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Non-Coarse Equivalent Subsets of a Metric Space

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Abstract

In this paper the large scale structure of R is investigated and two kinds of non-coarse equivalent subsets of R are characterized. In addition L-spaces and H-spaces in arbitrary metric spaces are introduced and it is shown that they are not coarse equivalent.

Keywords: *Metric space, Coarse structure, Coarse map, Coarse equivalent spaces.*

1 Introduction

Coarse geometry [4, 5, 1, 2, 3] is the study of spaces from a "large scale" point of view. Two spaces that look the same from a great distance are equivalent in coarse geometry. This point of view is useful because it is often true that the geometric properties of metric spaces are determined by their coarse geometry. When one defines continuity of a function on a metric space, one neglects a great deal of information about the metric d. For example the topology of the metric d and $d' = min\{d, 1\}$ are the same. But d' erases all information about d-distance greater than 1.
Coarse geometry studies the dual case. Instead of focusing on the small scale structure defined on a metric space we will focus on the large scale structure. In this paper we investigate the large scale structure of subsets of real numbers. Let us recall some definitions from coarse geometry.

Definition 1.1 *[4] Let X and Y be metric spaces, and let $f : X \to Y$ be a* *map (not necessarily continuous).*

- *The map f is (metrically) proper if the inverse image of each bounded subset of Y is a bounded subset of X.*
- *The map f is (uniformly) bornologous if for every $R > 0$ there is $S > 0$ such that*
$$d(x, y) < R \Rightarrow d(f(x), f(y)) < S.$$
- *The map f is coarse if it is proper and bornologous.*

For example, if $X = Y = N$, the natural numbers, then the map $n \mapsto 14n+78$ is coarse, but the map $n \mapsto 1$ is not coarse (it fails to be proper) and the map $n \mapsto n^2$ is not coarse either (it fails to be brornologous).

Definition 1.2 *[4] Two maps f, g from X into a metric space Y are close if $d(f(x), f(y))$ is bounded, uniformly in X. We say that the metric spaces X and Y are coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g; Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and X, respectively.*

Definition 1.3 *[4] One says that $f : X \to Y$ is large-scale Lipschitz if there are positive constants c and A such that*
$$d(f(x), f(y)) \leq cd(x, y) + A.$$
Clearly a large-scale Lipschitz map is bornologous. In general the converse is not true.

Lemma 1.4 *[4] Let X be a lenght space and Y any metric space. Then the following properties of a map $f : X \to Y$ are equivalent.*

- *f is large-scale Lipschitz;*
- *f is bornologous;*
- *There exist $R, S > 0$ such that $d(x, y) < R$ implies $d(f(x), f(y)) < S$.*

Example 1.5 *The inclusion map $i : Z \to R$ and the map $[.] : R \to Z$ (which assigns to each real number x the greatest integer less than or equal to x) are coarse maps. In addition $i \circ [.]$ and $[.] \circ i$ are close to the identity maps on R and Z. Consequently, R and Z are coarsely equivalent.*

Remark 1.6 *1) finite sets are coarse equivalent.*
2) If X is a set with more than one element and $a \in X$, then X and $X - \{a\}$ are coarse equivalent.

Lemma 1.7 *Let $A_2 = \{2^n, -2^n : n \in N\}$. A_2 and Z are not coarse equivalent.*

Proof: Suppose by contradiction that the coarse maps $f : Z \to A_2$ and $g : A_2 \to Z$ exist such that $f \circ g$ and $g \circ f$ are close to id_{A_2} and id_Z, respectively. Based on definition 1.1, there exists $c > 0$ such that $|z_1 - z_2| < 2$ implies $|f(z_1) - f(z_2)| < c$. Hence $|f(n+1) - f(n)| < c$, for every $n \in Z^+$. We choose $m > 0$ such that $2^{m+1} - 2^m > c$. Hence for every $n \in N$, $f(n) = f(n+1)$ or $f(n), f(n+1) \in K_m$, where $K_m = \{n : |n| \leq 2^m\}$.
Claim: For every $n_0 \in N$, there is $n > n_0$ such that $f(n) \in K_m$.
Proof of the claim. If $f(n) \notin K_m$ for all $n > n_0$, then we have

$$f(n) = f(n+1) = ... = d.$$

Since $g \circ f$ is close to the identity map there is $a > 0$ such that

$$|p - g \circ f(p)| = |p - g(d)| < a.$$

This is a contradiction if we choose p sufficiently large.
Since K_m is finite $g(K_m)$ is finite too. In addition, since for every $n_0 \in N$ there exists $n > n_0$ such that $f(n) \in K_m$, $g(f(n)) \in g(K_m)$ and we have $|n - g(f(n))| < a$, which is a contradiction.

2 Coarse Equivalent Subsets of N

Let $A_2^+ = \{2^n, n \in N\}$.

Theorem 2.1 *Suppose that $\{x_n : n \in N\}$ and $\{d_m = |x_{m+1} - x_m|, m \in N\}$ are increasing sequences in N.*

- *$\{x_n\}$ and N are coarse equivalent if $lim_{m\to\infty} d_m < \infty$;*
- *$\{x_n\}$ and A_2^+ are coarse equivalent if $lim_{m\to\infty} d_m = \infty$.*

Proof: Let $lim_{m\to\infty} d_m < \infty$ Suppose that the map $f : N \to \{x_n\}$ is defined by $n \mapsto x_n$. We prove that f and f^- are coarse maps.
Let $|n - n_1| < R$, for $R > 0$. By assumption, there is $m \in N$ such that $d_i = d$, for $i \geq m$. If $c = max\{(R+1)d, |x_i - x_j|, 1 \leq i, j \leq m\}$ then $|x_n - x_{n_1}| < c$, so f is a coarse map.
Similarly let $|x_n - x_k| < R$, for $R > 0$. if $c = max\{m, (R/d) + 1\}$ then $|n - k| < c$ and consequently g is a coarse map too.
If $lim_{m\to\infty} d_m = \infty$ then consider the natural maps $f : A_2^+ \to \{x_n\}$, $f(2^n) = x_n$ and f^{-1}. Let $|2^n - 2^k| < R$ for $R > 0$. We choose $m \in N$ such that $|2^{m+1} - 2^m| = 2^m > R$. If $c = x_m$ then $|2^n - 2^k| < R$ implies that $|x_n - x_k| < c$

and consequently f is a coarse map. If $|x_n - x_k| < R$ then we choose $m \in N$ such that $x_{m+1} - x_m > R$. Now if $c = 2^m$ the proof is complete.
The following example shows that the coarse equivalence of A_1 with A_2 and B_1 with B_2 doesn't imply the coarse equivalence of $A_1 \cup A_2$ with $B_1 \cup B_2$.

Example 2.2 *1) A_2^+ and $\{n! : n \in N\}$ are coarse equivalent.*
2) Let $A_1 = A_2 = N$, $B_1 = \{n - 1/n : n \in N\}$ and $B_2 = \{-1, -2, -3, ...\}$. It is easy to check that A_1 is coarse equivalent with A_2 and B_1 is coarse equivalent with B_2.
$A_1 \cup B_1 = \{1, 1-1, 2, 2-1/2, 3, 3-1/3, 4, 4-1/4, ...\}$ is coarse equivalent with N. But $A_2 \cup B_2 = Z - \{0\}$ which is not coarse equivalent with $A_2 \cup B_2$.

Theorem 2.3 *Let X be a metric space and $A_i,\ B_i \subseteq X$, for $i = 1, 2$, be such that $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$. In addition, suppose that A_1 is coarse equivalent with A_2, Using the coarse maps $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_1$, and B_1 is coarse equivalent with B_2, Using the coarse maps $g_1 : B_1 \to B_2$ and $g_2 : B_2 \to B_1$. If there is $M > 0$ such that*

$$|d(a_i, b_i) - d(f_i(a_i), g_i(b_i))| < M,$$

for $a_i \in A_i$ and $b_i \in B_i$, then $A_1 \cup B_1$ is coarse equivalent with $A_2 \cup B_2$.

Proof: Suppose that the map $f_1 \cup g_1 : A_1 \cup B_1 \to A_2 \cup B_2$ ($f_2 \cup g_2 : A_2 \cup B_2 \to A_1 \cup B_1$) is defined by $f_1 \cup g_1(x) = f_1(x)$ if $x \in A_1$ ($f_2 \cup g_2(x) = f_2(x)$ if $x \in A_2$) and $f_1 \cup g_1(x) = g_1(x)$, $x \in B_1$ ($f_2 \cup g_2(x) = g_2(x)$, $x \in B_2$).
For every $R > 0$ there is s_1 and s_2 such that
$d(x, y) < R$ implies $d(f_1(x), f_1(y)) < s_1$, for $x,\ y \in A_1$,
and
$d(x, y) < R$ implies $d(g_1(x), g_1(y)) < s_1$, for $x,\ y \in B_1$.
Let $x \in A_1$, $y \in B_1$ and $d(x, y) < R$ then
$d(f_1(x), g_1(x)) < M + d(x, y) < M + R$.
This implies that $d(f_1 \cup g_1(x), f_1 \cup g_1(y)) < M + R$. Let $s = max\{s_1, s_2, M + R\}$. $d(x, y) < R$ implies $d(f_1 \cup g_1(x), f_1 \cup g_1(y)) < s$ and consequently $f_1 \cup g_1$ is bornolougous. In a similar way one can prove that $A_1 \cup B_1$ is coarse equivalent with $A_2 \cup B_2$.

3 L-Spaces and H-Spaces

The properties of N and Z helps us to define two kinds of subspaces which are not coarse equivalent.

Definition 3.1 *The metric space Y is called a L-space if for each N_R^y there are subsets $A, B \neq \emptyset$ of Y such that $A \cup B = (N_R^y)^c$, $d(A, B) \geq R$ and $sup_{a \in A} d(a, N_R^y) = sup_{b \in B} d(b, N_R^y) = \infty$, where $N_R^y = \{z \in Y : d(y, z) < R\}$ and $d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$*

Example 3.2 *The metric space Z is a L-space.*

Definition 3.3 *The metric space X is called a H- space if there is $c > 0$ such that for each $x_0 \in X$ there is $R_0 \in N$ that:*

If $x, z \in (N_R^{x_0})^c$, for every $R \geq R_0$, then there is a net $\{x_1 = x, x_2, ..., x_n = z\} \subseteq N_R^{x_0 c}$ that $d(x_i, x_{i+1}) \leq c$ for all $i = 1, 2, ..., n-1$.

Example 3.4 *The metric space N is a H-space. The Euclidean space R^n, $n \neq 2$, is a H-spaces.*

Theorem 3.5 *Let Y be a L-space and $X \subset Y$ be a H-space. Then X and Y are not coarse equivalent.*

Proof: Since X is a H-space we choose $c > 0$ as in definition 2.2. Suppose by contradiction that X and Y, using the coarse maps $f : X \to Y$ and $g : Y \to X$, are coarse equivalent. There is $R > 0$ such that for $x, x_1 \in X$, $d(x, x_1) < c$ implies $d(f(x), f(x_1)) < R$ and for $y, y_1 \in Y$, $d(y, y_1) < c$ implies $d(g(y), g(y_1)) < R$. There is $a > 0$ such that

$$d(x, g \circ f(x)) < a \; \forall x \in X$$

and

$$d(y, f \circ g(y)) < a \; \forall y \in Y,$$

since $f \circ g$ and $g \circ f$ are close to i_Y and i_X, respectively.
We choose $T > max\{a, R\}$. If $y_0 \in Y$ then $f^{-1}(N_T^{y_0})$ is bounded. Hence there are $x_0 \in X$ and $M > 0$ such that $f^{-1}(N_T^{y_0}) \subseteq N_M^{x_0}$. By hypothesis there are subsets $A, B \neq \emptyset$ of Y such that $A \cup B = (N_T^{y_0})^c$ and $d(A, B) \geq T$.
Let $R_0 > 0$ be such that if $x, z \in (N_R^{x_0})^c$, for every $R \geq R_0$, then there is a net $\{x_1 = x, x_2, ..., x_n = z\} \subseteq N_R^{x_0 c}$ that $d(x_i, x_{i+1}) \leq c$ for all $i = 1, 2, ..., n-1$.
Let $m_0 > \{R_0, M\}$.

Claim 1: $f((N_{m_0}^{x_0})^c) \subseteq A$ or $f((N_{m_0}^{x_0})^c) \subseteq B$.

Proof of the Claim 1: Suppose that $x, z \in (N_{m_0}^{x_0})^c$. If $f(x) \in A$ we prove that $f(z) \in A$.
Suppose that $f(z) \notin A$, since $f(z) \notin N_T^{y_0}$ $f(z) \in B$ and there is a net $\{x_1 = x, x_2, ..., x_n = z\} \subseteq (N_{m_0}^{x_0})^c$ such that $d(x_i, x_{i+1}) \leq c$ for all $i = 1, 2, ..., n-1$. This implies that $d(x, x_2) < c$ and consequently $d(f(x), f(x_2)) < R$. Hence $f(x_2) \in A$. By induction $f(x_n) = f(z) \in A$ and this is a contradiction. Indeed $f(X) \subseteq f(N_{m_0}^{x_0}) \cup A$.
We note that if K is a bounded subset of X then $f(K)$ is a bounded subset of Y.

Now let $f(N_{m_0}^{x_0}) \subseteq N_k^{y_0}$. Let $b \in B$. If $f(g(b)) \in A$ then $d(b, f(g(b))) < a$ so $d(A, B) < a < T$ which is a contradiction. If $f(g(b)) \in f(N_{m_0}^{x_0})$ Since B is not bounded $f(g(B))$ is not bounded too but by the pervious claim $f(N_{m_0}^{x_0})$ is bounded and hence $f(g(B))$ is bounded which is again a contradiction.

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