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ON SPACES OF THE SAME STRONG n-TYPE

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Abstract

Let X be a connected CW complex and $[X]$ be its homotopy type. As usual, $\mathrm{SNT}(X)$ denotes the pointed set of homotopy types of CW complexes Y such that their n^{th}-Postnikov approximations $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent for all n. In this paper we study a particularly interesting subset of $\mathrm{SNT}(X)$, denoted $\mathrm{SNT}_\pi(X)$, of strong n type; the n^{th}-Postnikov approximations $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent by homotopy equivalences satisfying an extra condition at the level of homotopy groups. First, we construct a CW complex X such that $\mathrm{SNT}_\pi(X) \neq \{[X]\}$ and we establishe a connection between the pointed set $\mathrm{SNT}_\pi(X)$ and sub-groups of homotopy classes of self-equivalences via a certain $\varprojlim^1$ set. Secondly, we prove a conjecture of Arkowitz and Maruyama concerning subgroups of the group of self equivalences of a finite CW complex and we use this result to establish a characterization of simply connected CW complexes with finite dimensional rational cohomology such that $\mathrm{SNT}_\pi(X) = \{[X]\}$.

1. Introduction

Let X be a connected CW complex and $[X]$ be its homotopy type. As usual, $\mathrm{SNT}(X)$ denotes the pointed set of homotopy types of CW complexes Y such that their n^{th}-Postnikov approximations $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent (then X and Y have the same n type for all $n \geqslant 1$).

The first example of a space X with $\mathrm{SNT}(X) \neq \{[X]\}$ has been given by J.F. Adams in 1957 ([**1**]). In 1966 Brayton Gray, ([**7**]), found one with finite type. In general, the determination of spaces X such that $\mathrm{SNT}(X)$ is a singleton remains an open problem. For instance, in ([**12**]), C.A. McGibbon and J. Möller conjecture that $\mathrm{SNT}(\Omega X) = \{[\Omega X]\}$ for a simply connected finite CW complex X and prove this conjecture when X is an H_0-space, and in fact using, [**13**], their proof works as well in order to prove the conjecture for any rational elliptic space, [**6**].

In this paper we consider a particular subset of $\mathrm{SNT}(X)$ denoted $\mathrm{SNT}_\pi(X)$ that we now describe. Denote by P_X the set of pairs $(Y,(f_n)_{n\geqslant 1})$, where $f_n : Y^{(n)} \to X^{(n)}$ is a homotopy equivalence and such that the following diagrams commute for $n \geqslant 2$.

$$\begin{array}{ccc}
\pi_*(Y^{(n)}) & \xrightarrow{\pi_*(f_n)} & \pi_*(X^{(n)}) \\
\downarrow{\scriptstyle \pi_*(p_Y^{(n)})} & & \downarrow{\scriptstyle \pi_*(p_X^{(n)})} \\
\pi_*(Y^{(n-1)}) & \xrightarrow{\pi_*(f_{n-1})} & \pi_*(X^{(n-1)})
\end{array}$$

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We put $\mathrm{SNT}_\pi(X) = \{ [Y] \,|\, (Y, (f_n)) \in P_X \}$.

When $[Y]$ belongs to $\mathrm{SNT}_\pi(X)$, we say that X and Y *have the same strong* n *type*.

It follows from the works of Adams ([**1**]), Wilkerson ([**18**]), McGibbon and Möller ([**12**], [**10**]) that

$$SNT(X) = \mathrm{SNT}_\pi(X) = \{[X]\}$$

in each of the following cases

(a) if $\pi_i(X)$ is a finite group for all $i > 0$;

(b) if X is a simply connected finite type rational space;

(c) if X is a simply connected, $\mathbb{Q}$-finite type, H_0-space such that the natural map $\mathrm{Aut}(X) \to \mathrm{Aut}\, H^{\leqslant n}(X;\mathbb{Z})$ has a finite cokernel for $n \geqslant 1$. Here $\mathrm{Aut}\ H^{\leqslant n}(X;\mathbb{Z})$ denotes the group of ring homomorphisms.

Example 1. $\mathrm{SNT}_\pi(X) \neq \{[X]\}$) when X is the 0-localization of the CW complex

$$Z = (S^2_a \times S^2_b \times \bigvee_{p\geqslant 1} S^{6p}) \vee (\bigvee_{n\geqslant 1} S^4_n) \cup_{(\gamma_{n,p})} \bigcup_{\substack{p\geqslant 1\\ n>p}} e^{4+6p}_n\,,$$

where $\gamma_{n,p}$ is the Whitehead bracket $[S^4_n, S^{6p}]$. Observe that X is not of finite type (see (b) above !). More precisely we prove the following equality (Theorem 4 below)

$$\mathrm{SNT}_\pi(X) \cong \prod_{n\geqslant 1} \mathbb{Q} / \bigoplus_{n\geqslant 1} \mathbb{Q}\,.$$

Our main result establishes a connection between the pointed set $\mathrm{SNT}_\pi(X)$ and sub-groups of homotopy classes of self-equivalences via a certain $\varprojlim{}^1$ set. We will state the precise results with its first consequences after some definitions.

If X is a based topological space, $\mathrm{Aut}\ (X)$ denotes the group of homotopy classes of self homotopy equivalences of X, and $\mathrm{Aut}_\pi(X)$ the subgroup of homotopy classes which induce the identity on the homotopy groups of X. A tower of groups

$$\cdots \to \ \mathrm{Aut}\ X^{(n)} \ \stackrel{q_n}{\to}\ \mathrm{Aut}\ X^{(n-1)} \ \to \cdots \to\ \mathrm{Aut}\ X^{(1)}\,,$$

is defined by the homomorphisms $q_n([f]) = [f^{(n-1)}]$ where $f^{(n-1)}$ denotes the map induced up to homotopy by f :

$$\begin{array}{ccc} X^{(n)} & \stackrel{f}{\longrightarrow} & X^{(n)} \\ p^{(n)}_Y \downarrow & & \downarrow p^{(n)}_X \\ X^{(n-1)} & \stackrel{f^{n-1}}{\longrightarrow} & X^{(n-1)}\,. \end{array}$$

In 1975, C. Wilkerson ([**18**]) establishes the existence of a natural pointed set bijection

$$\theta^X : \mathrm{SNT}(X) \to \varprojlim{}^1\ \mathrm{Aut}\ X^{(n)}\,.$$

where $\varprojlim{}^1 G_n$ is defined for a tower of groups

$$\cdots \to G_n \stackrel{q_n}{\to} G_{n-1} \to \cdots \stackrel{q_2}{\to} G_1\,,$$

as the orbit set for the action of the group $\prod_n G_n$ acting on the set $\prod_n G_n$ by

$$(\gamma_n)\cdot(\alpha_n) = (\gamma_n \alpha_n q_{n+1}(\gamma_{n+1})^{-1})\,.$$

The sequence of injections $\mathrm{Aut}_\pi X^{(n)} \to \mathrm{Aut}\ X^{(n)}$ induces a natural map

$$j_X : \varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)} \to \varprojlim{}^1\ \mathrm{Aut}\ X^{(n)}\,.$$

Theorem 1. *Let X be a simply connected CW complex. Then there is a pointed set bijection $SNT_\pi(X) \to \mathrm{Image}\ j_X$.*

Example 2. Assume X is a simply connected finite type CW-complex that has the rational homotopy type of a bouquet of spheres. By ([**12**], Lemma 1), the groups $\mathrm{Aut}_\pi X^{(n)}$ are finite, so that $\varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)} = \{*\}$ and $\mathrm{SNT}_\pi(X) = \{[X]\}$, see [**12**].

Example 3. Denote by $f : \mathbb{C}P^\infty \to S^3$ a phantom map and by $H : S^3 \to S^2$ the Hopf map. The homotopy cofibre Z, of $H \circ f$, and the space $S^2 \vee \Sigma\mathbb{C}P^\infty$ have the same n type for all n. D. Stanley ([**16**]) has proved that the Lusternik-Schnirelmann category of Z is two whereas the category of $S^2 \vee \Sigma\mathbb{C}P^\infty$ is one. Therefore the spaces Z and $S^2 \vee \Sigma\mathbb{C}P^\infty$ do not have the same homotopy type and thus not the same strong n type because they are rational suspensions.

Theorem 2. *Let X be a simply connected finite type CW complex with finite dimensional rational cohomology, then $SNT_\pi(X) = \{[X]\}$ if and only if for some integer N, the morphism $\mathrm{Aut}_\pi(X) \to \mathrm{Aut}_\pi(X^{(N)})$ has a finite cokernel.*

2. The six term exact sequence

Let X be a simply connected finite type CW complex. Denote by G_n the image of the natural morphism $\mathrm{Aut}X^{(n)} \to \mathrm{Aut}\pi_{\leqslant n}(X)$. Then the short exact sequence of towers

$$1 \to \mathrm{Aut}_\pi X^{(n)} \to \mathrm{Aut}X^{(n)} \to G_n \to 1$$

induces a 6-term exact sequence, [**8**]:

$$1 \to \varprojlim\ \mathrm{Aut}_\pi X^{(n)} \to \varprojlim \mathrm{Aut}X^{(n)} \to \varprojlim G_n \xrightarrow{\delta} \varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)} \xrightarrow{j_X}$$

$$\xrightarrow{j_X} \varprojlim{}^1 \mathrm{Aut}X^{(n)} \to \varprojlim{}^1 G_n \to *$$

The image of j_X is, by Theorem 1, in bijection with the subset $\mathrm{SNT}_\pi(X)$ of $SNT(X)$ so that we obtain a 6-term exact sequence

$$1 \to \mathrm{Aut}_\pi X \to \mathrm{Aut}X \to \varprojlim G_n \xrightarrow{\delta} \varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)} \xrightarrow{j_X} SNT(X) \to \varprojlim{}^1 G_n \to *\,.$$

In particular, in the case $\mathrm{SNT}_\pi(X) = \{[X]\}$ we have a bijection

$$SNT(X) \xrightarrow{\cong} \varprojlim{}^1 G_n\,.$$

This 6-term exact sequence provides us with anaother description of the set $\mathrm{SNT}_\pi X$. The group $\varprojlim G_n$ is the subgroup of $\mathrm{Aut}\,\pi_*(X)$ consisting of those automorphisms φ such that, for each $n \geqslant 1$, the restriction of φ to $\pi_{\leqslant n}$ can be realized by some automorphism $\alpha_n \in \mathrm{Aut}X^{(n)}$. The connecting map δ associates to φ the class of the element $(\alpha_n(\alpha_{n+1})^{-1})_{n\geqslant 1}$ in $\varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)}$.

The group $\varprojlim G_n$ acts on the set $\varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)}$ in the following way : Let φ be an element in $\varprojlim G_n$ whose restriction to $\pi_{\leqslant n}$ is realized by an automorphism $\alpha_n \in \mathrm{Aut}X^{(n)}$, and let $\beta = [(\beta_n)] \in \varprojlim{}^1\ \mathrm{Aut}_\pi X^{(n)}$, then

$$(\varphi \cdot \beta)_n = \alpha_n \beta_n (\alpha_{n+1})^{-1}\,.$$

Therefore we deduce:

Proposition 1. *The set* $SNT_\pi(X)$ *is the orbit space of* $\varprojlim^1 \mathrm{Aut}_\pi X^{(n)}$ *for this action.*

Let us precise here another interesting fact concerning the above 6-term exact sequence. Let $\mathrm{aut}_\pi X$ denote the monoid of self equivalences which induce the identity map at the level of homotopy groups and recall that $\pi_0(\mathrm{aut}_\pi X) = \mathrm{Aut}_\pi X$. From the work of Bousfield and Kan ([**4**]), we have a diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \to & \varprojlim^1 \pi_1(\mathrm{aut}_\pi X^{(n)}) & \to & \mathrm{aut}_\pi X & \xrightarrow{\rho_1} & \varprojlim \mathrm{Aut}_\pi X^{(n)} & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & \varprojlim^1 \pi_1(\mathrm{aut}_\pi X^{(n)}) & \to & \mathrm{Aut} X & \xrightarrow{\rho} & \varprojlim \mathrm{Aut}\, X^{(n)} & \to & 1
\end{array}$$

Moreover the kernel of ρ is the subgroup of automorphisms φ whose restrictions to $X^{(n)}$ are homotopic to the identity for each $n \geqslant 1$. This subgroup of $\mathrm{Aut}_\pi X$ is called the *group of weak identities of* X. Observe that if X is an H-space, the set of homotopy classes of phantom maps from X into X, $Ph(X, X)$, is a group, and that the map $\theta : Ph(X, X) \to Ker\,\rho$ defined by $\theta(f) = id + f$ is a group isomorphism ([**15**]).

There is another interesting subgroup of $\mathrm{Aut}_\pi X^{(n)}$, the subgroup $\mathrm{Aut}_\Omega X^{(n)}$ formed by the automorphisms φ such that $\Omega\varphi$ is homotopic to the identity. By ([**5**]), the maps $\mathrm{Aut}_\Omega X^{(n)} \to \mathrm{Aut}_\pi X^{(n)}$ are injections of finitely generated nilpotent groups. Since these injections become isomorphisms after Malcev completion, the quotients $H_n = \mathrm{Aut}_\pi X^{(n)} / \mathrm{Aut}_\Omega X^{(n)}$ are finite groups. Therefore $\varprojlim^1 H_n = 0$, and we have a surjection

$$\varprojlim^1 \mathrm{Aut}_\Omega X^{(n)} \to \varprojlim^1 \mathrm{Aut}_\pi X^{(n)} .$$

This shows that

Proposition 2.

$$SNT_\pi(X) = Image(\varprojlim^1 \mathrm{Aut}_\Omega X^{(n)} \to \varprojlim^1 \mathrm{Aut} X(n)) .$$

In other words, the set $\mathrm{SNT}_\pi(X)$ can also be described as the subset of $SNT(X)$ formed by the spaces Y for which there exist maps $f_n : Y^{(n)} \to X^{(n)}$ such that the following diagrams commute for $n \geqslant 2$.

$$\begin{array}{ccc}
\Omega Y^{(n)} & \xrightarrow{\Omega f_n} & \Omega X^{(n)} \\
\downarrow{\scriptstyle \Omega p_Y^{(n)}} & & \downarrow{\scriptstyle \Omega p_X^{(n)}} \\
\Omega Y^{(n-1)} & \xrightarrow{\Omega f_{n-1}} & \Omega X^{(n-1)}
\end{array}$$

3. Proof of Theorem 1

Recall that P_X denotes the set of pairs $(Y, (f_n)_{n\geqslant 1})$, where $f_n : Y^{(n)} \to X^{(n)}$ is an homotopy equivalence and such that the following diagrams commute

$$\begin{array}{ccc}
\pi_*(Y^{(n)}) & \xrightarrow{\pi_*(f_n)} & \pi_*(X^{(n)}) \\
\downarrow{\scriptstyle \pi_*(p_Y^{(n)})} & & \downarrow{\scriptstyle \pi_*(p_X^{(n)})} \\
\pi_*(Y^{(n-1)}) & \xrightarrow{\pi_*(f_{n-1})} & \pi_*(X^{(n-1)})
\end{array}$$

We introduce an equivalence relation on the set P_X. Two pairs $(Y,(f_n))$ and $(Z,(g_n))$ in P_X are equivalent if there is an homotopy equivalence $\varphi : Z \to Y$ such that the maps $f_n \circ \varphi \circ g_n^{-1}$ belongs to $\mathrm{Aut}_\pi(X^{(n)})$ for all n. For instance, for any homotopy equivalence $m : Y \to Z$ and each element $(Z,(g_n))$, the elements $(Y,(g_n \circ m))$ and $(Z,(g_n))$ are equivalent. We denote by E_X the quotient set $P_X/\sim$. To each pair $(Y,(f_n)) \in P_X$ we associate the homotopy type of Y into $\mathrm{SNT}_\pi(X)$. We obtain in this way a well defined map $\rho : E_X \to \mathrm{SNT}_\pi(X)$.

The following proposition implies then clearly Theorem 1.

Proposition 3. *Let X be a simply connected CW complex. There is a natural pointed set bijection $\theta_1^X : E_X \to \varprojlim{}^1 \mathrm{Aut}_\pi X^{(n)}$. Moreover, the bijections θ^X and θ_1^X fit together in a commutative diagram*

$$\begin{array}{ccccc}
\varprojlim{}^1 \mathrm{Aut}_\pi X^{(n)} & \longrightarrow & \mathrm{Image}\ j_X & \hookrightarrow & \varprojlim{}^1 \mathrm{Aut} X^{(n)} \\
\uparrow \theta_1^X & & \uparrow \cong & & \uparrow \theta^X \\
E_X & \xrightarrow{\rho} & SNT_\pi(X) & \hookrightarrow & SNT(X)\,.
\end{array}$$

Proof. Since the simplicial Postnikov decomposition is functorial, we consider Kan complexes instead of CW complexes. The n^{th}-Postnikov complex of a simplicial complex X is the quotient $X^{(n)} = X/\sim_n$ where $x \sim_n y$ if the corresponding faces of x and y of dimensions less than or equal to n are equal. The quotient $X^{(n)}$ is a Kan complex, and the natural projections

$$p_X^{(n)} : X^{(n)} \to X^{(n-1)}$$

are Kan fibrations with Eilenberg-MacLane complexes $K(\pi_n(X), n)$ as fibres.

We suppose that X is a minimal Kan complex. Therefore each $X^{(n)}$ is also minimal. As usual, $\mathrm{Aut}X^{(n)}$ denotes the group of simplicial isomorphisms and $\mathrm{Aut}_\pi X^{(n)}$ denotes the subgroup of automorphisms inducing the identity map on the homotopy groups.

If $\alpha_n \in \mathrm{Aut}X^{(n)}$, the composite $\alpha_n \circ p^{(n+1)}$ is again a Kan fibration and a Postnikov section. Let $\bar\alpha = (\alpha_0, \cdots)$ denotes the sequence of the α_n in $\prod_n \mathrm{Aut}_\pi(X^{(n)})$, and define

$$X_{\bar\alpha} = \varprojlim (X^{(n)}, \alpha_{n-1} \circ p^{(n)})\,.$$

Then $X_{\bar\alpha}$ is again a minimal Kan complex and the commutativity of the square

$$\begin{array}{ccc}
X^{(n)} & = & X^{(n)} \\
p_X^{(n)} \downarrow & & \downarrow \alpha_{n-1} \circ p^{(n)} \\
X^{(n-1)} & \xrightarrow{\alpha_{n-1}} & X^{(n-1)}
\end{array}$$

shows that for each n there are homotopy equivalences $f_n^{\bar\alpha} : X^{(n)} \to X_{\bar\alpha}^{(n)}$ such that the following diagram commutes

$$\begin{array}{ccc}
\pi_*(X^{(n)}) & \xrightarrow{\pi_*(f_n^{\bar\alpha})} & \pi_*(X_{\bar\alpha}^{(n)}) \\
\pi_*(p_X^{(n)}) \downarrow & & \downarrow \pi_*(\alpha_{n-1} \circ p^{(n)}) \\
\pi_*(X^{(n-1)}) & \xrightarrow{\pi_*(\alpha_{n-1})} & \pi_*(X^{(n-1)})\,.
\end{array}$$

In particular, $(X_{\bar\alpha}, (f_n^{\bar\alpha}))$ defines an element in P_X and in E_X.

Suppose that $\bar\alpha$ and $\bar\beta$ define the same element in $\varprojlim{}^1 \mathrm{Aut}_\pi X^{(n)}$. Then there is a sequence $\bar\gamma = (\gamma_n) \in \prod_n \mathrm{Aut}_\pi X^{(n)}$ such that $\alpha_n = \gamma_n \circ \beta_n \circ (q_{n+1}(\gamma_{n+1}))^{-1}$. We then have maps

$\gamma_n : X^{(n)}_{\bar\beta} \longrightarrow X^{(n)}_{\bar\alpha}$ such that the following diagrams commute up to homotopy

$$\begin{array}{ccc} X^{(n)} & \xrightarrow{\gamma_n} & X^{(n)} \\ {\scriptstyle \beta_{n-1}\circ p^n_X} \downarrow & & \downarrow {\scriptstyle \alpha_{n-1}\circ p^n_X} \\ X^{(n-1)} & \xrightarrow{\gamma_{n-1}} & X^{(n-1)} \end{array}$$

and thus the homotopy class $[(f^{\bar\alpha}_n)^{-1}\circ\gamma_n\circ f^{\bar\beta}_n]$ belongs to $\mathrm{Aut}_\pi X^{(n)}$. Here, if f is a homotopy equivalence, then f^{-1} means any representative for the class $[f]$. The correspondance $\bar\alpha \to (X_{\bar\alpha},(f^{\bar\alpha}_n))$ induces therefore a well defined map

$$\Phi : \underleftarrow{\lim}^1 \ \mathrm{Aut}_\pi X^{(n)} \to E_X \,,$$

Which respects base points.

In order to show that Φ is a bijection we construct the inverse map. The map

$$\theta : P_X \to \underleftarrow{\lim}^1 \ \mathrm{Aut}_\pi X^{(n)}\,, \qquad \theta(Y,(f_n)) = ([f_n\circ(f^{(n)}_{n+1})^{-1}])$$

factors to give a map $\theta^X_1 : E_X \to \underleftarrow{\lim}^1\ \mathrm{Aut}_\pi X^{(n)}$. Indeed, suppose that $(Y,(f_n))$ and $(Z,(g_n))$ are equivalent, then there exists a map φ such that, for each integer $n \geqslant 1$, the class $\alpha_n = [g_n\circ\varphi^{(n)}\circ f_n^{-1}] \in \mathrm{Aut}_\pi X^{(n)}$. This shows that

$$(\alpha_n)\cdot[f_n\circ(f^{(n)}_{n+1})^{-1}] = [g_n\circ(g^{(n)}_{n+1})^{-1}]\,.$$

The relation $\Phi\circ\theta^X_1 = id$ results from the commutativity, up to homotopy, of the diagram

$$\begin{array}{ccc} Y^{(n)} & \xrightarrow{f_n} & X^{(n)} \\ p^n_Y \downarrow & & \downarrow {\scriptstyle f_{n-1}\circ(f^{n-1}_n)^{-1}p^n_X} \\ Y^{(n-1)} & \xrightarrow{f^{(n-1)}_n} & X^{(n-1)} \end{array}$$

The relation $\theta^X_1\circ\Phi = id$ is trivially satified. □

4. A conjecture of Arkowitz and Maruyama

For a simply connected CW complex Z we denote by $\mathrm{Aut}^n_\#(Z)$, $n \geqslant 2$, the kernel of the natural morphism $\mathrm{Aut}(Z) \to \mathrm{Aut}\,\pi_{\leqslant n}(Z)$. In ([**3**]) Arkowitz and Maruyama conjecture that for a finite simply connected CW complex there is an integer N such that the restriction map $\mathrm{Aut}_\pi \to \mathrm{Aut}^N_\# Z$ is an isomorphism. Here we prove the conjecture as a corollary of its rational version.

Theorem 3. *Let Z be a simply connected finite type CW complex and let Z_0 its rationalization. Suppose there $H^{>M}(Z;\mathbb{Q}) = 0$ for some M, then there is some integer N such that the restriction map*

$$\mathrm{Aut}_\pi Z_0 \to \mathrm{Aut}^N_\# Z_0\,,$$

is an isomorphism.

Moreover if $H^{>M}(Z;\mathbb{Z}) = 0$, then there is some integer N such that $\mathrm{Aut}_\pi Z \cong \mathrm{Aut}^N_\# Z$.

Example 3. Suppose W is the fat wedge of the three spheres S^2, S^2 and S^4, ie. $S^2\times S^2\times S^4 = W\cup_\omega D^8$ where $\omega\in\pi_7(W)$ denotes a triple Whitehead product. Then the rational homotopy Lie algebra of W is generated by elements of degrees less than or equal to 6 :

$$\pi_*(\Omega W)\otimes\mathbb{Q} \cong Ab(x_1,y_1,z_3)\coprod \mathbb{L}(t_6)\,.$$

The cokernel of the morphism $\mathrm{Aut}_{\#}^{7}(W) \to \mathrm{Aut}_{\#}^{6}(W)$ is not finite, but for $n \geqslant 8$, the cokernel of the correponding injection $\mathrm{Aut}_{\#}^{n}(W) \to \mathrm{Aut}_{\#}^{n-1}(W)$ is finite.

Example 4. The integer N depends on the space and not only on the dimension of the space. Consider for instance the 6-dimensional rational space X whose minimal Lie model is given by the following differential graded Lie algebra

$$L = \left(\mathbb{L}(x_i, y_i, z_i, a_{i,j}, b_{i,j}, u, v, w)_{i,j=1,\dots,n;\ i \leqslant j}, d\right)$$

$|x_i| = |y_i| = |z_i| = 2$, $|u| = 1$, $|v| = 2$, $|a_{i,j}|, |b_{i,j}|, |w| = 5$,
$d(u) = d(v) = d(x_i) = d(y_i) = d(z_i) = 0$, $d(a_{i,j}) = [x_i, z_j]$, $d(b_{i,j}) = [y_i, z_j]$, and

$$d(w) = [x_1, y_1] + \ldots + [x_n, y_n]\,.$$

The autoomorphism φ equal to the identity on all generators except w and defined on w by $\varphi(w) = w + [v,[v,u]]$ belongs to $\mathrm{Aut}_{\#}^{5+2n}(X)$ but not to $\mathrm{Aut}_{\pi}(X)$.

Proof of Theorem 3. Denote by $(\wedge V, d)$ the Sullivan minimal model of Z ([**17**]). We denote by $\mathrm{Aut}(\wedge V, d)$ the group of homotopy classes of automorphisms of $(\wedge V, d)$. Each automorphism of $(\wedge V, d)$ induces an isomorphism on the vector space of indecomposable elements $Q(\wedge V) = \wedge V / \wedge^{\geqslant 2} V \cong V$. We denote by $\mathrm{Aut}_{\pi}(\wedge V, d)$ the kernel of the representation morphism $\mathrm{Aut}(\wedge V, d) \to \mathrm{Aut}(Q(V))$ and by $\mathrm{Aut}_{\#}^{N}(\wedge V, d)$ the kernel of the morphism $\mathrm{Aut}(\wedge V, d) \to \mathrm{Aut}(Q(V)^{\leqslant N})$. We denote also by $Iso(\wedge V, d)$ the group of automorphisms of $(\wedge V, d)$, and by $Iso_1(\wedge V, d)$ the subgroup of autoomorphisms of the form $id + \varphi$, with $\varphi : V \to \wedge^{\geqslant 2} V$.

From the Sullivan theory of minimal models one deduce the following group isomorphisms

$$\begin{cases} \mathrm{Aut} Z_0 = \mathrm{Aut}(\wedge V, d) \\ \mathrm{Aut}_{\#}^{N} Z_0 = \mathrm{Aut}_{\#}^{N}(\wedge V, d) \\ \mathrm{Aut}_{\pi} Z_0 = \mathrm{Aut}_{\pi}(\wedge V, d) \end{cases}$$

where Z_0 denote the 0-localization of Z.

Since $H^{>M}(\wedge V, d) = 0$, each automorphism of $(\wedge V^{\leqslant M}, d)$ extends to an automorphism of $(\wedge V, d)$. This means that the restriction map

$$\rho : \mathrm{Aut}(\wedge V, d) \to \mathrm{Aut}(\wedge V^{\leqslant M}, d)$$

is onto. Let $\theta : (\wedge V, d) \to (A, d)$ be a quasi-isomorphism with $A^{>M} = 0$. Suppose now that $f, g \in \mathrm{Aut}(\wedge V)$. If $\rho(f) \sim \rho(g)$, ($\sim$ means homotopy in the category of c.d.g.a.'s), then $\theta \circ f \sim \theta \circ g$, and since θ is a quasi-isomorphism, $f \sim g$. This shows that ρ is also injective.

We form the vector space $Hom^0(V^{\leqslant M}, \wedge^{\geqslant 2} V)$, consisting of the degree zero linear maps. We choose an homogeneous basis, $(x_i)_{i=1,\dots,N_1}$, of $V^{\leqslant M}$, and an homogeneous basis, $(y_j)_{j=1,\dots,N_2}$, for $(\wedge^{\geqslant 2} V)^{\leqslant M}$. Then for any $\varphi \in Hom^0(V^{\leqslant M}, \wedge^{\geqslant 2} V)$ we write

$$\varphi(x_i) = \sum_j \alpha_i^j y_j\,.$$

Of course $id + \varphi$ is an automorphism of $(\wedge V^{\leqslant M}, d)$ if and only if it commutes with the differential, i.e. if and only if, for $i = 1, \dots, N_1$, we have

$$(id + \varphi) d(x_i) = d(x_i + \varphi(x_i))\,.$$

By expressing these conditions in terms of the α_i^j, we obtain a bijective correspondance between $Iso_1(\wedge V^{\leqslant M}, d)$ and some sub-algebraic variety W in $\mathbb{Q}^{N_1 N_2}$.

We choose a linear section, $\sigma : (d(\wedge^{\geqslant 2} V))^{>M} \to (\wedge V)^{\geqslant M}$, of the differential d. Then each $\psi = id + \varphi$ in $Iso_1(\wedge V^{\leqslant M}, d)$ extends in a natural way to an automorphism of $(\wedge V, d)$ by the following rule : Let $z \in V^{M+k}$, $k \geqslant 1$, then ψdz is a coboundary so that we put $\psi(z) = \sigma \psi dz$.

Performing this construction on a homogeneous basis of each V^{n+k} we construct the required homomorphism. Moreover, ψ is an isomorphism since it is a quasi-isomorphism and $(\wedge V, d)$ is minimal ([**17**]). We put $\Gamma_M = Iso_1(\wedge V^{\leqslant M}, d)$, and for $m \geqslant M$,

$$\Gamma_m = \{\, \psi \in \Gamma \,,\, \psi(z) - z \in \wedge^{\geqslant 2} V \,, \quad \text{for } z \in V^{\leqslant m} \,\}\,.$$

the choice of the section isomorphisms of $(\wedge V, d)$. We ψ such that in their extension
Now the sequence

$$\ldots \subset \Gamma_{M+2} \subset \Gamma_{M+1} \subset \Gamma_M$$

is a sequence of inclusions of algebraic varieties. Since $\mathbb{Q}[\alpha_i^j]$ is noetherian, this sequence stabilizes : there is an integer N such that

$$\Gamma_N = \Gamma_{N+r} \qquad \forall r \geqslant 1\,.$$

Denote by Γ_{id} the normal subgroup of Γ consisting of the automorphisms homotopic to the identity. Then $f \sim g$ if and only if $fg^{-1} \in \Gamma_{id}$. Thus $\mathrm{Aut}(\wedge V^{\leqslant M}, d)$ is isomorphic to the quotient $Iso_1(\wedge V^{\leqslant M}, d))/\Gamma_{id}$. This yields the equality

$$\mathrm{Aut}_{\#}^{N}(\wedge V, d) = \mathrm{Aut}_{\#}^{N+r}(\wedge V, d) \qquad \forall r \geqslant 1\,.$$

Since $\mathrm{Aut}_{\#}^{k}(\wedge V, d) \cong \mathrm{Aut}_{\#}^{k}(Z_0)$, ([**17**]), we obtain finally

$$\mathrm{Aut}_{\#}^{N} Z_0 = \mathrm{Aut}_{\#}^{N+r} Z_0 \qquad \forall r \geqslant 1$$

Suppose now that Z is a simply connected finite CW complex. Then $\mathrm{Aut}_\pi(Z) \subset \mathrm{Aut}_{\#}^{N}(Z)$ is an normal inclusion of finitely generated nilpotent groups ([**9**]) with same Malcev completions. It is then easy to see by induction on the nilpotency index that the quotient is a finite group. Therefore there is an integer $P \geqslant N$ such that $\mathrm{Aut}_\pi(Z) \cong \mathrm{Aut}_{\#}^{P}(Z)$. □

5. Proof of Theorem 2

The proof of Theorem 2 is based on a sequence of 4 lemmas.

Lemma 1. *There is a rational homotopy equivalence $K : W \to X$ where W is a finite CW complex.*

Henceforth we consider $K : W \to X$ as in lemma 1, and by Theorem 3, we fix an integer N such that $\mathrm{Aut}_{\#}^{N} X_0 = \mathrm{Aut}_\pi X_0$ and $H^{\geqslant N}(X; \mathbb{Q}) = 0$.

Lemma 2. *The morphism $\mathrm{Aut}_\pi W^{(n+1)} \xrightarrow{f \to f^{(n)}} \mathrm{Aut}_\pi W^{(n)}$ is injective and has a finite cokernel for each $n > N$.*

Lemma 3. *The morphism $\mathrm{Aut}_\pi X^{(n+1)} \xrightarrow{f \to f^{(n)}} \mathrm{Aut}_\pi X^{(n)}$ has a finite kernel and a finite cokernel for $n \geqslant N$.*

Theorem 2 follows from previous lemmas and the following lemma due to McGibbon and Möller ([**12**], Lemma 3).

Lemma 4. *Let $G_n \xrightarrow{q_n} G_{n-1}$ be a tower of groups. Suppose that $q_n(G_n)$ has finite index into G_{n-1}, for $n > N$. Then $\varprojlim{}^1 G_n = \{*\}$ if and only if $\varprojlim G_n \to G_N$ has a finite cokernel.*

Proof of Lemma 1.
Suppose that $H^{>M}(X; \mathbb{Q}) = 0$. We consider the cellular chain complex of X,

$$C_{M+1}(X; \mathbb{Z}) \xrightarrow{\partial} C_M(X; \mathbb{Z}) \to C_{M-1}(X; \mathbb{Z}) \to \cdots$$

where each $C_k(X;\mathbb{Z})$ is freely generated by the k-cells of X as a $\mathbb{Z}$-module. In particular, $C_{M+1}(X;\mathbb{Z}) = \oplus_{i=1}^{r}\mathbb{Z}e_i$. By an elementary argument in linear algebra, there exists elements $i_1,\cdots,i_s \in \{1,\cdots,r\}$ such that $\partial\otimes\mathbb{Q}$ sends isomorphically $\oplus_{j=1}^{s}\mathbb{Q}e_{i_j}$ to the image of $\partial\otimes\mathbb{Q}$ in $C_M(X;\mathbb{Q})$.

Denote by W the subcomplex of X obtained by adding to the M-skeleton of X the $(M+1)$-cells $e_{i_1},\cdots,e_{i_s}$. The injection $K: W\to X$ induces an isomorphism in the homology of the rational cellular chain complexes and is therefore a rational homotopy equivalence. □

Proof of Lemma 2. Since $n > N > \dim W$, and since $W^{(n)} = W\cup_{\varphi_\alpha} e^{n_\alpha}$, $n_\alpha > n+2$, the restriction maps

$$\begin{cases} \mathrm{Aut}^n_\# W^{(n+1)} \to \mathrm{Aut}^n_\# W^{(n)} \\ \mathrm{Aut}W \to \mathrm{Aut}W^{(n+1)} \to \mathrm{Aut}W^{(n)} \end{cases}$$

are isomorphisms. This gives directly the first part of the lemma.

Consider the following diagram obtained by restriction and rationalization

$$\begin{array}{ccccc} \mathrm{Aut}^{n+1}_\# W^{(n+1)} & \overset{\rho}{\hookrightarrow} & \mathrm{Aut}^n_\# W^{(n+1)} & \xrightarrow{rep} & \mathrm{Aut}\pi_{n+1}(W) \\ {\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle r} & & \\ \mathrm{Aut}^{n+1}_\# W_0^{(n+1)} & = & \mathrm{Aut}^n_\# W_0^{(n+1)} & \xrightarrow{rep_0} & \mathrm{Aut}\pi_{n+1}(W)\otimes\mathbb{Q} \end{array}$$

By hypothesis, (cf. Theorem 3), the representation map rep_0 is trivial, the image of the representation $rep : \mathrm{Aut}^n_\# W^{(n+1)} \to \mathrm{Aut}\,\pi_{n+1}(W)$ is finite and thus its kernel $\mathrm{Aut}^{n+1}_\# W^{(n+1)}$ is a subgroup of finite index.

Therefore the composite

$$\mathrm{Aut}_\pi W^{(n+1)} = \mathrm{Aut}^{n+1}_\# W^{(n+1)} \hookrightarrow \mathrm{Aut}^n_\# W^{(n+1)} \to \mathrm{Aut}^n_\# W^{(n)} = \mathrm{Aut}_\pi W^{(n)}$$

has a finite cokernel. □

Proof of Lemma 3. We use the same argument as in ([**12**]). We define

$$\Delta(f^{(n)}) = \{(\alpha,\beta)\in \mathrm{Aut}_\pi W^{(n)} \times \mathrm{Aut}_\pi X^{(n)} \mid f^{(n)}\alpha = \beta f^{(n)}\}$$

Then $\Delta(f^{(n)})$ is a group and since $f : W \to X$ is a rational equivalence, by a result of Wilkerson ([**19**]) the projections from $\Delta(f^{(n)})$ to $\mathrm{Aut}_\pi W^{(n)}$ and to $\mathrm{Aut}_\pi X^{(n)}$ have finite kernels and cokernels.

Now, using Lemma 2, a simple diagram chasing yields the result.

$$\begin{array}{ccccc} \mathrm{Aut}_\pi W^{(n+1)} & \leftarrow & \Delta(f^{(n+1)}) & \rightarrow & \mathrm{Aut}_\pi X^{(n+1)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}_\pi W^{(n)} & \leftarrow & \Delta(f^{(n)}) & \rightarrow & \mathrm{Aut}_\pi X^{(n)} \end{array}$$

□

6. Example of a space X with $\mathbf{SNT}_\pi(X) \neq \{[X]\}$

Consider the CW complex

$$Z = (S^2_a \times S^2_b \times \bigvee_{p\geqslant 1} S^{6p}) \vee (\bigvee_{n\geqslant 1} S^4_n) \cup_{(\gamma_{n,p})} \bigcup_{\substack{p\geqslant 1\\ n>p}} e_n^{4+6p},$$

where $\gamma_{n,p}$ is the Whitehead bracket $[S^4_n, S^{6p}]$.

The space T is constructed as a deformation of the space Z.

$$T = \big((S_a^2 \vee S_b^2) \times (\vee_{p\geqslant 1} S^{6p})\big) \cup_{[S_a^2, S_b^2]} e^4 \vee (\bigvee_{n\geqslant 1} S_n^4) \cup_{(\gamma_{n,p})} \bigcup_{\substack{p\geqslant 1 \\ n>p}} e_n^{4+6p} \cup_{\overline{\omega}_p} \bigcup_{p\geqslant 1} e^{4+6p},$$

where $\overline{\omega}_p = \omega_p + \sum_{j=1}^{p} [S^{6p}, S_j^4]$. Here the element ω_p is a triple Whitehead product in the π_{6p+3} of the fat wedge $T(S_a^2, S_b^2, S^{6p})$.

The spaces X and Y are respectively the rationalizations of Z and T. We show that X and Y do not have the same homotopy type, and that there exists a sequence of homotopy equivalences between the n^{th} skeleton

$$f_n : X_n \to Y_n$$

such that

$$\pi_r(f_n) = \pi_r(f_m),$$

when $r \leqslant n \leqslant m$. In particular f_n induces an homotopy equivalence $f_n^{(n-1)} : X^{(n-1)} \to Y^{(n-1)}$, so that $\mathrm{SNT}_\pi(X) \neq \{X\}$.

By Quillen rational homotopy theory ([**14**]), there is an equivalence of homotopy category between the homotopy category of simply connected CW complexes and the homotopy category of connected differential graded Lie algebras. It thus suffices to prove the corresponding results into the category of differential graded Lie algebras.

The Quillen models $\mathcal{L}_X$ and $\mathcal{L}_Y$ of X and Y are respectively given by :

$$\mathcal{L}_X = (\mathbb{L}(a, b, c, x_i, y_i, v_i, w_i, t_i, z_{i,j}\,; \quad i \geqslant 1, j > i), d),$$

with $|a| = |b| = 1$, $|c| = 3$, $|x_i| = 6i - 1$, $|y_i| = 3$, $|z_{i,j}| = 6i + 3$, $|v_i| = 6i + 1$, $|w_i| = 6i + 1$, $|t_i| = 6i + 3$, $d(c) = [a, b]$, $d(x_i) = 0$, $d(y_i) = 0$, $d(z_{i,j}) = [x_i, y_j]$, $d(v_i) = [a, x_i]$, $d(w_i) = [b, x_i]$, $d(t_i) = \omega_i = [v_i, b] + [c, x_i] + [w_i, a]$.

$$\mathcal{L}_Y = (\mathbb{L}(a, b, c, x_i, y_i, v_i, w_i, t_i, z_{i,j}\,; \quad i \geqslant 1, j > i), d),$$

with same degrees and differentials, except that

$$d(t_i) = \omega_i + \sum_{j=1}^{i} [x_i, y_j].$$

Suppose that $\varphi : \mathcal{L}_X \to \mathcal{L}_Y$ is a quasi-isomorphism and denote by $\mathbb{L}^p$ the ideal of $\mathcal{L}_X$ or $\mathcal{L}_Y$ generated by iterated brackets of lenght $\geqslant p$.

By composition with an automorphism of $\pi_2(X) \otimes \mathbb{Q} = a\mathbb{Q} \oplus b\mathbb{Q}$ we can suppose that $\varphi(a) = a$ and $\varphi(b) = b$. Then we have

$$\begin{cases} \varphi(c) = c + \sum_{k\geqslant 1} \gamma_k y_k \\ \varphi(x_i) = \mu x_i \mod \mathbb{L}^2, \mu \neq 0 \\ \varphi(v_i) = \mu v_i \mod \mathbb{L}^2 \\ \varphi(w_i) = \mu w_i \mod \mathbb{L}^2 \\ \varphi(t_i) = \delta t_i + \sum_{j>i} \beta_{i,j} z_{i,j} \end{cases}$$

and,

$$(d\varphi - \varphi d)(t_i) = (\delta - \mu)\omega_i + \sum_{j=1}^{i} \delta[x_i, y_j] + \sum_{j>i} \beta_{i,j}[x_i, y_j] - \sum_{k\geqslant 1} \gamma_k \mu [y_k, x_i] \mod \mathbb{L}^3.$$

Therefore $\gamma_k \neq 0$ for $k \leqslant i$. Since this has to be true for any $i \geqslant 1$, we have

$$\varphi(c) = c + \sum_{k \geqslant 1} \gamma_k y_k$$

with all $\gamma_k \neq 0$, which is impossible.

On the other hand, by taking $\mu = \delta = \gamma_k = 1$ for $k \leqslant p$, and $\gamma_k = 0$ for $k > p$, we obtain an homotopy equivalence $f_n : X^n \to Y^n$ with $n = 4 + 6p$.

Remark now that the composition of injections

$$\big((S^2 \times S^2 \times (\vee_{p \geqslant 1} S^{6p})) \vee (\vee_{n \geqslant 1} S^4_n)\big)_0 \xrightarrow{i} X \to \big([(S^2 \times S^2) \vee (\vee_{n \geqslant 1} S^4_n)] \times [\vee_{p \geqslant 1} S^{6p}]\big)_0$$

induces a surjective map on homotopy groups. It results then from results of Anick ([**2**]) that $\pi_*(i)$ is a surjective map, and using this it is clear that $\pi_r(f_m) = \pi_r(f_n)$ when $r \leqslant m \leqslant n$.

Theorem 4. *Let X be the rationalizations of the CW complex*

$$Z = (S^2_a \times S^2_b \times \bigvee_{p \geqslant 1} S^{6p}) \vee (\bigvee_{n \geqslant 1} S^4_n) \cup_{(\gamma_{n,p})} \bigcup_{\substack{p \geqslant 1 \\ n > p}} e_n^{4+6p} ,$$

where $\gamma_{n,p}$ is the Whitehead bracket $[S^4_n, S^{6p}]$. Then

$$\mathrm{SNT}_\pi(X) \cong \varprojlim{}^1 \mathrm{Aut}_\pi X^{(n)} \cong \prod_{n \geqslant 1} \mathbb{Q} / \bigoplus_{n \geqslant 1} \mathbb{Q}.$$

Proof. Since $\pi_*(\Omega X) \otimes \mathbb{Q}$ is generated as an algebra by the elements a, b, x_i and y_i, $i \geqslant 1$, with the notations of section 4, the restriction map Aut $X \to \varprojlim G_n$ is surjective. Therefore $\mathrm{SNT}_\pi(X) = \varprojlim{}^1 \mathrm{Aut}_\pi X^{(n)}$.

Denote by $\mathcal{L}_X^{(n)}$ the quotient of $\mathcal{L}_X$ by the ideal $\mathcal{J} = (\mathcal{L}_X)^{>n+1} \oplus S$, where $S = d(\mathcal{L}_X^{n+2})$. Clearly Aut ${}_\pi X^{(n)}$ can be identified with the group of homotopy classes of automorphisms φ of $\mathcal{L}_X^{(n)}$ such that φ is the identity on the vector space generated by the elements a, b, x_i, y_i, $i \geqslant 1$.

To each $\varphi \in$ Aut ${}_\pi X^{(n)}$, $n \geqslant 4$, we can associate the element $\varphi(c) - c \in \oplus_{i \geqslant 1} \mathbb{Q} y_i$. We define in this way a linear map Aut ${}_\pi X^{(n)} \to \pi_3(\Omega X)$ whose kernel and cokernel are respectively denoted by K_n and I_n;

$$0 \to K_n \to \text{ Aut } {}_\pi X^{(n)} \to I_n \to 0 .$$

For $\varphi \in K_n$ and $n \geqslant 6i + 3$, the elements $\varphi(v_i) - v_i$ and $\varphi(w_i) - w_i$ belong to the sub Lie algebra L generated by a, b and the x_j. This define a map

$$\psi : K_n \to \bigoplus_{i \,|\, 6i+3 \leqslant n} (L_{6i+3} \oplus L_{6i+3}) ,$$

and this gives a short exact sequence

$$0 \to K'_n = Ker\psi \to K_n \xrightarrow{\psi} \bigoplus_{i \,|\, 6i+3 \leqslant n} (L_{6i+3} \oplus L_{6i+3}) \to 0 .$$

Since the induced map $\psi : \varprojlim K_n \to \bigoplus_{i \geqslant 1} (L_{6i+3} \oplus L_{6i+3})$ is surjective, we have

$$\varprojlim{}^1 K_n \cong \varprojlim{}^1 K'_n .$$

Now remark that the sequence $K'_n \to K'_{n-1}$ is a sequence of surjective maps. It follows that $\varprojlim{}^1 K'_n = 0$. This gives the isomorphism

$$\varprojlim{}^1 \mathrm{Aut}\,_\pi X^{(n)} \cong \varprojlim{}^1 I_n\,.$$

Now, clearly, $I_{6i+4} = \bigoplus_{j>i} \mathbb{Q}y_j$. We consider then the short exact sequence of towers

$$0 \to \bigoplus_{j\leqslant i} \mathbb{Q}y_j \to \bigoplus_{j\geqslant 1} \mathbb{Q}y_j \to \bigoplus_{j>i} \mathbb{Q}y_j \to 0\,.$$

The associated six-terms exact sequence reduces to the exact sequence

$$0 \to \bigoplus_{j\geqslant 1} \mathbb{Q}y_j \to \prod_{j\geqslant 1} \mathbb{Q}y_j \to \varprojlim{}^1 I_n \to 0\,.$$

□

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