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A MAGNUS–WITT TYPE ISOMORPHISM FOR NON-FREE GROUPS

G. ELLIS

Dedicated to Professor Hvedri Inassaridze

Abstract. We use the theory of nonabelian derived functors to prove that certain Baer invariants of a group G are torsion when G has torsion second integral homology. We use this result to show that if such a group has torsion-free abelianisation then the Lie algebra formed from the quotients of the lower central series of G is isomorphic to the free Lie algebra on G_{ab}. We end the paper with some related remarks about precrossed modules and partial Lie algebras.

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1. Introduction

The lower central series of a group G is defined by setting $\gamma_1 G = G$ and $\gamma_{n+1}G = [\gamma_n G, G]$ for $n \geqslant 1$. The quotients of this series form a Lie algebra (over $\mathbf{Z}$)

$$\Gamma G = \bigoplus_{n\geqslant 1} \gamma_n G/\gamma_{n+1}G,$$

where the Lie bracket is induced by the commutator maps $\gamma_m G \times \gamma_n G \to \gamma_{m+n}G, (x,y) \mapsto [x,y] = xyx^{-1}y^{-1}$.

Also associated to the group G is the free Lie algebra $L(G_{ab})$ on the abelian group $G_{ab} = G/\gamma_2 G$. This is defined, up to isomorphism, by the following universal property: there is an additive homomorphism $\iota\colon G_{ab} \to L(G_{ab})$ such that, for any other Lie algebra M and additive homomorphism $\alpha\colon G_{ab} \to M$, there exists a unique Lie homomorphism $\overline{\alpha}\colon L(G_{ab}) \to M$ making the triangle

$$\begin{tikzcd} G_{ab} \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & L(G_{ab}) \arrow[d, "\overline{\alpha}"] \\ & M \end{tikzcd}$$

commute. The existence of $L(G_{ab})$ follows from general considerations, and the universal property implies that ι must be injective.

The canonical inclusion $\phi\colon G_{ab} \to \Gamma G$ induces a (surjective) homomorphism $\overline{\phi}\colon L(G_{ab}) \twoheadrightarrow \Gamma G$. A result of Magnus [10] and Witt [13] asserts that $\overline{\phi}$ is an isomorphism when G is a free group. In this paper we shall prove the following.

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Theorem 1. *If the second integral homology $H_2(G,\mathbf{Z})$ is a torsion group then $\ker\overline{\phi}$ lies in the torsion subgroup of the additive group of $L(G_{ab})$. In particular $\overline{\phi}\colon L(G_{ab}) \twoheadrightarrow \Gamma G$ is an isomorphism under the additional assumption that G_{ab} is torsion free.*

A variant of this theorem was proved by Strebel [12]. His proof involves the Poincaré-Birkhoff-Witt theorem and techniques from classical homological algebra. Although Theorem 1 could be deduced from Strebel's work, we shall give an alternative proof involving the theory of nonabelian derived functors developed by Hvedri Inassaridze [6] and others. We shall derive the theorem from the following result on the Baer invariants

$$M^{(n)}(G) = \frac{R\cap\gamma_{n+1}F}{\gamma_{n+1}(R,F)}\,, \quad n\geqslant 1,$$

which are defined in terms of a free presentation $G\cong F/R$. Here F is a free group with normal subgroup R, $\gamma_1(R,F)=R$ and $\gamma_{n+1}(R,F)=[\gamma_n(R,F),F]$ for $n\geqslant 1$. The invariants $M^{(n)}(G)$ are abelian groups and are independent of the choice of free presentation (*cf.* [9]). Note that Hopf's formula states $M^{(1)}(G)\cong H_2(G,\mathbf{Z})$.

Theorem 2. *If $M^{(1)}(G)$ is a torsion group then $M^{(n)}(G)$ is a torsion group for all $n\geqslant 1$.*

We end the paper with some related remarks about the partial Lie algebra formed from the lower Peiffer central series of a precrossed module.

2. The Proof of Theorem 1

For all $n\geqslant 1$ a normal inclusion $N\leqslant G$ gives rise to a natural five term exact sequence (*cf.* [8])

$$M^{(n)}(G)\to M^{(n)}(G/N)\to N/\gamma_{n+1}(N,G)\to G/\gamma_{n+1}G\to G/N\gamma_{n+1}G\to 1\,.$$

On taking $N=\gamma_2 G$ we obtain the exact sequence

$$M^{(n)}(G)\xrightarrow{\lambda} M^{(n)}(G_{ab})\to \gamma_{n+1}G/\gamma_{n+2}G\to 0\,. \tag{1}$$

It is explained in [3] that there is a natural isomorphism of abelian groups

$$L(G_{ab})\cong\bigoplus_{n\geqslant 1} M^{(n)}(G_{ab})\,. \tag{2}$$

When $G=F$ is a free group this isomorphism is precisely the Magnus-Witt isomorphism. The isomorphism is obtained for arbitrary G by describing $L((F/R)_{ab})$ as a quotient of $L(F_{ab})$. We can combine (1) and (2) to obtain an exact sequence

$$\bigoplus_{n\geqslant 1} M^{(n)}(G)\xrightarrow{\oplus\lambda} L(G_{ab})\xrightarrow{\overline{\phi}}\Gamma G\to 1.$$

The first assertion of Theorem 1 follows from this sequence and Theorem 2. The second assertion then follows from the following claim applied to $A=G_{ab}$.

Claim 1. If A is a torsion-free abelian group, then the additive group of the free Lie algebra $L(A)$ is torsion-free.

To prove the claim suppose that A is a torsion-free abelian group. Let $T(A) = \oplus_{n\geqslant 1}(\otimes^n A)$ denote the tensor algebra on A. Thus $T(A)$ is an associative algebra (over $\mathbf{Z}$) formed by taking the direct sum of iterated tensor products of the abelian group A. Since the tensor product of torsion-free groups is torsion-free, the abelian group underlying $T(A)$ is torsion-free. The Poincaré-Birkhoff-Witt theorem for $\mathbf{Z}$-modules [5] implies that the free Lie algebra $L(A)$ imbeds into $T(A)$. This proves the claim.

3. The Proof of Theorem 2

Recall that a free simplicial resolution of G consists of a simplicial group $F_* = \{F_m\}_{m\geqslant 0}$ with $\pi_0(F_*) \cong G$, $\pi_m(F_*) = 0$ for $m \geqslant 1$, and with the F_m free groups whose bases are preserved by the boundary and degeneracy morphisms $d_i^m\colon F_m \to F_{m-1}$, $s_i^m\colon F_m \to F_{m+1}$, $0 \leqslant i \leqslant m$. In this context the homotopy groups of F_* can be defined as $\pi_m(F_*) = \ker(\partial_m)/\mathrm{im}(\partial_{m+1})$ where $\partial_m\colon \cap_{i=1}^m \ker d_i^m \longrightarrow \cap_{i=1}^{m-1} \ker d_i^{m-1}$ is the restriction of $d_0^m\colon F_m \to F_{m-1}$. In other words, $\pi_m(F_*)$ is the mth homology group of the Moore complex

$$N(F_*) : \cdots N(F_*)_m \xrightarrow{\partial_m} N(F_*)_{m-1} \to \cdots \to N(F_*)_0 \to 1$$

with $N(F_*)_m = \cap_{i=1}^m \ker d_i^m$.

Given a functor $T\colon$ (Groups) $\to$ (Groups), we define its left derived functors as

$$L_m^T(G) = \pi_m(T(F_*)),\ m \geqslant 0.$$

The groups $L_m^T(G)$ are independent of the choice of free simplicial resolution. For more details see for instance [6].

We shall consider the derived functors of the functor

$$\tau_n\colon \text{(Groups)} \to \text{(Groups)},\ G \mapsto G/\gamma_{n+1}G$$

for $n \geqslant 1$. In [4] it was observed that there are natural isomorphisms

$$L_0^{\tau_n}(G) \cong \tau_n G = G/\gamma_{n+1}G\,,$$
$$L_1^{\tau_n}(G) \cong M^{(n)}(G)\,.$$

Let F_* be a free simplicial resolution of G. The short exact sequence of simplicial groups

$$1 \to \gamma_{n+1}(F_*) \to F_* \to \tau_n(F_*) \to 1$$

gives rise to a long exact sequence of homotopy groups part of which is

$$\pi_1(F_*) \to \pi_1(\tau_n(F_*)) \to \pi_0(\gamma_{n+1}(F_*)) \to \pi_0(F_*) \to \pi_0(\tau_n(F_*)) \to 1\,.$$

Using the various isomorphisms we can rewrite this as an exact sequence

$$0 \to M^{(n)}(G) \to \pi_0(\gamma_{n+1}(F_*)) \to G \to \tau_n(G) \to 1$$

from which we obtain the natural isomorphism

$$M^{(n)}(G) \cong \pi_0(\gamma_{n+1}(F_*)) \tag{3}$$

The short exact sequence of simplicial groups

$$1 \to \gamma_{n+2}(F_*) \to \gamma_{n+1}(F_*) \to \gamma_{n+1}(F_*)/\gamma_{n+2}(F_*) \to 1$$

and (3) yield the exact sequence

$$\pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*)) \to M^{(n+1)}(G) \to M^{(n)}(G)\,. \tag{4}$$

Theorem 2 follows by induction from (4) and the following.

Claim 2. If $M^{(1)}(G)$ is a torsion group, then so too is $\pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*))$ for $n \geqslant 1$.

To prove the claim we introduce the functors

$$\Gamma_n\colon (\text{Groups}) \to (\text{Groups}),\ G \mapsto \gamma_n G/\gamma_{n+1}G,\ n \geqslant 1.$$

Thus, as an abelian group, ΓG is equal to $\bigoplus_{n\geqslant 1}\Gamma_n G$. Define $L_n(G_{ab})$ to be the preimage of $\Gamma_n G$ under the canonical surjection $L(G_{ab}) \twoheadrightarrow \Gamma G$. The result of Magnus and Witt gives a natural isomorphism $L_n(F_{ab}) \cong \Gamma_n F$ for any free group F.

An abelian group is torsion if and only if it is trivial when tensored by the rationals. Tensoring the Lie algebra $L(F_{ab})$ by the rationals yields a Lie algebra $L(F_{ab}) \otimes \mathbf{Q}$ over $\mathbf{Q}$. Let $L(F_{ab} \otimes \mathbf{Q})$ denote the free lie algebra (over $\mathbf{Q}$) on the vector space $F_{ab} \otimes \mathbf{Q}$. By verifying the appropriate universal property we see that there is a natural isomorphism $L(F_{ab}) \otimes \mathbf{Q} \cong L(F_{ab} \otimes \mathbf{Q})$. Let $L_n(F_{ab} \otimes \mathbf{Q})$ denote the vector subspace of $L(F_{ab} \otimes \mathbf{Q})$ generated by the canonical image of $L_n(F_{ab})$. So $L_n(F_{ab} \otimes \mathbf{Q}) \cong L_n(F_{ab}) \otimes \mathbf{Q}$.

We have

$$\begin{aligned}
\pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*)) \otimes \mathbf{Q} &= \pi_1(\,\Gamma_{n+1}(F_*)\,) \otimes \mathbf{Q} \\
&\cong \pi_1(\,L_{n+1}((F_*)_{ab})\,) \otimes \mathbf{Q} \\
&= H_1(\,N(\,L_{n+1}((F_*)_{ab}))\,)\,) \otimes \mathbf{Q} \\
&\cong H_1(\,N(\,L_{n+1}((F_*)_{ab})\,) \otimes \mathbf{Q}\,) \qquad (5) \\
&\cong H_1(\,N(\,L_{n+1}((F_*)_{ab}) \otimes \mathbf{Q}\,)\,) \qquad (6) \\
&= \pi_1(\,L_{n+1}((F_*)_{ab}) \otimes \mathbf{Q}\,) \\
&\cong \pi_1(\,L_{n+1}((F_*)_{ab} \otimes \mathbf{Q})\,).
\end{aligned}$$

Isomorphism (5) follows from the Universal Coefficient Theorem for chain complexes of free abelian groups (see for instance [11]). Isomorphism (6) follows from the Dold-Kan theorem [7] (which expresses a simplicial abelain group A_* in terms of its Moore complex $N(A_*)$) and the fact that tensor products commute with direct sums.

By the Universal Coefficient Theorem, $\pi_1((F_*)_{ab} \otimes \mathbf{Q}) \cong \pi_1((F_*)_{ab}) \otimes \mathbf{Q} \cong M^{(1)}(G) \otimes \mathbf{Q}$. Suppose that $M^{(1)}$ is a torsion group. Then $M^{(1)}(G) \otimes \mathbf{Q} = 0$ and in low dimensions ($\leqslant 2$) the simplicial abelian group $(F_*)_{ab} \otimes \mathbf{Q}$ coincides with a free simplicial resolution of the vector space $G_{ab} \otimes \mathbf{Q}$. Since vector spaces are 'free objects', the simplicial group $(F_*)_{ab} \otimes \mathbf{Q}$ admits a contracting homotopy in low dimensions. Therefore $\pi_1(L_{n+1}((F_*)_{ab} \otimes \mathbf{Q})) = 0$. This proves the claim.

4. Peiffer Commutators and Partial Lie Algebras

The isomorphism of Magnus and Witt has been generalised by H. J. Baues and D. Conduché [1]. In the generalisation free groups are replaced by 'free precrossed modules' and Lie algebras are replaced by "partial Lie algebras". We suspect that Theorem 1 might also admit a generalisation in this direction.

Recall that a *precrossed module* is a homomorphism of groups $\partial\colon M \to P$ with an action of P on M satisfying $\partial({}^p m) = p(\partial m)p^{-1}$ for all $m \in M, p \in P$. An element in M of the form

$$mm'm^{-1}({}^{\partial m}m')^{-1}$$

with $m, m' \in M$ is called a *Peiffer commutator* and denoted by $\langle m, m'\rangle$. A precrossed module is called a *crossed module* if all Peiffer commutators are trivial. For any subgroup N in M let us denote by $\langle N, M\rangle$ the subgroup of M generated by those Peiffer commutators with either $m \in N$ or $m' \in N$. The lower Peiffer central series is then defined by setting $M_1 = M$ and $M_{n+1} = \langle M_n, M\rangle$ for $n \geqslant 1$. Each term M_n is a normal subgroup of M closed under the action of P. For $n \geqslant 2$ the quotient group M_n/M_{n+1} is abelian. It is natural to set $C = (M/M_2)_{ab}$ and form the direct sum

$$\Gamma M = C \;\oplus\; \bigoplus_{n \geqslant 2} M_n/M_{n+1}\,.$$

The Peiffer commutator map $M_m \times M_n \to M_{m+n}$, $(m, m') \mapsto \langle m, m'\rangle$ induces a bilinear mapping $[-,-]\colon \Gamma M \times \Gamma M \to \Gamma M$. The structure $(\Gamma M, [-,-])$ is not in general a Lie algebra. It is however the motivating example of what is called a 'partial Lie algebra' in [1]. The main result in [1] asserts that ΓM is a free partial Lie algebra if $\partial\colon M \to P$ is a free precrossed module with P a free group. The Magnus-Witt isomorphism corresponds to the case when P is the trivial group (for in this case the lower Peiffer central series of M coincides with the lower central series of M considered as a group).

Integral homology groups $H_m(M)_P$ for a precrossed module were introduced in [2] for dimensions $m = 1, 2$. Theorem 1 in [2] with the main result in [1] immediately imply the following.

Proposition 1. *Let $\partial\colon M \to P$ be a precrossed module such that:* 1) *P is a free group;* 2) *the induced map $\partial\colon M/M_2 \to P$ is a free crossed module;* 3) *$H_2(M)_P$ is trivial. Then ΓM is a free partial Lie algebra.*

This proposition with P equal to the trivial group is a special case of Theorem 1 (since in this case: hypothesis (2) asserts that M_{ab} is a free abelian group; $H_2(M)_1$ is the second integral homology of the group M; and ΓM is a Lie algebra). It would be interesting to know whether Proposition 1 can be generalized in the direction of Theorem 1 when P is not trivial.

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Author's address:
Mathematics Department
National University of Ireland Galway
Ireland
E-mail: 0002319S@bodkin.nuigalway.ie