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ERRATUM TO "ON SPACES OF THE SAME STRONG n-TYPE" [HHA, V. 1 (1999) NO. 10, PP. 205-217]

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(*communicated by Lionel Schwartz*)

On the Arkowitz-Maruyama conjecture.

The main purpose to this short note is to make a correction to one of the result of the article : *On spaces of the same strong n-type* which has been published in [**7**]. We want to thank Ken-Ichi Maruyama who kindly reports to us our mistake. We also add some comments about the Arkowitz-Maruyama conjecture.

1) The AM-conjecture.

Let (X, x_0) be a based path connected space and let $\mathrm{Aut}X$ be the group of based homotopy classes of homotopy self-equivalences of (X, x_0). We denote by $\mathrm{Aut}^n_\pi X$ the subgroup of homotopy classes that induce the identity on the homotopy groups $\pi_i(X, x_0)$ for $i \leqslant n$. Then we obtain the normal series

$$\mathrm{Aut}X \supset \mathrm{Aut}^1_\pi X \supset ...\mathrm{Aut}^{n-1}_\pi X \supset \mathrm{Aut}^n_\pi X \supset ...$$

and we denote by $\mathrm{Aut}_\pi Z$ the inverse limit:

$$\lim_{\leftarrow} \mathrm{Aut}^n_\pi X \cong \bigcap_{n\geqslant 1} \mathrm{Aut}^n_\pi X\,.$$

M. Arkowitz and K.I. Maruyama, [**2**] have conjectured that:

A-M. conjecture. *Let Z be a simply connected finite complex. There exists an integer N such that the natural monomorphism*

$$\rho_N : Aut_\pi Z \to Aut^N_\pi Z$$

is an isomorphism, ie. $Aut^N_\pi Z = Aut^n_\pi Z$ for all $n \geqslant N$.

At our knowledge, the AM-conjecture is still unsolved for general complexes. It is trivially true for any finite complex Z which admits a finite Postnikov decomposition. In this case, if $Z^{(n)}$ denotes the n^{th}-Postnikov section of $Z = Z^{(k)}$ then for $n \geqslant k$

$$\mathrm{Aut}Z = \mathrm{Aut}Z^{(n)} = \mathrm{Aut}^n_\pi Z^{(n)} \cong \mathrm{Aut}_\pi Z\,.$$

The conjecture is also known for products of spheres [**2**] and if Z is an H_0-space [**6**].

2) The localization conjecture.

Now recall that if $n \geqslant \dim Z$ then $\mathrm{Aut}^n_\pi Z$ is a finitely presented nilpotent group [**3**]. Let P be any set of prime numbers. Given a localization $l_P : Z \to Z_P$, the natural homomorphism $l_P : \mathrm{Aut}^n_\pi Z \to \mathrm{Aut}^n_\pi(Z_P), [f] \mapsto [f_P]$ and the localization homomorphism $\lambda_p : \mathrm{Aut}^n_\pi Z \to$

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$(\mathrm{Aut}^n_\pi Z)_P$ coincides, up to a natural isomorphism [**4**]:

$$\begin{array}{ccc} & \mathrm{Aut}^n_\pi Z & \\ \lambda^n_P \swarrow & & \searrow l^n_P \\ (\mathrm{Aut}^n_\pi Z)_P & \overset{\theta^n_P}{\underset{\cong}{\longrightarrow}} & \mathrm{Aut}^n_\pi(Z_P) \end{array}$$

Thus, each $\mathrm{Aut}^n_\pi(Z_P)\,, n \geqslant \dim Z$ is P-local and the group $\mathrm{Aut}_\pi(Z_P) = \lim\limits_{\leftarrow} \mathrm{Aut}^n_\pi(Z_P)$ is also P-local. Universal property of localization defines the natural homomorphisms θ_P in the diagram below:

$$\begin{array}{ccc} & \mathrm{Aut}_\pi Z & \\ L_P \swarrow & & \searrow \lim\limits_{\leftarrow} l^n_P = \phi_P \\ (\mathrm{Aut}_\pi Z)_P & \overset{\theta_P}{\longrightarrow} & \mathrm{Aut}_\pi(Z_P) \end{array}$$

Localization does not necessarily respect inverse limit, nonetheless we conjecture:

P-LOCAL CONJECTURE. *Let Z be a nilpotent finite complex. Then the natural map ϕ_P : $Aut_\pi Z \to Aut_\pi(Z_P)$ is a P-localization, ie. θ_P is an isomorphism.*

As usual we denote by Z_0, instead of $Z_\emptyset$, the rationalization of the space Z and more generally the subscript $_\emptyset$ is replaced by subscript $_0$. In a recent preprint, [**5**], K-I. MARUYAMA proves:

If X is a finite nilpotent complex and if $Aut_\pi(X_0) = \{1\}$ then $Aut_\pi X_P \cong (Aut_\pi X)_P$ for any set of primes P.

3) Equivalence of the AM-conjecture and of the $\emptyset$-local conjecture.

In [**7**]-(first part of theorem 3), we have proved:

THEOREM A. *Let Z be a simply connected CW complex of finite type and let Z_0 its rationalization. If $H^{>M}(Z;\mathbb{Q}) = 0$ for some M then there exists an integer N such that the natural map $\rho^N_0 : Aut_\pi(Z_0) \to Aut^N_\pi(Z_0)$ is an isomorphism.*

Recently K-I. MARUYAMA [**5**] has proved theorem A for finite nilpotent complexes.

A consequence of theorem A is

THEOREM B. *Let Z be a simply connected finite complex. The space Z satisfies the AM-conjecture iff Z satisfies the $\emptyset$-conjecture.*

Proof. Let N as in theorem A and consider the commutative diagram,

$$\begin{array}{ccc} (\mathrm{Aut}_\pi Z)_0 & \overset{\theta_0}{\longrightarrow} & \mathrm{Aut}_\pi(Z_0) \\ \left(\rho^N\right)_0 \downarrow & & \cong\downarrow \rho^N_0 \\ \left(\mathrm{Aut}^N_\pi Z\right)_0 & \overset{\theta^n_0}{\underset{\cong}{\longrightarrow}} & \mathrm{Aut}^N_\pi(Z_0) \end{array}$$

If the AM-conjecture holds then $\left(\rho^N\right)_0$ is an isomorphism and so is θ_0. Thus the $\emptyset$-conjecture is satisfied. Conversely, suppose that θ_0 is an isomorphism then the monomorphism ρ^N has finite cokernel $C^N(Z)$. If $C^N(Z) = C^n(Z)$ for all $n \geqslant N$ then $\mathrm{Aut}^N_\pi Z = \mathrm{Aut}^n_\pi Z$ and the AM-conjecture is proved. If for some $N_1 \geqslant N$, $C^N(Z) \neq C^{N_1}(Z)$ then $C^{N_1}(Z)$ is strictly included in $C^N(Z)$. Again with N_1 playing the role of N the AM-conjecture is satisfied or there exists N_2 such that ... and so on. At the end we have a sequence $N_1, N_2, ..., N_k$ with $C^{N_k}(Z) = \{1\}$ and the AM-conjecture is proved for Z.

4) Composition of homotopy classes.

THEOREM C. *The AM-conjecture is true for simply connected finite complexes Z satisfying: for each element $[a] \in \pi_m(Z)$ there exists a non torsion element $[b] \in \pi_r(Z)$ and a continuous map $g : S^m \to S^r$ such that $[bg] = [a]$.*

Proof. Let us denote by ${\rm Aut}^n{}_{\pi/\tau}Z$ the subgroup of ${\rm Aut}Z$ which consists of elements inducing the identity on each quotient $\pi_i(Z)/\tau(\pi_i(X))\,, i \leqslant n$ where $\tau(\pi_i(Z))$ denotes the torsion subgroup of $\pi_i(Z)$. By our assumption,

$$\mathrm{Aut}^n_\pi Z = \mathrm{Aut}^n_{\pi/\tau} Z\,.$$

This subgroup $\mathrm{Aut}_{\pi/\tau}Z$ have been considered in [**5**]. I.K. MARUYAMA has observed that these groups are not nilpotent in general and proves (Th. 1.2) that the natural map

$$\rho^N_\tau : \mathrm{Aut}_{\pi/\tau}Z \to \mathrm{Aut}^N_{\pi/\tau}Z$$

is an isomorphism for some N. Then theorem C is a consequence of theorem A and of the following commutative diagram:

$$\begin{array}{ccccc}
(\mathrm{Aut}_\pi Z)_0 & = & \big(\mathrm{Aut}_{\pi/\tau}Z\big)_0 & \xrightarrow{\theta_{0,\tau}} & \mathrm{Aut}_{\pi/\tau}(Z_0) \\
\big(\rho^N\big)_0 \downarrow & & \downarrow \big(\rho^N_\tau\big)_0 & & \cong\downarrow \rho^N_0 \\
\big(\mathrm{Aut}^N_\pi Z\big)_0 & = & \Big(\mathrm{Aut}^N_{\pi/\tau}Z\Big)_0 & \xrightarrow[\cong]{\theta^N_{0,\tau}} & \mathrm{Aut}^N_{\pi/\tau}(Z_0)\,.
\end{array}$$

5) Correction to the last assertion of the theorem 3 in [7].

The proof of the last assertion of theorem 3 in [**7**]:

"*Moreover if $H^{>M}(Z;\mathbb{Z}) = 0$, then there exists an integer N such that the natural map $Aut_\pi Z \to Aut^N_\pi Z$ is an isomorphism*"

is false, since in fact we have assumed the $\emptyset$-local conjecture to be true in our proof.

6) The Ω-conjecture.

Denote by $\mathrm{Aut}^n_\Omega X$ the group of homotopy classes of self-homotopy equivalences f of X such that the restriction of Ωf to $(\Omega X)^{(n-1)}$ is homotopic to the identity.

Clearly, each $\mathrm{Aut}^n_\Omega X$ is a subgroup of $\mathrm{Aut}^n_\pi X$.

If Z is a finite simply connected complex then $\mathrm{Aut}^n_\pi Z\,, n \geqslant \dim Z$ is a finitely generated nilpotent group and thus $\mathrm{Aut}^n_\Omega Z$ is a nilpotent group for $n \geqslant \dim Z$.

We denote by $\mathrm{Aut}_\Omega X$ the inverse limit :

$$\varprojlim \mathrm{Aut}^n_\Omega X \cong \bigcap_{n\geqslant 2} \mathrm{Aut}^n_\Omega X\,.$$

Ω-CONJECTURE. *Let Z be a simply connected finite complex. There exists an integer N such that the natural map*

$$\rho^N_\Omega : Aut_\Omega Z \to Aut^N_\Omega Z$$

is an isomorphism.

If Z is a finite simply connected complex, the natural injections $\mathrm{Aut}^n_\Omega Z \hookrightarrow \mathrm{Aut}^n_\pi Z$ induce isomorphisms

$$(\mathrm{Aut}^n_\Omega Z)_0 \cong (\mathrm{Aut}^n_\pi Z)_0\,,$$

for any $n \geqslant \dim Z$. Indeed, if $[f] \in \mathrm{Aut}_\pi Z$ there are only finitely many obstructions for $[f]$ being in $\mathrm{Aut}_\Omega Z$.

We do not know if there exists a simply connected finite complex Z such that $(\mathrm{Aut}_\Omega Z)_0 \ncong (\mathrm{Aut}_\pi Z)_0$. Clearly, we obtain:

THEOREM D. *Let Z be a simply connected finite complex such that*

$$(Aut_\Omega Z)_0 \cong (Aut_\pi Z)_0 \ .$$

Then Z satisfies the AM-conjecture iff Z satisfies Ω-conjecture.

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