_n]$ (see, e.g., [24, 49, 77]). In fact, computer programs exist which can compute such resolutions, e.g. Macaulay 2 [25].

We are specifically interested in algorithms for computing $\mathrm{Tor}^A(M, \Bbbk)$ where M is a graded right A-module. Recall [15, 62] that $\mathrm{Tor}^A(M, \Bbbk)$ may be computed as follows. Let

$$X \longrightarrow \Bbbk \longrightarrow 0$$

be a free A-module resolution of $\Bbbk$. By definition, $\mathrm{Tor}^A(M, \Bbbk)$ is the homology of the complex $M \otimes_A X$. It is well-known that Tor is independent of the resolution used to compute it.

In fact, one also has $\mathrm{Tor}^A(\Bbbk, N)$ for a graded left A-module N. It can be computed by finding a free A-module resolution

$$Y \longrightarrow N \longrightarrow 0$$

of N. One has that $\mathrm{Tor}^A(\Bbbk, N)$ is the homology of the complex $Y \otimes_A N$. Again, this is independent of the resolution used. Furthermore, it is also well-known that as vector spaces over $\Bbbk$, one has

$$\mathrm{Tor}^A(M, \Bbbk) \cong \mathrm{Tor}^A(\Bbbk, M) \tag{4.1}$$

where if M is a right A-module, it has the left A-module structure given by $am = ma$ [15, 62]. Similar remarks apply if M is a left A-module. This gives quite a bit of freedom in finding resolutions for computing Tor. For example consider the following six resolutions.

4.1. **Schreyer Type Resolutions.**

Using the notion of a Gröbner basis, Schreyer [74] essentially proved the following (we follow the exposition in [44, Section 2]).

Proposition 4.1. *Let M be generated by $\{m_1, \ldots, m_s\}$ over A and let $f : A^s \longrightarrow M$ be the A-linear map given by $f(e_i) = m_i$ (e_i is the standard basis vector in A^s). There is an explicit algorithm that computes a finite generating set for* $\ker(f)$.

The interested reader should see [24] or [44] (which is given in a more general context) for details. By iterating this construction, one obtains a resolution

$$\cdots \longrightarrow S_i \longrightarrow S_{i-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow M \longrightarrow 0$$

which is well-known to be a finite sequence of finite-dimensional free A-modules S_i. We call this the Schreyer resolution S of M over A. Variations of this construction are implemented in the Macaulay 2 program [25] and other variations in the context of involutive bases can be found in [79].

The purpose of the following sections is to prepare for a description of a novel explicit algorithm given in [49] for computing resolutions of M over A.

4.2. **The Koszul Resolution.** The Koszul resolution K of $\Bbbk$ over A is the complex $K = A \otimes_{\Bbbk} \bar{K}$ where $\bar{K}$ is the exterior algebra $E[u_1, \dots, u_n]$ (cf. Section 2.11). We add the relation that $u_I f = f u_I$ for $f \in A$. The differential in K is given by extending the map $d(u_i) = x_i$ A-linearly as a derivation:

$$d(u_I u_J) = d(u_I)u_J + (-1)^{|I|} u_I d(u_J).$$

Clearly, one has

$$d(u_{j_1} \cdots u_{j_k}) = \sum_{i=1}^{k} (-1)^{k+1} x_{j_i} u_{j_1} \cdots \widehat{u_{j_i}} \cdots u_{j_k} \tag{4.2}$$

where $\widehat{u_{j_i}}$ denotes omission. Thus, for any right A-module M, we have that

$$\operatorname{Tor}^A(M, \Bbbk) = H(M \otimes_A K) = H(M \otimes \bar{K}) = H(M \otimes E[u_1, \dots, u_n]). \tag{4.3}$$

If M is a graded module, $\operatorname{Tor}^A(M, \Bbbk)$ inherits a bigrading as follows. Let $(M \otimes \bar{K})_{i,j} = M_i \otimes \bar{K}_j$. From (2.3) and (4.2) above, it follows that

$$(M \otimes \bar{K})_{i,j} \xrightarrow{1_M \otimes d} (M \otimes \bar{K})_{i+1,j-1}. \tag{4.4}$$

and hence $\operatorname{Tor}^A(M, \Bbbk)$ inherits the bigrading.

Remark 4.2. For simplicity in the exposition, we assume that all modules over A are of the form $M = A/I$ where I is an ideal in A in the next two resolutions which are valid only for monomial ideals. The more general case of finitely presented modules $M = A^m/N$ where N is a monomial submodule follows by using what we present coordinate-wise.

4.3. **The Taylor Resolution.** For a given set $\{m_1, \dots, m_k\}$ of monomials and for any subset $J = \{j_1, \dots, j_s\} \subseteq \{1, \dots, k\}$, let $m_J = \operatorname{lcm}(m_{j_1}, \dots, m_{j_s})$, and $J^i = J \setminus \{j_i\}$.

For a monomial ideal $N = (m_1, \dots, m_k)$, the Taylor resolution [85] of $M = A/N$ over A is given by the following A-linear differential d on the free A-module $A \otimes E[u_1, \dots, u_k]$:

$$d(u_J) = \sum_{i=1}^{|J|} (-1)^{i-1} \frac{m_J}{m_{J^i}} u_{J^i}.$$

This resolution will be denoted by T throughout this paper. An explicit contracting homotopy for T was given in [27]. We recall it here.

For a monomial $\mathbf{x}^\alpha$ and a basis element u_J of E, let

$$\iota(\mathbf{x}^\alpha u_J) = \min\{i \mid m_i \preceq \mathbf{x}^\alpha m_J\} \tag{4.5}$$

(recall $\preceq$ from 2.4). Note that $\iota(\mathbf{x}^\alpha u_J) \le j_1$. Define a k-linear map

$$\psi(x^\alpha u_J) = [\iota < j_1] \frac{\mathbf{x}^\alpha m_J}{m_{\{\iota\} \cup J}} u_{\{\iota\} \cup J} \tag{4.6}$$

where $\iota = \iota(\mathbf{x}^\alpha u_J)$, and $[p]$ is the Kronecker-Iverson symbol [33] which is zero if p is false and one otherwise. It is straightforward to calculate that ψ is indeed a contracting homotopy, i.e. we have $d\psi + \psi d = 1$ on elements of positive degree.

One can show [81] that the Taylor resolution is a special case of the resolutions obtainable via Schreyer's construction. The contracting homotopy ψ is then related to normal form computations with respect to a Gröbner basis.

Thus one can compute $\mathrm{Tor}^A(\Bbbk, M)$ as

$$\mathrm{Tor}^A(\Bbbk, A/N) = H(\Bbbk \otimes_A T) = H(E[u_1, \ldots, u_k]). \tag{4.7}$$

4.4. **The Lyubeznik Resolution.** Suppose again that $N = \{m_1, \ldots, m_k\}$ is a monomial ideal in A. A subcomplex L of the Taylor resolution which is itself a resolution of $M = A/N$ over A was given in [61]. We recall it here. For a given $I \subseteq \{1, \ldots, k\}$ and positive integer s between 1 and k, let $I_{>s} = \{i \in I \mid i > s\}$. L is generated by those basis elements u_I which satisfy the following condition for all $1 \leq s < k$:

$$m_s \not\preceq m_{I_{>s}}. \tag{4.8}$$

In [81] it was shown that this corresponds to repeated applications of Buchberger's chain criterion [12] for avoiding redundant S-polynomials in the construction of Gröbner bases.

As usual, we denote $\Bbbk \otimes_A L$ by $\bar{L}$. Thus one can also compute $\mathrm{Tor}^A(\Bbbk, M)$ as

$$\mathrm{Tor}^A(\Bbbk, M) = H(\Bbbk \otimes_A L) = H(\bar{L}). \tag{4.9}$$

4.5. **The Bar Resolution.** The two sided bar construction $B(A, A)$ [15, 62] is defined as follows.

$$\begin{aligned} \bar{B}_0(A) &= k \\ \bar{B}_k(A) &= \otimes^k \bar{A}, \qquad k > 0, \end{aligned}$$

and $\bar{A} = \mathrm{coker}(\sigma)$ where $k \xrightarrow{\sigma} A$ is the unit. The usual convention is to abbreviate the product $a \otimes a_1 \otimes \cdots a_k \otimes a'$ as $a[a_1|\cdots|a_k]a'$ and we will follow that convention. The differential in $B(A, A)$ is given by the A-linear map induced by

$$\begin{aligned} \partial\big([a_1|\cdots|a_k]a'\big) &= a_1[a_2|\cdots|a_k]a' \\ &+ \sum_{i=1}^{k-1}(-1)^i[a_1|\cdots a_i a_{i+1}\cdots|a_k]a' \\ &+ (-1)^k[a_1|\cdots|a_{k-1}]a_k a' \end{aligned}$$

The k-linear map $B(A, A) \xrightarrow{s} B(A, A)$ is defined by

$$s\big(a[a_1|\cdots|a_k]a'\big) = [a|a_1|\cdots|a_k]a'.$$

The map $A \xrightarrow{\sigma} B(A, A)$ is given by

$$\sigma(a) = [\ ]a$$

and the map $B(A, A) \xrightarrow{\epsilon} A$ is given by

$$\begin{aligned} \epsilon\big(a[\ ]a'\big) &= aa' \\ \epsilon\big(a[a_1|\cdots|a_k]a'\big) &= 0, \qquad k \geq 1. \end{aligned}$$

The map s is a contracting homotopy for $B(A, A)$.

4.6. **Bar resolution of $\Bbbk$.** It is not hard to see that $B(A, \Bbbk) = B(A, A) \otimes_A \Bbbk$ is a resolution of $\Bbbk$ over A. Here $\Bbbk$ is given the A-module structure defined by the augmentation map $\epsilon(p) = p(0)$ (cf. Section 2.9).

Define $\Bbbk \xrightarrow{\sigma_{\Bbbk}} B(A, \Bbbk)$ by

$$\sigma_{\Bbbk}(r) = [\ ]r,$$

$B(A, \Bbbk) \xrightarrow{\epsilon_{\Bbbk}} \Bbbk$ by

$$\begin{aligned} \epsilon_{\Bbbk}\big(a[\ ]r\big) &= ar \\ \epsilon_{\Bbbk}\big(a[a_1|\cdots|a_k]x\big) &= 0, \qquad k \geq 1, \end{aligned}$$

and $B(A, \Bbbk) \xrightarrow{s_{\Bbbk}} B(A, \Bbbk)$ by

$$s\big(a[a_1|\cdots|a_k]\big) = [a|a_1|\cdots|a_k].$$

One then has that $s_{\Bbbk}$ is a contracting homotopy.

Thus one can also compute $\mathrm{Tor}^A(M, \Bbbk)$ as

$$\mathrm{Tor}^A(M, \Bbbk) = H\big(M \otimes_A B(A, \Bbbk)\big) = H\big(M \otimes \bar{B}(A)\big). \tag{4.10}$$

4.7. **Bar resolution of M.** For any left A-module M, one has a free A-complex given by $B(A, M) = B(A, A) \otimes_A M$. As an A-module, note that

$$B(A, M) = A \otimes_k \bar{B}(A) \otimes_k M.$$

The differential is given by $\partial_M = \partial \otimes 1_M$. Thus,

$$\begin{aligned} \partial_M\big(a[a_1|\cdots|a_k]x\big) &= aa_1[a_2|\cdots|a_k]x \\ &+ \sum_{i=1}^{k-1}(-1)^i a[a_1|\cdots a_i a_{i+1}\cdots|a_k]x \\ &+ (-1)^k a[a_1|\cdots|a_{k-1}]a_k x \end{aligned}$$

for $a, a_i \in A$ and $x \in M$. Define $M \xrightarrow{\sigma_M} B(A, M)$ by

$$\sigma_M(x) = [\ ]x,$$

$B(A, M) \xrightarrow{\epsilon_M} M$ by

$$\begin{aligned} \epsilon_M\big(a[\ ]x\big) &= ax \\ \epsilon_M\big(a[a_1|\cdots|a_k]x\big) &= 0, \qquad k \geq 1, \end{aligned}$$

and $B(A, M) \xrightarrow{s_M} B(A, M)$ by

$$s\big(a[a_1|\cdots|a_k]x\big) = [a|a_1|\cdots|a_k]x.$$

One then has that s is a contracting homotopy.

Thus one can also compute $\mathrm{Tor}^A(\Bbbk, M)$ as

$$\mathrm{Tor}^A(\Bbbk, M) = H\big(\Bbbk \otimes_A B(A, M)\big) = H\big(\bar{B}(A) \otimes M\big). \tag{4.11}$$

Remark 4.3. The last five resolutions will be involved in the derivation of a new explicit algorithm for computing resolutions of M over A using homological perturbation theory (Section 7). This will involve explicit comparisons between the Lyubeznik complex, the bar complexes and the Koszul complex. The standard proof that $\mathrm{Tor}^A(M, \Bbbk)$ and $\mathrm{Tor}^A(\Bbbk, M)$ are isomorphic does not produce an explicit map between the resolutions involved. However, it is quite easy to see that the chain map

$$\varphi(m[a_1|\cdots|a_k]) = [a_1|\cdots|a_k]m \tag{4.12}$$

induces an explicit isomorphism of $\mathrm{Tor}^A(M, \Bbbk)$ and $\mathrm{Tor}^A(\Bbbk, M)$. We will make use of this isomorphism in the following sections.

4.8. **Strong Deformation Retracts.** Let X and Y be chain complexes over k, $\nabla : X \longrightarrow Y$, $f : Y \longrightarrow X$ be chain maps and let $\phi : Y \longrightarrow Y$ be a degree one k-linear map such that $f\nabla = 1_X$ and $d\phi + \phi d = 1 - \nabla f$, i.e. ϕ is a chain homotopy between the identity and ∇f (cf. Section 2.2). Such a collection of data is said to form a *strong deformation retraction* (SDR). We denote this situation by the diagram

$$X \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, \phi)\,. \tag{4.13}$$

The so called *side conditions* [60] are the equations

$$\phi^2 = 0, \quad \phi\nabla = 0, \quad \text{and,} \quad f\phi = 0. \tag{4.14}$$

In fact, we may always assume that the side conditions hold [60].

4.9. **Relatively Free Resolutions.** We will consider resolutions of N over A which have form $X = A \otimes \overline{X}$ where $\overline{X}$ is a vector space over $\Bbbk$ [15, 62]. Such complexes are called *relatively free* [62]. The elements of $\overline{X}$ above are called *reduced* elements. An even stronger condition is that there exists an explicit contracting homotopy ψ which forms an SDR

$$N \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} (X, \psi)$$

where ϵ is an A-linear map, but generally, σ and ψ are only $\Bbbk$-linear. Here N is given the zero differential. As we will see, each of the five resolutions in Section 4 are of this form. In fact, using the maps defined in Sects. 4.6 and 4.7, it is clear that $B(A, \Bbbk)$ and $B(A, M)$ are relatively free resolutions. We claim that we also have SDRs

$$\Bbbk \underset{\epsilon_{\Bbbk}}{\overset{\sigma_{\Bbbk}}{\rightleftarrows}} (B(A, \Bbbk), s) \tag{4.15}$$

and

$$M \underset{\epsilon_M}{\overset{\sigma_M}{\rightleftarrows}} (B(A, M), s) \tag{4.16}$$

where the maps are also from Sects. 4.6 and 4.7.

Remark 4.4. Recall from Section 4.7 that $\epsilon(a[\,]x) = ax$. Thus, in order to make *explicit* calculations, we need a unique representative $b \in A$ for each class in $N = A/M$. In other words, we need a *normal form* for elements of M. This can be obtained by the standard normal form algorithm from Gröbner basis theory [6, 28]. The algorithm depends on a given term order (cf. Section 2.1) and a given set $G = \{g_1, \dots, g_r\} \subset A$. Here is a recursive algorithm for computing the normal form, denoted by $\mathrm{rem}_G(p)$ of a given polynomial p: if there exists a minimal i such that $\mathrm{LT}(g_i) \preceq \mathrm{LT}(p)$, then

$$\mathrm{rem}_G(p) = \mathrm{rem}_G\left(p - \frac{\mathrm{LT}(p)}{\mathrm{LT}(g_i)} g_i\right)$$

and

$$\mathrm{rem}_G(p) = \mathrm{LT}(p) + \mathrm{rem}_G(p - \mathrm{LT}(p))$$

otherwise. Here we have used the notation $\mathrm{LT}(p)$ for the leading term of a polynomial p with respect to the given term order. In our case, we have that G is a (minimal) generating set for M.

Consider again the Koszul resolution K (Section 4.2). We need an explicit contracting homotopy for K. For this, we exploit once more the fact that $A = \Bbbk[x_1, \dots, x_n] = \otimes^n \Bbbk[x]$. In the case of one variable, $K = \Bbbk[x] \otimes E[u]$ is just the complex with $d(p) = 0, p \in A$, and $d(pu) = px$. Given $\epsilon(p) = p(0)$ and $\epsilon(pu) = 0$, and $\sigma(r) = r \otimes 1$ for $r \in \Bbbk$, we need to solve the equation

$$d\psi(y) + \psi d(y) = y - \sigma\epsilon(y)$$

for all $y \in K$. Clearly, we can (and must) take $\psi(pu) = 0$ (the degree of ψ is $+1$), and so we only need consider ψ on x^i for $i \geq 1$ (ψ is $\Bbbk$-linear, but not A-linear). But the equation

$$d\psi(x^i) = x^i$$

is easily seen to be satisfied by

$$\psi(x^i) = x^{i-1}u. \tag{4.17}$$

Now it is clear that the Koszul complex over $\Bbbk[x_1, \dots, x_n]$ is just the tensor product complex $\otimes^n \Bbbk[x] \otimes E[u]$. But as is well-known, chain homotopies can be tensored as well. In this case, we can use

$$\psi_n = \psi \otimes 1 \otimes \cdots \otimes 1 + \sigma\epsilon \otimes \psi \otimes 1 \otimes \cdots \otimes 1 + \cdots + \sigma\epsilon \otimes \sigma\epsilon \otimes \cdots \otimes \sigma\epsilon \otimes \psi. \tag{4.18}$$

It is now easy to see that along with the maps $\sigma_n = \otimes^n \sigma$ and $\epsilon_n = \otimes^n \epsilon$, we have an SDR

$$\Bbbk \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} (K, \psi)\,. \tag{4.19}$$

4.10. **Splitting Off Of the Bar Construction.** The well-known *comparison theorem* [15, 62] in homological algebra states that any two projective resolutions are chain homotopy equivalent. For the relatively free resolutions of the last section, we will need some explicit comparisons that actually yield SDRs. For relatively free resolutions X and Y over A with explicit SDR data

$$N \underset{\epsilon_X}{\overset{\sigma_X}{\rightleftarrows}} (X, \psi_X) \tag{4.20}$$

and

$$N \underset{\epsilon_Y}{\overset{\sigma_Y}{\rightleftarrows}} (Y, \psi_Y)$$

there are inductive procedures for obtaining explicit chain equivalences $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. As was shown in [16], these procedures essentially follow from the requirements that f and g are A-linear and are chain maps. One uses the explicit contracting homotopies to construct them. This was used in [49, 55, 56, 57] for example. In addition, there are inductive procedures for obtaining explicit chain homotopies of fg and the identity and with gf and the identity. Generally, the maps defined in this way do not form an SDR. However, the following lemma was given in [49].

Lemma 4.5. *Suppose that X is a relatively free resolution as above and that the contracting homotopy ψ_X satisfies $\psi_X(\overline{X}) = 0$ and $d(\bar{X}) \cap \bar{X} = 0$ and the homotopy ψ_Y satisfies $\psi_Y(Y) \subseteq \overline{Y}$, then the inductive constructions mentioned above give an SDR*

$$X \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, \phi)\,.$$

It is well known (e.g. [56]) that $X = K$ (the Koszul resolution) and $Y = B(A, \Bbbk)$ satisfy the hypotheses of the lemma and so we have an SDR

$$K \underset{f_K}{\overset{\nabla_K}{\rightleftarrows}} (B(A, \Bbbk), \phi_K)\,. \tag{4.21}$$

We emphasize that these SDRs are explicitly given in terms of the explicit SDRs of the form (4.20) for the objects involved as given above. In fact, the first author has implemented these SDRs using computer algebra (specifically the system described in [47]) and these implementations will be used for all calculations that follow.

4.11. **Twisting Cochains.** A twisting cochain is a degree minus one map $C \xrightarrow{\tau} A$ where C is a differential graded coalgebra and A is a differential graded algebra and τ satisfies

$$d\tau + \tau d = \tau \cup \tau \tag{4.22}$$

where the map $\tau \cup \tau$ is given by

$$(\tau \cup \tau)(c) = \tau(c_{(1)})\tau(c_{(2)})$$

in H-S notation. The following fact is a fundamental property of twisting cochains.

Proposition 4.6. *Let $C \xrightarrow{\tau} A$ be a degree minus one map from a (differential) algebra to a (differential) coalgebra. Let*

$$A \otimes C \xrightarrow{d_\tau} A \otimes C$$

be the map defined by the composite

$$\begin{tikzcd} A \otimes C \arrow[r, "d_\tau"] \arrow[d, "1 \otimes \Delta"'] & A \otimes C \\ A \otimes C \otimes C \arrow[r, "1 \otimes \tau \otimes 1"'] & A \otimes A \otimes C \arrow[u, "m \otimes 1"'] \end{tikzcd}$$

then $(A \otimes C, d_\tau)$ is a chain complex, if and only if τ is a twisting cochain.

The complex $(A \otimes C, d_\tau)$ is called a *twisted tensor product* complex. The interested reader should see, e.g., [9, 34, 35] for details.

Note that an analogous result holds for degree one maps from C to A in which case d_τ will have degree one. Note also that in H-S notation, d_τ is given by

$$d_\tau(a \otimes c) = a\tau(c_{(1)}) \otimes \tau(c_{(2)}).$$

5. A Homological Approach to Involution

5.1. **A Theorem by Serre.** The key to a homological interpretation of involution is a theorem by Serre given in an appendix to [39]. It gives a criterion for the vanishing of $\mathrm{Tor}^A(M, \Bbbk)$ where M is a finitely generated graded module over the polynomial algebra $A = \Bbbk[x_1, \dots, x_n]$. As shown in Section 4.2, $\mathrm{Tor}^A(M, \Bbbk)$ is bigraded in this case.

Theorem 5.1. *Let M be a finitely generated graded module over the polynomial algebra $A = \Bbbk[x_1, \dots, x_n]$. Then $\mathrm{Tor}^A_{p,q}(M, \Bbbk) = 0$ for all $p \geq 1$, and all $q \geq 0$, if and only if a basis $\{y_1, \dots, y_n\}$ of the component A_1 exists such that for all $0 \leq i < n$ and all $p \geq 0$ the maps*

$$m_{i+1} : M_{p+1}/(y_1, \dots, y_i)M_p \longrightarrow M_{p+2}/(y_1, \dots, y_i)M_{p+1} \tag{5.1}$$

induced by the multiplication with y_{i+1} are injective.

Serre's proof is quite lucent and we encourage the reader to read it.

It is customary to call a sequence $(y_1, \dots, y_n) \subset A_1$ satisfying the condition of the theorem *quasiregular*. We will see below that for the polynomial modules of interest for us this notion coincides with the notion of δ-regularity introduced in Section 3.6.

5.2. **An Alternative Criterion for Involution.** In Section 3.4 we introduced the symbol comodule N which was a comodule over the polynomial coalgebra C. By Prop. 2.6 we have an isomorphism $M = A^m/N^\perp \cong N^*$ where M is now a module over the polynomial algebra $A = C^*$. The Cartan test (3.8) yields a criterion for an involutive symbol directly in terms of the comodule N. Now we provide a criterion in terms of the module M.

Theorem 5.2. *The symbol $\mathscr{N}_q$ is involutive, if and only if there exist local coordinates $\mathbf{x}$ of the base manifold $\mathscr{B}$ such that for all $r \geq 0$ and all $0 \leq i < n$ the maps*

$$m_{i+1} : M_{q+r+1}/(x_1, \dots, x_i)M_{q+r} \longrightarrow M_{q+r+2}/(x_1, \dots, x_i)M_{q+r+1} \tag{5.2}$$

induced by the multiplication with x_{i+1} are injective.

For technical reasons, it is easier to consider first the submodule $N^\perp \subseteq A^m$ and move later to the factor module M. Furthermore, the following lemma shows the direct link between the approach via multiplicative variables used in Section 3.6 and the new approach presented in this section.

Lemma 5.3. *The symbol $\mathscr{N}_q$ is involutive, if and only if there exist local coordinates $\mathbf{x}$ of the base manifold $\mathscr{B}$ such that for all $r \geq 0$ and all $0 \leq i < n$ the maps*

$$\hat{m}_{i+1} : N^\perp_{q+r+1}/(x_1, \dots, x_i)N^\perp_{q+r} \longrightarrow N^\perp_{q+r+2}/(x_1, \dots, x_i)N^\perp_{q+r+1} \tag{5.3}$$

induced by the multiplication with x_{i+1} are injective.

Proof. Let us assume first that $\mathscr{N}_q$ was involutive. Following the discussion in Section 3.6, this implies the existence of local coordinates $\mathbf{x}$ of $\mathscr{B}$ such that we may construct a triangular (vector space) basis of $N^\perp_q$ of the form

$$\mathscr{B}_q = \left\{h_{k,\ell} \mid 1 \leq k \leq n,\ \operatorname{cls} h_{k,\ell} = k,\ 0 \leq \ell \leq \ell_k\right\}, \tag{5.4}$$

i. e. we sort the elements of the basis according to their classes. The elements of this basis correspond to the rows of the matrix $\mathbf{N}_q$ appearing in Prop. 3.8. For an involutive symbol, a basis of $N^\perp_{q+r}$ is given by

$$\mathscr{B}_{q+r} = \left\{x_1^{i_1} \cdots x_k^{i_k} h_{k,\ell} \mid 1 \leq k \leq n,\ 0 \leq \ell \leq \ell_k,\ i_1 + \cdots + i_k = r\right\}, \tag{5.5}$$

i. e. by multiplying each element of $\mathscr{B}_q$ by r multiplicative variables. Now we may straightforwardly construct explicit bases of the quotient spaces $N^\perp_{q+r+1}/(x_1, \dots, x_i)N^\perp_{q+r}$, as they are isomorphic to the linear spans of the subsets $\mathscr{B}^{(i)}_{q+r}$ consisting of only those generators whose class is greater than i. The assertion follows trivially from these bases, as x_{i+1} is multiplicative for all generators in $\mathscr{B}^{(i)}_{q+r}$.

For the converse, we use an indirect proof: we show that if $\mathscr{N}_q$ is not involutive, then it is not possible that all the maps $\hat{m}_i$ are injective. As we consider the maps $\hat{m}_i$ for all $r \geq 0$, we may assume without loss of generality that already $\mathscr{N}_{q+1}$ is involutive. We will prove now that this implies that at least one of the maps $\hat{m}_i$ is not injective for $r = 0$.

We use again the basis $\mathscr{B}_q$ for $N_q^\perp$. However, as $\mathscr{N}_q$ is not involutive, $\mathscr{B}_{q+1}$ generates only a subset of $N_{q+1}^\perp$, even if we use a δ-regular coordinate system $\mathbf{x}$ on $\mathscr{B}$. Thus there exist values i, k, ℓ with $i \geq k$ such that $x_{i+1}h_{k,\ell} \in N_{q+1}^\perp$ is not contained in the span of $\mathscr{B}_{q+1}$. The equivalence class $[x_{i+1}h_{k,\ell}] \in N_{q+1}^\perp/(x_1, \dots, x_i)N_q^\perp$ is by construction nonzero and linearly independent of the equivalence classes of all elements in $\mathscr{B}_{q+1}$ with a class greater than i. We consider now the action of $\hat{m}_{i+1}$ on this element.

Obviously, $\operatorname{cls}(x_{i+1}h_{k,\ell}) = k$ and thus x_{i+1} is non-multiplicative for this element. By assumption, $\mathscr{N}_{q+1}$ is involutive and thus the non-multiplicative product $x_{i+1} \cdot (x_{i+1}h_{k,\ell})$ can be expressed as a linear combination of other multiplicative products. It is obvious that all these multiplicative products can only be with respect to the variables $x_1, \dots, x_{i+1}$ and this implies immediately that $\hat{m}_{i+1}$ cannot be injective. □

Example 5.4. In Lemma 5.3 it is important that the injectivity holds for all $r \geq 0$. Even if all maps $\hat{m}_i$ are injective for $r = 0$, we cannot conclude that $\mathscr{N}_q$ is involutive, as the following simple example demonstrates. Consider the differential system $u_{xxx} = u_{yyy} = 0$ where for notational simplicity we write $x_1 = x$ and $x_2 = y$. Then the module $N^\perp$ is generated by the two monomials x^3 and y^3. It is trivial that $\hat{m}_x : N_4^\perp \to N_5^\perp$ is injective. For $\hat{m}_y$ we note that $N_4^\perp/xN_3^\perp \cong \langle x^3y, y^4\rangle$ and thus it is again easy to see that $\hat{m}_y$ is injective.

Nevertheless, the symbol $\mathscr{N}_3$ is *not* involutive. Indeed, consider the non-multiplicative prolongation $D_y u_{xxx} = u_{xxxy}$; it is obviously independent of all multiplicative prolongations and thus the criterion (3.9) is not satisfied. Similarly, the symbol $\mathscr{N}_4$ is not involutive, as the non-multiplicative prolongation $D_y u_{xxxy} = u_{xxxyy}$ is again independent of all multiplicative prolongations. In contrast, $\mathscr{N}_5$ and all higher symbols are trivially involutive, as they vanish.

If we consider the map $\hat{m}_y : N_5^\perp/xN_4^\perp \to M_6/xM_5$, then we find (using the identification $N_5^\perp/xN_4^\perp \cong \langle x^3y^2, y^5\rangle$) that $\hat{m}_y([x^3y^2]) = [x^3y^3] = 0$ so that $\hat{m}_y$ is not injective. This was to be expected by our proof of Lemma 5.3. The observation that at some lower degree the maps $\hat{m}_x$ and $\hat{m}_y$ are injective may be understood by looking at the syzygies of M_3. The syzygy module is generated by the single element $(y^3, -x^3) \in \mathbb{k}[x, y]^2$. As it is of degree 3, nothing happens with the maps $\hat{m}_i$ before we encounter M_6 and the equation $\hat{m}_y([x^3y^2]) = 0$ is a trivial consequence of this syzygy. More on the relation between involution and syzygies may be found in [79].

The proof of Thm. 5.2 consists now of a simple homological argument and two applications of Serre's Thm. 5.1.

Proof (*of Thm.* 5.2). It is a classical result in homological algebra that the short exact sequence $0 \to N^\perp \to A^m \to M \to 0$ where the first map is the inclusion and the second one the canonical projection induces a long exact sequence for the torsion modules. As $\operatorname{Tor}^A(A^m, \mathbb{k})$ trivially vanishes in positive degree, $\operatorname{Tor}^A(M, \mathbb{k}) \cong \operatorname{Tor}^A(N^\perp, \mathbb{k})$ in positive degree.

Using Lemma 5.3 and applying Serre's Theorem 5.1 to the polynomial module $N^\perp$ yields that involution of $\mathscr{N}_q$ is equivalent to the vanishing of $\mathrm{Tor}^A(N^\perp, \Bbbk)$ in positive degree. By the argument above this implies that $\mathrm{Tor}^A(M, \Bbbk)$ vanishes in positive degree. Applying again Serre's Theorem 5.1, this time to the polynomial module M yields the assertion. □

5.3. **Spencer Cohomology.** The Spencer cohomology was originally introduced in a completely different context [82] and only later related to involution. In this section we present the classical approach to it; in the sequel, we will give a completely different derivation of the relevant complex. Note that in line with Remark 3.2 we use again the vector space $\mathfrak{S}_q(T^*\mathscr{B})$ instead of the more common $\mathscr{S}_q(T^*\mathscr{B})$.

Consider the vector space homomorphism $\delta : \mathfrak{S}_{r+1}(T^*\mathscr{B}) \to T^*\mathscr{B} \otimes \mathfrak{S}_r(T^*\mathscr{B})$ defined by the composition of the natural inclusion map $\mathfrak{S}_{r+1}(T^*\mathscr{B}) \hookrightarrow T^*\mathscr{B} \otimes \bigotimes^r T^*\mathscr{B}$ with the canonical projection $T^*\mathscr{B} \otimes \bigotimes^r T^*\mathscr{B} \to T^*\mathscr{B} \otimes \mathfrak{S}_r(T^*\mathscr{B})$. By wedging both sides with $E_s(T^*\mathscr{B})$ (the s-fold exterior product of $T^*\mathscr{B}$) and tensoring with the vertical bundle $V\mathscr{E}$ we extend δ to a map $E_s(T^*\mathscr{B}) \otimes \mathfrak{S}_{r+1}(T^*\mathscr{B}) \otimes V\mathscr{E} \to E_{s+1}(T^*\mathscr{B}) \otimes \mathfrak{S}_r(T^*\mathscr{B}) \otimes V\mathscr{E}$.

In local coordinates $(\mathbf{x}, \mathbf{u})$ of $\mathscr{E}$, we obtain the following picture. Let $I = (i_1, \dots, i_{r+1})$ be an arbitrary sequence of integers $1 \le i_k \le n$ and $J = (j_1, \dots, j_s)$ an *ascending* sequence with $1 \le j_1 < j_2 < \cdots < j_s \le n$. Then we denote by $d\mathbf{x}^{(I)}$ the symmetric product $dx_{i_1} \cdots dx_{i_{r+1}}$ and by $d\mathbf{x}^{\langle J\rangle}$ the antisymmetric product $dx_{j_1} \wedge \cdots \wedge dx_{j_s}$. A basis of $E_s(T^*\mathscr{B}) \otimes \mathfrak{S}_{r+1}(T^*\mathscr{B}) \otimes V\mathscr{E}$ consists now of elements of the form $d\mathbf{x}^{\langle J\rangle} \otimes d\mathbf{x}^{(I)} \otimes \partial_{u_\alpha}$ and we obtain

$$\delta(d\mathbf{x}^{\langle J\rangle} \otimes d\mathbf{x}^{(I)} \otimes \partial_{u_\alpha}) = \sum_{k=1}^{r+1} \mathrm{sgn}(J, i_k) d\mathbf{x}^{\langle \mathrm{sort}(J \cup i_k)\rangle} \otimes d\mathbf{x}^{(I_k)} \otimes \partial_{u_\alpha} \,. \tag{5.6}$$

Here I_k denotes the sequence I without the element i_k. The sequence $\mathrm{sort}(J \cup i_k)$ is empty, if i_k already appears in J; otherwise i_k is sorted into place in the natural order. If t is the number of interchanges needed for this sorting, then $\mathrm{sgn}(J, i_k) = (-1)^t$.

Setting $\mathfrak{S}_i(T^*\mathscr{B}) = 0$ for $i < 0$, we may consider the δ-sequences

$$\begin{aligned} 0 \longrightarrow \mathfrak{S}_r(T^*\mathscr{B}) \otimes V\mathscr{E} \xrightarrow{\delta} T^*\mathscr{B} \otimes \mathfrak{S}_{r-1}(T^*\mathscr{B}) \otimes V\mathscr{E} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} E_s(T^*\mathscr{B}) \otimes \mathfrak{S}_{r-s}(T^*\mathscr{B}) \otimes V\mathscr{E} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} E_n(T^*\mathscr{B}) \otimes \mathfrak{S}_{r-n}(T^*\mathscr{B}) \otimes V\mathscr{E} \longrightarrow 0 \end{aligned} \tag{5.7}$$

where again $n = \dim \mathscr{B}$. The *formal Poincaré lemma* states that these sequences are exact for all $r \ge 0$.

Given the symbol $\mathscr{N}_q$ of a differential equation $\mathscr{R}_q \subseteq J_q\mathscr{E}$, we set $\mathscr{N}_i = 0$ for $i < 0$ and $\mathscr{N}_i = \mathfrak{S}_i(T^*\mathscr{B}) \otimes V\mathscr{E}$ for $0 \le i < q$. Then the δ-sequence (5.7) may be restricted to a sequence

$$0 \longrightarrow \mathscr{N}_{q+r} \xrightarrow{\delta} T^*\mathscr{B} \otimes \mathscr{N}_{q+r-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} E_n(T^*\mathscr{B}) \otimes \mathscr{N}_{q+r-n} \longrightarrow 0 \tag{5.8}$$

which is still a complex but in general no longer exact. Its (bigraded) cohomology is called the *Spencer cohomology* of the symbol $\mathscr{N}_q$. We denote by $H^{s,r}(\mathscr{N}_q)$ the cohomology group at $E_s(T^*\mathscr{B}) \otimes \mathscr{N}_r$.

Now we can give another homological characterization of an involutive symbol. As it is the only one of all our criteria for involution that does not require the choice of regular coordinates, it is often taken as definition of an involutive symbol (see, e.g., [18, 31, 65]).

Theorem 5.5. *The symbol $\mathscr{N}_q$ is involutive, if and only if its Spencer cohomology vanishes beyond degree q, i. e. $H^{s,r}(\mathscr{N}_q) = 0$ for all $r \geq q$.*

This will follow from Theorem 6.5 after which the easy proof will be given.

Remark 5.6. In Section 3.4 we introduced the symbol comodule N. Due to (2.19) it possesses non-trivial components of lower degree, whereas here we simply set $\mathscr{N}_i = \mathfrak{S}_i(T^*\mathscr{B}) \otimes V\mathscr{E}$ for $0 \leq i < q$. However, for the definition of an involutive symbol only the components of degree greater than or equal to q matters, so that this difference is of no importance for our purposes.

6. A New View of Spencer Cohomology

6.1. **Tor and Cotor.** We have seen that the Koszul resolution $K = A \otimes E[u_1, \ldots, u_n]$ is a "small model" of $B(A, \Bbbk)$ in the sense that there is the SDR (4.21). There is a dual result which we will now describe. First, we want to look at the bar construction another way.

For a given algebra A, we have $\bar{B}(A)$ as in Section 4.5. In fact, $\bar{B}(A)$ is a coalgebra with coproduct given by

$$\Delta[a_1|\cdots|a_n] = \sum_{i=0}^{n}[a_1|\cdots|a_i] \otimes [a_i|\cdots|a_n]$$

where the terms for $i = 0$ and $i = n$ are respectively, $[a_1|\cdots|a_n] \otimes [\,]$ and $[\,] \otimes [a_1|\cdots|a_n]$. It is well-known that the differential $\bar{\partial}$ in $B(\Bbbk, \Bbbk) \cong \bar{B}(A)$ is a *coderivation*. In fact, the coalgebra $\bar{B}(A)$ is a cofree coalgebra and as such any $\Bbbk$-linear map $\bar{B}(A) \longrightarrow \bar{A}$ can be coextended as a coderivation. The differential $\bar{\partial}$ is, in fact, the coextension as a coderivation of the map

$$\bar{A} \otimes \bar{A} \longrightarrow \bar{A}$$

given by multiplication. See [36, Section 2.2] for details.

Now define a degree minus one map $\bar{B}(A) \xrightarrow{\pi} A$ by

$$\pi([a]) = a, \quad \pi([a_1|\cdots|a_m] = 0, \text{ if } m \neq 1.$$

It is immediate that π is a twisting cochain and the twisted tensor product $(A \otimes \bar{B}(A), \bar{\partial}_\tau)$ is just the bar construction $B(A, \Bbbk)$. In fact, the Koszul complex is also a twisted tensor product complex. Giving the polynomial algebra trivial degrees (i.e. all elements are of degree zero), and zero differential, and giving elements of the exterior algebra degrees determined by $|u_i| = 1$ for all $i = 1 \ldots n$ and zero differential, we have that

$$E[u_1, \ldots, u_n] \xrightarrow{\kappa} \Bbbk[x_1, \ldots, x_n] \tag{6.1}$$

given by

$$\begin{aligned} \kappa(u_i) &= x_i, \qquad \text{for } i = 1, \ldots, n \\ \kappa(u_I) &= 0, \qquad \text{for } |I| > 1 \end{aligned} \tag{6.2}$$

is a twisting cochain. This is quite easy to see. First of all, since the differentials involved are zero, the twisting cochain condition reduces to $\kappa \cup \kappa = 0$. It is easy to see that

$$\Delta(u_I) = \sum_{J \subseteq I} \pm u_J u_{I-J}$$

from Section 2.11, so since any non zero term of $(\kappa \cup \kappa)(u_I)$ must be of the form $u_i \otimes u_j$ and for each such term the term $-u_j \otimes u_i$ must also occur. Thus since $x_i x_j = x_j x_i$, we must have $(\kappa \cup \kappa)(u_I) = 0$. The following proposition follows by an easy calculation.

Proposition 6.1. *Let A be the polynomial algebra and E be the exterior coalgebra and let $\kappa : E \longrightarrow A$ be the Koszul twisting cochain* (6.2)*. Let $K = (A \otimes E, d)$ be the Koszul resolution and let*

$$M \otimes A \xrightarrow{\mu} M$$

be an A-module. The differential d_M in the complex $M \otimes E$ which is suitable for computing $\mathrm{Tor}^A(M, \Bbbk)$ *is given by the following composite:*

$$\begin{tikzcd} M \otimes E \arrow[r, "d_M"] \arrow[d, "1 \otimes \Delta"'] & M \otimes E \\ M \otimes E \otimes E \arrow[r, "1 \otimes \kappa \otimes 1"'] & M \otimes A \otimes E \arrow[u, "\mu \otimes 1"'] \end{tikzcd}$$

Dually, one has the *loop algebra* construction [1, 23, 35, 38] for a coalgebra. Given a coalgebra C, let $\bar{C}$ be the kernel of the counit $\epsilon : C \longrightarrow \Bbbk$ and let $\bar{\Omega}(C)_n = \otimes^n \bar{C}$. Let $\widetilde{\Omega}(C) = \sum_{i=0}^{\infty} \bar{\Omega}(C)_n$. Elements of $\bar{\Omega}(C)_n$ will be written as $\langle c_1 | \cdots | c_n \rangle$. Note that we can think of $\widetilde{\Omega}(C)$ as the free algebra generated by $\bar{C}$ with the identity element given by $\langle\,\rangle$. Define $\widetilde{\delta} : \widetilde{\Omega}(C) \longrightarrow \widetilde{\Omega}(C)$ to be the unique derivation extending the map

$$\bar{C} \xrightarrow{\Delta} \bar{C} \otimes \bar{C}$$

given by the coproduct. The map $\iota : C \longrightarrow \widetilde{\Omega}(C)$ given by

$$\iota(c) = \langle c - \epsilon(c) \rangle$$

is in fact a twisting cochain as is easily verified. The twisted tensor product $(\widetilde{\Omega}(C) \otimes C, \widetilde{\delta}_\iota)$ is called the (one-sided) loop algebra construction.

Remark 6.2. Note that when C is of finite type, $\bar{B}(C^*)$ is the *graded* dual to $\widetilde{\Omega}(C)$. Also note that $\left(\widetilde{\Omega}(C) \otimes C, \widetilde{\delta}_\iota\right)$ is *not* in general suitable for computing Cotor (see, e.g., [38, pp. 15]). For that we would need $\prod_{i=0}^{\infty} \bar{\Omega}(C)_n$ (see [35,

§ 5] for a formal defiition of Cotor). In our case of interest, as will be seen, the smaller complex will be suitable for Cotor.

Given a left comodule N over C, the following composite map δ_N is a differential on $\widetilde{\Omega}(C) \otimes M$:

$$\begin{tikzcd} \widetilde{\Omega}(C) \otimes N \arrow[rr, "\delta_N"] \arrow[d, "1 \otimes \rho"'] & & \widetilde{\Omega}(C) \otimes N \\ \widetilde{\Omega}(C) \otimes C \otimes N \arrow[rr, "1 \otimes \iota \otimes 1"'] & & \widetilde{\Omega}(C) \otimes \widetilde{\Omega}(C) \otimes N \arrow[u, "m \otimes 1"'] \end{tikzcd}$$

As might be expected, (4.21) should have a dual. In fact, consider the dual of the Koszul twisting cochain κ given by (6.2). By Sects. 2.9.1 and 2.11.1, this is a map

$$\Bbbk[y_1, \ldots, y_n] \xrightarrow{\kappa^*} E[z_1, \ldots, z_n] \tag{6.3}$$

where the z_i are dual to the u_i and the y_i are dual to the x_i for $i = 1, \ldots, n$ and we consider $\Bbbk[y_1, \ldots, y_n]$ as a coalgebra and $E[z_1, \ldots, z_n]$ as an algebra. It is easy to see that κ^* is also a twisting cochain. Thus, we have a twisted tensor product

$$S = \big(E[z_1, \ldots, z_n] \otimes \Bbbk[y_1, \ldots, y_n], d_{\kappa^*}\big). \tag{6.4}$$

It is not difficult to work out the differential d_{κ^*} explicitly. We have

Proposition 6.3.

$$d_{\kappa^*}(u_I p) = \sum_{i=1}^{n} \operatorname{sgn}(I, i) u_{\operatorname{sort}(I \cup i)} \frac{\partial p}{\partial x_i} \tag{6.5}$$

where $I \cup i$ is zero if $i \in I$, $\operatorname{sort}(I \cup i)$ is $I \cup i$ with i sorted into place in the natural order, and $\operatorname{sgn}(I, i) = (-1)^s$ where s is the number of interchanges needed to sort $I \cup i$ when i is not in I.

The arguments leading up to (4.21) dualize completely to give the

Proposition 6.4. *Let S be the above complex, we have an SDR*

$$S \underset{f_S}{\overset{\nabla_S}{\rightleftarrows}} \Big(\big(\widetilde{\Omega}(C), \delta_E\big), \phi_S\Big).$$

For a comodule

$$N \xrightarrow{\rho} C \otimes N$$

over the polynomial coalgebra, give $E \otimes N$ the composite differential δ_N given by:

$$\begin{tikzcd} E \otimes N \arrow[r, "\delta_N"] \arrow[d, "1 \otimes \rho"'] & E \otimes N \\ E \otimes C \otimes N \arrow[r, "1 \otimes \kappa^* \otimes 1"'] & E \otimes E \otimes N \arrow[u, "m \otimes 1"'] \end{tikzcd}$$

Using Proposition 2.6 along with the observations of this section, and an observation of Gugenheim [35], we have the following.

Theorem 6.5. *Let C be the polynomial coalgebra and $N \subseteq C^m$ be a subcomodule. Let $A = C^*$ be the dual polynomial algebra and $M = A^m/N^\perp$. Let E denote the exterior bialgebra and $\kappa : E \longrightarrow A$ be the Koszul twisting cochain and κ^* its linear dual. A complex for computing $\mathrm{Tor}^A(M, \Bbbk)$ is given by the twisted tensor product*

$$(M \otimes E, d_\kappa) \tag{6.6}$$

and a complex for computing $\mathrm{Cotor}_C(\Bbbk, N)$ is given by

$$(E \otimes N, \delta_N) \tag{6.7}$$

Furthermore, there is an isomorphism

$$\mathrm{Cotor}_C(\Bbbk, N)^* \cong \mathrm{Tor}^A(A^m/N^\perp, \Bbbk). \tag{6.8}$$

Furthermore, $\mathrm{Cotor}_C(\Bbbk, N)$ is isomorphic to the cohomology of the Spencer complex [83].

Proof. The differential (6.5) is clearly 1-trivial [35, Section 6] and hence by Theorem 6.2 of that paper, the result on Cotor follows. Using the explicit formula for the differential, it is also clear that it is nothing more or less than the Spencer differential. □

At this point, we can easily give the proof of Proposition 3.13:

Proof. Theorems 5.1 and 5.2 show that a finitely generated comodule $N \subseteq C^m$ is involutive if and only if $\mathrm{Tor}^A(M, \Bbbk)$ vanishes in positive degrees where $M = A^m/N^\perp$ as usual. But it is well known that $\mathrm{Tor}^A(M, \Bbbk)$ is finite dimensional over $\Bbbk$ (a simple argument is given e.g. in [11]). Thus if we define the desuspension of a module by

$$s^{-1}(M)_n = M_{n+1}$$

we see that since Tor is finite dimensional over $\Bbbk$, it must be the case that $\mathrm{Tor}(s^{-r}(M), \Bbbk)$ is zero in positive degrees for some r where inductively, $s^{-r}(M) = s^{-1}(s^{-r+1}(M))$, but by Proposition 2.6, we have that $N^* \cong M$ so it is clear that desuspension corresponds exactly to prolongation. □

The proof of Theorem 5.5 is also an easy consequence of Theorems 5.1 and 5.2 and the above Theorem 6.5.

7. Perturbing Resolutions

In [49], it is shown that if I is an ideal of the polynomial algebra A and there is a given term order (cf. Section 2.1) on A and M is the ideal of leading terms of I, the Lyubeznik resolution of M over A can be "perturbed" into a resolution of I over A. An explicit algorithm for doing this was given and is based on *homological perturbation theory* (described below) and Gröbner basis theory. While we do not need the full thrust of [49], we briefly describe the results in a section below since we will present a corollary and we also want to elaborate on an algorithm for calculating minimal resolutions which was used in that paper without discussion.

7.1. **The Perturbation Lemma.** Given an SDR (4.13)

$$X \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, \phi)$$

and, in addition, a second differential d'_Y on Y, let $t = d'_Y - d_Y$. The *perturbation lemma*, [3, 10, 35, 54] states that if we set $t_n = (t\phi)^{n-1}t,\ n \geq 1$ and if we, for each n, define new maps on X,

$$\begin{aligned} \partial_n &= d + f(t_1 + t_2 + \cdots + t_{n-1})\nabla \\ \nabla_n &= \nabla + \phi(t_1 + t_2 + \cdots + t_{n-1})\nabla, \end{aligned}$$

and on Y:

$$\begin{aligned} f_n &= f + f(t_1 + t_2 + \cdots + t_{n-1})\phi \\ \phi_n &= \phi + \phi(t_1 + t_2 + \cdots + t_{n-1})\phi, \end{aligned}$$

then in the limits, provided they exist, we have new SDR data

$$(X, \partial_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} ((Y, d'_Y), \phi_\infty)\,.$$

Remark 7.1. The difference t of the differentials above is called the initiator in [3]. And the situation above is called a transference problem. Examples and more information can be found in [3, 4, 10, 35, 36, 37, 42, 43, 48, 49, 54, 55, 56, 57, 58, 60].

7.2. **Perturbing Resolutions of Monomial Modules.** Consider now the Lyubeznik resolution L (4.4). It was shown in [49] that the contracting homotopy ψ defined by (4.6) satisfies $\psi(L) \subseteq L$, so that ψ is also a homotopy for L. In order to get an SDR of A/N and L, we need a normal form as above. As in [49], by defining $\epsilon(a \otimes 1) = a + N$ and requiring it to vanish otherwise and defining $\sigma : A/N \longrightarrow L$ by $\sigma(a + N) = \mathrm{rem}_N(a) \otimes 1$ we obtain an SDR

$$N \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} (L, \psi) \tag{7.1}$$

It was noted in [49] that $X = L$ (the Lyubeznik resolution) and $Y = B(\Bbbk, M)$ satisfy the hypotheses of Lemma 4.5. Thus we have an explicit SDR

$$L \underset{f_L}{\overset{\nabla_L}{\rightleftarrows}} (B(\Bbbk, M), \phi_L)\,. \tag{7.2}$$

All this brings us to the main theorem of [49].

Theorem 7.2. *Let $A = \Bbbk[x_1, \dots, x_n]$ be the polynomial algebra and $I \subseteq A$ be an ideal. Let G be a Gröbner basis for I with respect to some term order. Let $N = \mathrm{LT}(I)$ be the ideal of leading terms of elements of I, so that N is generated by $\{\mathrm{LT}(g) \mid g \in G\}$. Gröbner basis theory gives that as vector spaces over $\Bbbk$,*

$$A/N \cong A/I.$$

Thus, $B(\Bbbk, A/N) \cong B(\Bbbk, A/I)$ as vector spaces (i.e. ignoring differentials). Using this isomorphism and the SDR (7.2)*, we obtain a transference problem. The perturbation formulae presented in Section* 7.1 *converge in this case and we therefore obtain a relatively free resolution (L, d_∞) of A/M over A of the form*

$$d_\infty = d + P$$

where d is the ordinary Lyubeznik differential and P is an explicit perturbation. There is furthermore an explicit SDR

$$(L, d_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} (B(\Bbbk, A/N), \phi_\infty)\,. \tag{7.3}$$

We now have the following corollary.

Corollary 7.3. *Let Φ be the composite chain map*

$$(\bar{L}, \bar{d}_\infty) \xrightarrow{\nabla_\infty} \bar{B}(A) \otimes A/N \xrightarrow{\tau} A/N \otimes \bar{B}(A) \xrightarrow{\bar{f}_K} A/N \otimes \bar{K}$$

where, as usual, $\bar{L} = \Bbbk \otimes_A L$ etc. and the second map is determined by (4.1) *while the third map is determined by* (4.21)*. Then Φ induces an isomorphism in homology.*

Remark 7.4. When the Taylor resolution involving the monomial ideal $I = (m_1, \dots, m_k)$ is *minimal*, it is the same as the Lyubeznik resolution and in that case, Fröberg [27] defined an explicit map ρ from $\bar{T}$ to the Koszul complex $M \otimes \bar{K}$ as follows. When T is minimal, the monomial generators $m_i = x_1^{\alpha_{i,1}} \dots x_n^{\alpha_{i,n}}$ can be arranged so that if $i \neq j$, $\alpha_{j,j} > \alpha_{i,j}$. With the monomials indexed this way, the map is given by

$$\rho(u_I) = \frac{m_I}{x_{i_1} \dots x_{i_r}} w_{i_1} \dots w_{i_r}$$

where $I = (i_1, \dots, i_r)$ and the w_j are the generators of the Koszul complex. The map Φ generalizes this chain equivalence to all cases.

7.3. **Finitely Presented Modules.** Consider modules M that are finitely presented over the polynomial algebra A. Thus, $M = A^r/I$ for some finitely generated submodule $I \subseteq A^r$. The notion of Gröbner bases extends to this case, i.e. one has the notion of a Gröbner basis for M with respect to a monomial order and there is a normal form algorithm with respect to a given monomial order. A good reference for all of this is [44].

A submodule $I \subseteq A^r$ is said to be a monomial submodule, if it has a basis B such that every element of B is of the form $(m_1, \ldots, m_r)$ where $m_i \in A$ is a monomial for all $i = 1, \ldots, r$. Fixing a given monomial order, every element of $m \in I$ has a leading term $\mathrm{LT}(m)$ just as in the case $r = 1$. We let $\mathrm{LT}(I) = \{\mathrm{LT}(m) \mid m \in I\}$. It is clear that $\mathrm{LT}(I)$ is a monomial submodule for any submodule I.

Note that for $M = A^r/I$, the resolution $B(A, M)$ is exactly as defined in Section 4.5. The resolutions K and $B(A, \Bbbk)$ are also exactly as defined in Sects. 4.2 and 4.5, so these may be used to compute $\mathrm{Tor}^A(\Bbbk, M)$ and $\mathrm{Tor}^A(M, \Bbbk)$ for any finitely generated module M. The Taylor and Lyubeznik resolutions can be applied coordinate-wise in this case. With this said, all of the results of this Section apply verbatim.

7.4. **Minimal Resolutions of Monomial Modules.** If one is only interested in computing $\mathrm{Tor}^A(M, \Bbbk)$, this algorithm may seem a bit tautological, but there are reasons that one might want actual resolutions in general, of course.

We need the notion of a *homology decomposition* [41]. An explicit algorithm for computing a homology decomposition of a chain complex X of finite type over $\Bbbk$ was given in [49, Section 7]. A homology decomposition of X is a direct sum decomposition of the form

$$X = K \oplus B \oplus H$$

where H is isomorphic to the homology of X, $B = \mathrm{im}\,(d)$ is the subspace of boundries and furthermore, there are explicit bases for K, B, and H for which the differential d is locally diagonal. Using the basis for H, we identify it with the homology. It is pointed out in [49, Section 7.2] that such a homology decomposition gives rise to an explicit SDR

$$H \underset{f}{\overset{\iota}{\rightleftarrows}} (X, \phi)$$

where ι is the inclusion and ϕ is essentially the inverse to d locally (where it is non-zero).

Note that there are very efficient algorithms for computing homology based on Smith normal form [19, 45]. Using the system described in [47], the first author implemented the homology decomposition of a chain complex and these programs were used for all the computations in Section 8.

If $X \longrightarrow M \longrightarrow 0$ is a minimal resolution then the complex $X \otimes_A \Bbbk$ for computing $\mathrm{Tor}^A(\Bbbk, M)$ has zero differential, i.e. $\mathrm{Tor}^A(\Bbbk, M) \cong \Bbbk \otimes_A X$. So suppose that the Lyubeznik (or some other) resolution is such that the homology decomposition of $\bar{L} = \Bbbk \otimes_A L$ can be efficiently computed. We then consider the corresponding homology SDR

$$H \underset{\bar{f}}{\overset{\bar{\iota}}{\rightleftarrows}} (\bar{L}, \bar{\phi}) .$$

given by a homology decomposition of $\bar{L}$. We would like to set up a transference problem (Remark 7.1) and "transfer" the differential from $L = A \otimes H$ to $A \otimes \bar{L}$. But it is clear how to obtain an initiator for this and we have the

Algorithm 7.5. Let L be the Lyubeznik resolution and

$$H \underset{\bar{f}}{\overset{\bar{\iota}}{\rightleftarrows}} (\bar{L}, \bar{\phi})$$

the corresponding homology SDR. Consider the tensor product SDR

$$A \otimes H \underset{f}{\overset{\iota}{\rightleftarrows}} (A \otimes \bar{L}, \phi) \tag{7.4}$$

where $\iota = 1_A \otimes \bar{\iota}$, $f = 1_A \otimes \bar{f}$, $\phi = 1_A \otimes \bar{\phi}$, and $A \otimes \bar{L}$ has differential $1_a \otimes \bar{d}$.

Let the initiator be $t = d - (1 \otimes \bar{d})$. Note that by the definition of the Lyubeznik differential, $\bar{d}(x)$, for $x \in \bar{L}$, consists of those terms of $d(x)$ which have constant coefficients. Thus, $t(x)$ will consist of those terms of $d(x)$ having non-trivial polynomial coefficients. Consider the perturbed differential given by the perturbation lemma:

$$d_\infty = ft + f(t\phi t) + \cdots + f\big((t\phi)^n t\big) + \dots \tag{7.5}$$

If d_∞ converges, we obtain the minimal resolution of M over A given by

$$(A \otimes H, d_\infty).$$

Remark 7.6. At this time, we have not been able to show that, in general, $t\phi$ is nilpotent in each degree, but we have examined many examples using a computer. In each example, the following is true. For each u_I, there is a non-negative integer n_I such that $(t\phi)^{n_I}(u_I)$ lands in $A \otimes (K \oplus H)$ (see the second paragraph of Section 7.4). It follows that $\phi\big((t\phi)^{n_I}(u_I)\big) = 0$ and so $t\phi$ is nilpotent in this degree. Examples will be given in Section 8.

8. Computations

8.1. **Example 1.** Consider the example $A = \Bbbk[x, y, z]$, $I = (x^2z^3, x^3z^2, xyz, y^2)$, and $M = A/I$ from [5, Example 3.4]. The Lyubeznik resolution L is the same as the Taylor resolution in this case and is given by

$$\begin{aligned} d(u_1) &= x^2z^3 \\ d(u_2) &= x^3z^2 \\ d(u_3) &= xyz \\ d(u_4) &= y^2 \end{aligned}$$

$$\begin{aligned} d(u_1u_2) &= -xu_1 + zu_2 \\ d(u_1u_3) &= -yu_1 + xz^2u_3 \\ d(u_1u_4) &= -y^2u_1 + x^2z^3u_4 \\ d(u_2u_3) &= -yu_2 + x^2zu_3 \\ d(u_2u_4) &= -y^2u_2 + x^3z^2u_4 \\ d(u_3u_4) &= -yu_3 + xzu_4 \end{aligned}$$

$$\begin{aligned} d(u_1u_2u_3) &= yu_1u_2 - xu_1u_3 + zu_2u_3 \\ d(u_1u_2u_4) &= y^2u_1u_2 - xu_1u_4 + zu_2u_4 \\ d(u_1u_3u_4) &= yu_1u_3 - u_1u_4 + xz^2u_3u_4 \\ d(u_2u_3u_4) &= yu_2u_3 - u_2u_4 + x^2zu_3u_4 \end{aligned}$$

$$d(u_1u_2u_3u_4) = -yu_1u_2u_3 + u_1u_2u_4 - xu_1u_3u_4 + zu_2u_3u_4$$

From this it is easy to see that

$$\begin{aligned} \mathrm{Tor}_1^A(M, \Bbbk) &= \Bbbk \langle u_1, u_2, u_3, u_4 \rangle, \\ \mathrm{Tor}_2^A(M, \Bbbk) &= \Bbbk \langle u_1u_2, u_1u_3, u_2u_3, u_3u_4 \rangle, \\ \mathrm{Tor}_3^A(M, \Bbbk) &= \Bbbk \langle u_1u_2u_3 \rangle. \end{aligned}$$

Using the algorithm from the last section, we consider a homology decomposition of $\bar{L}$ to obtain an SDR

$$H \underset{\bar{f}}{\overset{\bar{\nabla}}{\rightleftarrows}} (\bar{L}, \bar{\phi})$$

where $H = \mathrm{Tor}^A(M, \Bbbk) = H(\bar{L})$ and corresponding SDR

$$A \otimes H \underset{f}{\overset{\nabla}{\rightleftarrows}} (A \otimes \bar{L}, \phi)$$

where $\nabla = 1_A \otimes \bar{\nabla}$, $f = 1_A \otimes \bar{f}$, and $L = A \otimes \bar{A}$ has differential $1_A \otimes \bar{d}$. Taking the initiator to be $t = d = \bar{d}$, we have a transference problem.

In this case, the homology decomposition is so simple that it can be seen by inspection. We have

$$\begin{aligned}\bar{L}_1 &= H_1\\ \bar{L}_2 &= B_2 \oplus H_2\\ \bar{L}_3 &= K_3 \oplus B_3 \oplus H_3\\ \bar{L}_4 &= K_4\end{aligned}$$

where

$$\begin{aligned}H_1 &= \Bbbk\langle u_1, u_2, u_3, u_4\rangle\,,\ B_2 = \Bbbk\langle -u_1u_4, -u_2u_4\rangle\,,\\ H_2 &= \Bbbk\langle u_1u_2, u_1u_3, u_2u_3, u_3u_4\rangle\,,\ K_3 = \Bbbk\langle u_1u_2u_3, u_2u_3u_4\rangle\,,\\ B_3 &= \Bbbk\langle u_1u_2u_4\rangle\,,\ H_3 = \Bbbk\langle u_1u_2u_3\rangle\,,\ K_4 = \langle u_1u_2u_3u_4\rangle\,.\end{aligned}$$

From this it follows that

$$\bar{\phi}(u_1u_2u_4) = u_1u_2u_3u_4$$

and $\bar{\phi}$ vanishes on all other elements. Thus, the perturbation is zero in this case and we have the minimal resolution given by

$$\begin{aligned}d(u_1) &= x^2z^3\\ d(u_2) &= x^3z^2\\ d(u_3) &= xyz\\ d(u_4) &= y^2\\ d(u_1u_2) &= -xu_1 + zu_2\\ d(u_1u_3) &= -yu_1 + xz^2u_3\\ d(u_2u_3) &= -yu_2 + x^2zu_3\\ d(u_3u_4) &= -yu_3 + xzu_4\\ d(u_1u_2u_3) &= yu_1u_2 - xu_1u_3 + zu_2u_3.\end{aligned}$$

Note that this also gives $b_{2,1} = 1$, $b_{3,1} = 1$, $b_{5,1} = 2$, $b_{4,2} = 1$, $b_{6,2} = 3$, and $b_{7,3} = 1$ where $b_{i,j} = \dim_{\Bbbk}(Tor^A_{i,j}(M, \Bbbk)$ where the bigrading is the one given in Section 4.2.

8.2. **Example 2.** Consider the ideal

$$I = (x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3)$$

in $A = \Bbbk[x, y, z]$. The Taylor resolution has dimension $2^{10} = 1024$ and so is not very good for computations. The Lyubeznik resolution however has dimension 207 and contains "forms" only up to degree 7 (there is only one 7-form, viz. $u_1u_2u_4u_7u_8u_9u_{10}$. Using the algorithm given in [49, Section 7] (*and* a computer), it is quite easy to compute a homology decomposition of $\bar{L}$. We find that

$$K_1 = 0,\ B_1 = 0,\ H_1 = \Bbbk^{10},$$

$$K_2 = 0,\ B_2 = \Bbbk^{20},\ H_2 = \Bbbk^{15},$$

$$K_3 = \Bbbk^{20},\ B_3 = \Bbbk^{35},\ H_3 = \Bbbk^{6},$$

$$K_4 = \Bbbk^{35},\ B_4 = \Bbbk^{24},\ H_4 = 0,$$
$$K_5 = \Bbbk^{24},\ B_5 = \Bbbk^{8},\ H_5 = 0,$$
$$K_6 = \Bbbk^{8},\ B_6 = \Bbbk,\ H_6 = 0,$$
$$K_7 = \Bbbk,\ B_7 = 0,\ H_7 = 0.$$

Since H_i for $i \geq 4$ are all zero, we need only consider the map $\phi_2 : L_2 \longrightarrow L_3$. We have that

$$H_1 = \Bbbk \langle u_1, \dots, u_{10} \rangle$$

and the differential is as in L. We also have that

$$H_1 = \Bbbk \langle u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_2u_5, u_3u_6, u_4u_5, u_4u_8, u_5u_6, u_5u_9, u_6u_{10}, u_7u_8, u_8u_9, u_9u_{10} \rangle$$

and the differential is as in L. Finally,

$$H_3 = \Bbbk \langle u_1u_2u_3, u_2u_3u_6 - u_2u_5u_6, u_2u_4u_5, u_4u_5u_9, u_4u_8u_9, u_4u_7u_8, u_5u_6u_{10}, -u_5u_9u_{10} \rangle$$

and while ϕ_2 is generally non-zero, it vanishes on each of $t(x)$ for $x \in H_3$. In fact, each $t(x)$ for $x \in H_3$ is a linear combination of elements in H_2, so the perturbation vanishes once more and the differential is as it is in L.

8.3. **Example 3.** Consider the ideal $I = (x^4y, x^3yz, xy^3, xy^2z, xyz^2, y^3z)$ in $A = \Bbbk[x, y, z]$. The Lyubeznik resolution has dimension 39 in this case. The non-zero differentials in the reduced complex are

$$\begin{aligned}
\bar{d}(u_1u_2u_4) &= -u_1u_4 \\
\bar{d}(u_1u_2u_5) &= -u_1u_5 \\
\bar{d}(u_1u_2u_6) &= -u_1u_6 \\
\bar{d}(u_1u_3u_6) &= -u_1u_6 \\
\bar{d}(u_2u_3u_4) &= u_2u_3 \\
\bar{d}(u_2u_3u_6) &= u_2u_3 - u_2u_6 \\
\bar{d}(u_3u_4u_5) &= -u_3u_5
\end{aligned}$$

$$\begin{aligned}
\bar{d}(u_1u_2u_3u_4) &= -u_1u_2u_3 - u_1u_3u_4 \\
\bar{d}(u_1u_2u_3u_5) &= -u_1u_3u_5 \\
\bar{d}(u_1u_2u_3u_6) &= -u_1u_2u_3 + u_1u_2u_6 - u_1u_3u_6 \\
\bar{d}(u_1u_2u_4u_5) &= -u_1u_4u_5 \\
\bar{d}(u_1u_3u_4u_5) &= u_1u_3u_5 \\
\bar{d}(u_2u_3u_4u_5) &= u_2u_3u_5
\end{aligned}$$

$$\bar{d}(u_1u_2u_3u_4u_5) = -\, u_1u_2u_3u_5 - u_1u_3u_4u_5$$

Again, a straightforward calculation gives the homology decomposition

$$K_1 = 0,$$
$$B_1 = 0,$$
$$H_1 = \Bbbk \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle ,$$

$$K_2 = 0,$$
$$B_2 = \Bbbk \langle -u_1u_4, -u_1u_5, -u_1u_6, u_2u_3, u_2u_3 - u_2u_6, -u_3u_5 \rangle ,$$
$$H_2 = \Bbbk \langle u_1u_2, u_1u_3, u_2u_4, u_2u_5, u_3u_4, u_3u_6, u_4u_5 \rangle ,$$

$$K_3 = \Bbbk \langle u_1u_2u_4, u_1u_2u_5, u_1u_3u_6, u_2u_3u_4, u_2u_3u_6, u_3u_4u_5 \rangle ,$$
$$B_3 = \Bbbk \langle -u_1u_2u_3 - u_1u_3u_4, -u_1u_3u_5, -u_1u_2u_3 + u_1u_2u_6 - u_1u_3u_6, -u_1u_4u_5, u_2u_3u_5 \rangle ,$$
$$H_3 = \Bbbk \langle u_1u_2u_3, u_2u_4u_5 \rangle ,$$

$$K_4 = \Bbbk \langle -u_1u_3u_4u_5, u_1u_2u_3u_6, u_1u_2u_4u_5, u_2u_3u_4u_5 \rangle ,$$
$$B_4 = \Bbbk \langle -u_1u_2u_3u_5 - u_1u_2u_3u_4 \rangle ,$$
$$H_4 = 0,$$
$$K_5 = \Bbbk \langle u_1u_2u_3u_4u_5 \rangle ,$$
$$B_5 = 0,$$
$$H_5 = 0.$$

The splitting homotopy is zero in degrees 0 and 1. Thus, the differential in degrees 1 and 2 are as they are in L. In degree 3, we have

$$\begin{aligned} t(u_1u_2u_3) &= y^2u_1u_2 - zu_1u_3 + xu_2u_3 \\ t(u_2u_4u_5) &= zu_2u_4 - yu_2u_5 + x^2u_4u_5. \end{aligned}$$

Note that since u_2u_3 is not in H_2, so we expect some non-trivial action using our method in this case. In fact, our calculations show that ϕ vanishes on all terms involved in the right hand sides above except in one case, viz.

$$\phi(u_2u_3) = u_2u_3u_4.$$

We thus consider

$$\begin{aligned} \alpha &= t\phi(y^2u_1u_2 - zu_1u_3 + xu_2u_3) \\ &= xt(u_2u_3u_4) \\ &= x(-yu_2u_4 + x^2u_3u_4) \\ &= -xyu_2u_4 + x^3u_3u_4. \end{aligned}$$

We now note that ϕ vanishes on both u_2u_4 and u_3u_4, so the perturbation converges, i.e. $\phi t\phi(y^2u_1u_2 - zu_1u_3 + xu_2u_3) = 0$. We now note that

$$(\bar{d}\phi + \phi\bar{d})(u_2u_3) = \bar{d}(u_2u_3u_4) = u_2u_3$$

and so $f(u_2u_3) = 0$. Thus, we have

$$\begin{aligned} d_\infty(u_1u_2u_3) &= f(y^2u_1u_2 - zu_1u_3 + xu_2u_3 + xyu_2u_4 - x^3u_3u_4) \\ &= y^2u_1u_2 - zu_1u_3 + xyu_2u_4 - x^3u_3u_4. \end{aligned}$$

Finally, we note that ϕ vanishes on $t(u_2u_4u_5)$ and so we have derived the minimal resolution of M over A given by

$$\begin{aligned} d_\infty(u_1) &= x^4y \\ d_\infty(u_2) &= x^3yz \\ d_\infty(u_3) &= xy^3 \\ d_\infty(u_4) &= xy^2z \\ d_\infty(u_5) &= xyz^2 \\ d_\infty(u_6) &= y^3z \end{aligned}$$

$$\begin{aligned} d_\infty(u_1u_2) &= -zu_1 + xu_2 \\ d_\infty(u_1u_3) &= -y^2u_1 + x^3u_3 \\ d_\infty(u_2u_4) &= -yu_2 + x^2u_4 \\ d_\infty(u_2u_5) &= -zu_2 + x^2u_5 \\ d_\infty(u_3u_4) &= -zu_3 + yu_4 \\ d_\infty(u_3u_6) &= -zu_3 + xu_6 \\ d_\infty(u_4u_5) &= -zu_4 + yu_5 \end{aligned}$$

$$\begin{aligned} d_\infty(u_1u_2u_3) &= y^2u_1u_2 - zu_1u_3 + xyu_2u_4 - x^3u_3u_4 \\ d_\infty(u_2u_4u_5) &= zu_2u_4 - yu_2u_5 + x^2u_4u_5 \end{aligned}$$

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