Homology, Homotopy and Applications, vol.3(2), 2001, pp. vii–xv

PROBLEM SESSION

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These problems were submitted by the participants in the workshop on equivariant homotopy theory held at Stanford University during the period August 23-27, 2000. They reflect the wide range of mathematics which is included within equivariant homotopy theory. We hope they will be useful for researchers in the field, as well as for graduate students entering the subject.

1. Landweber cohomology

Mike Hopkins and Haynes Miller

Long ago Pierre Cartier asked whether one could speak of "cohomology with coefficients in a formal group". To be more precise we make a definition.

Definition. A *periodic ring spectrum* is a homotopy commutative and associative ring spectrum whose homotopy ring vanishes in odd dimensions and contains a unit in degree 2.

For such a theory E, the Atiyah-Hirzebruch spectral sequence collapses for any spectrum with evenly graded cohomology. It follows that $E^0(\mathbb{C}P^\infty)$ is a power series ring over E^0 on one generator, and the H-space structure on $\mathbb{C}P^\infty$ determines a formal group G_E over E^0, and an isomorphism from the E^0-module ω_G of invariant differentials to E^{-2}.

Cartier was asking whether there was a construction of a periodic ring spectrum E from a formal group G/R, together with isomorphisms $E^0 \cong R$, $G_E \equiv G$. One scenario might be to produce a symmetric monoidal category of "representations" of the formal group, and consider the associated spectrum. Any such construction should provide a functor from formal groups to spectra and not just to the homotopy category of spectra.

We still know virtually nothing about this question. Peter Landweber gave a condition on a formal group G/R which guarantees that a corresponding spectrum may be constructed algebraically out of MU in the homotopy category. The connection between MU and formal groups of course goes back to Quillen's observation that MU^* is the Lazard ring, supporting the universal formal group law.

To recall Landweber's condition, pick a coordinate t for the formal group so the endomorphism $a \mapsto na$ of G is represented by a power series $[n](t) \in R[[t]]$. Write $I_{p,n}(G/R)$ for the ideal in R of coefficients in $[p](t)$ through that of t^{p^n-1}. It is easy to check that these ideals are independent of choice of coordinate, and that for each $n \geqslant 0$ there is an element $v_n \in R$ such that

$$I_{p,n}(G/R) = I_{p,n-1}(G/R) + v_n R.$$

The formal group G/R is *regular* if for each prime p the sequence $v_0,\ v_1,\ \ldots$ is a regular sequence. This condition is clearly independent of choice of v_n's. Landweber's theorem, augmented by a recent theorem of Hovey and Strickland, is

Theorem *There is a fully faithful functor E from the category of regular formal groups to the category of periodic ring spectra, together with a natural isomorphism*

$$G_{E_{G/R}}/E^0_{G/R} \cong G/R.$$

Recall that one may "pull back" a formal group G/R along a ring homomorphism $f : R \to S$ to a formal group fG/S. Formal groups over arbitrary rings form a category **FG** in which a morphism from G/R to H/S is a ring homomorphism $R \to S$ together with an isomorphism $fG \to H$ of formal groups over S.

Here is an updated form of Cartier's question.

Conjecture. *The functor E lifts to a functor to the category of E_∞ ring spectra whose underlying ring spectra are periodic.*

We have no reason to expect a negative answer beyond the failure of current technique. We do have several examples of functors $\pi : \mathbf{C} \to \mathbf{FG}$ for which the composite $E\pi$ does lift.

Definition. A formal group G/R is a *universal deformation* if

(1) R is a complete local ring with maximal ideal m and perfect residue field K of characteristic p,

(2) G reduces to a formal group G_0 of finite height n over K, and

(3) G/R is regular and $I_{p,n}(G) = m$.

Let π be the inclusion of the universal deformations into all formal groups.

Theorem. (**Hopkins-Miller, Hopkins-Goerss**) *The composite $E\pi$ lifts to a functor to the category of periodic E_∞ ring spectra. Moreover, given universal deformations G/R and H/S, the functor establishes a homotopy equivalence from Hom$(G/R,\ H/S)$ to the subspace of E_∞ morphisms from $E_{G/R}$ to $E_{H/S}$ which induce local morphisms in π_0.*

The work of Hopkins and Miller on elliptic spectra gives an analogous lift from a suitable stack of elliptic curves.

2. Localization and completion theorems for MU_G.

John Greenlees and Peter May

If G is a compact Lie group, tom Dieck has defined equivariant homotopical complex cobordism $MU^*_G(\cdot)$: the representing spectrum is constructed using Thom spaces and it can be regarded as a stabilization of geometrical bordism which allows transversality. There are a number of basic questions about this theory and its relation to non-equivariant complex bordism.

Let $I = ker(MU^*_G \to MU^*)$ denote the augmentation ideal. Given any MU^*_G-module M, for any finitely generated ideal I' we may form the local cohomology $H^*_{I'}(M)$ in the sense of Grothendieck and the local homology $H^{I'}_*(M)$ and there is a comparison map associated to a containment $I' \subseteq I''$ of ideals.

The following problems were all answered affirmatively when the identity component G_e is a torus [J. P. C. Greenlees and J. P. May, *Localization and completion theorems for MU-spectra*, Annals of Math. 146 (1997), 509-544.]

Problem. *Is the local cohomology $H^*_{I'}(M)$ constant for all sufficiently large finitely generated ideals I'?*

Below we assume the answer is yes, and use H^*_I and H^I_* to denote the stable value.

Problem. *Is there a local cohomology theorem and a local completion theorem for equivariant bordism?*

This would give spectral sequences

(i) starting with local cohomology and terminating with the MU homology of BG^{ad} and

(ii) starting with local homology and terminating with the MU cohomology of BG

(Here BG^{ad} denotes the Thom complex of the vector bundle over BG determined by the adjoint representation of G.)

Problem. Is $H_0^I(MU_G^*) = (MU_G^*)^\wedge_I$ and is $H_i^I(MU_G^*) = 0$ for $i > 0$?

3. Equivariant formal group laws

John Greenlees

If G is a compact Lie group, a complex orientation on a G-equivariant cohomology theory $E^*_G(\cdot)$ is a ring map $MU \to E$ of G-spectra. If G is abelian it is known how to express this in terms of an element $y(\varepsilon) \in E^*_G(BU(1))$ as usual, where ε denotes the one dimensional trivial representation of G. A G-equivariant formal group law is an algebraic construct designed to capture the formal properties of the system of topological rings $E^*_G(BU(n))$ for $n \geqslant 0$, where $BU(n)$ is the classifying G-space for n-dimensional equivariant complex bundles, together with their universal Euler class $y_n(n\varepsilon)$. The definition includes the structure coming from direct sum, tensor product and exterior powers. If G is abelian, the entire structure is determined by $R = E^*_G(BU(1))$ and $y(\varepsilon) = y_1(\varepsilon)$, and may be formulated as follows, see [M. M. Cole, J. P. C. Greenlees, I. Kriz, *Equivariant formal group laws*, Proc. LMS 81(2000), 355-386]

If A is a finite abelian group, an *A-equivariant formal group law* over a commutative ring k is a complete topological Hopf k-algeb ra R with a homomorphism $\theta : R \to k^{A^*}$ of Hopf k-algebras whose kernel defines the topology together with an element $y(\varepsilon) \in R$ which is (i) regular and (ii) generates the kernel of the εth component, θ_ε of θ.

Problem. *Does every complex oriented cohomology theory give an equivariant formal group law, and for which G is there a universal G-Lazard ring?*

Problem. *For which G is MU^*_G in even degrees, and when is it the G-Lazard ring?*

For further details, see the article in these proceedings by J. P. C. Greenlees.

4. The equivariant Conner-Floyd-Sullivan isomorphisms

John Greenlees, Ib Madsen

The classical Conner-Floyd-Sullivan isomorphisms relate unitary and symplectic bordism to K-theory, namely:

$$\Omega_*^U(X) \otimes_{\Omega_*^U} K_* = K_*(X)$$

$$\Omega_*^{Sp}(X) \otimes_{\Omega_*^{Sp}} KO_* = KO_*(X),$$

where the action of $\Omega_*^U = \Omega_*^U(pt)$ on $K_* = K_*(pt)$ is via the K-theory Thom class of MU, and similarly for the symplectic case.

[P. E. Conner, E. E. Floyd, *The relation of cobordism to K-theories*, LNM vol. 28 (1966), Springer]. There is a similar theorem for spin bordism.

$$\Omega^{\mathrm{Spin}_*}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} KO_* = KO_*(X).$$

[M. Hopkins, M. Hovey, *Spin cobordism determines real K-theory*, Math, Z. 210, 181-196 (1992)]

In the equivariant setting where a finite group or more generally a compact Lie group G acts, one lacks tranversality in general, and there is a difference between the geometrically defined theories and the homotopy theoretically defined ones. The latter are denoted $MU_*^G(X)$, $M\mathrm{Spin}_*^G(X)$, etc. The G-version of the unitary Conner-Floyd-Sullivan isomorphism holds in the form

$$MU_*^G(X) \otimes_{MU_*^G} K_*^G = K_*^G(X).$$

[S. Costenoble, *The equivariant Conner-Floyd isomorphism for general compact Lie groups*, Trans. Amer. Math. Soc. 304, 1987,no.2, 801-818]

[C. Okonek, *Der Conner-Floyd isomorphisms f ur abelsches gruppen*, Math. Z. 179 (1982) 210-212]

Problem. *Is*

$$M\mathrm{Spin}_*^G(X) \otimes_{M\mathrm{Spin}_*^G} KO_*^G = KO_*^G(X),$$

and is there a similar formula involving $M\mathrm{Sp}_*^G(-)$*?*

One may ask the similar questions for the equivariant geometric bordism theories, see e.g. [I. Madsen, *Geometric equivariant bordism and K-theory*, Topology 25 (1986), 217-227].

5. Ring structures on spectra associated with complex bordism

J. Peter May

Let E be one of the spectra usually denoted BP, $BP\langle n\rangle$, $E(n)$, cf. [D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York, 1986].

Problem. *Are the spectra above E_∞ ring spectra?*

Problem. *Calculate the space $A_\infty(E,\ E)$ of A_∞-endomorphisms. Is it homotopically discrete?*

6. Equivariant surfaces and cobordism.

Ib Madsen, Mel Rothenberg

Let G be a finite group, and let Ω_2^G denote the geometric cobordism classes of oriented G-surfaces. There are partial calculations of Ω_2^G, e.g. for cyclic groups. The G-signature theorem of Atiyah and Bott gives strong relations on the local fixed point data.

Problem. *Determine Ω_2^G and relate the structure to the G-signature theorem.*

7. Equivariant infinite loop space theory.

J. Peter May

Let C_n be the little cubes operad in $\mathbb{R}^n$ with $C_n(k)$ the space of k little cubes in $\mathbb{R}^n$. Define for a pointed space X

$$C_nX = \coprod_{k=0}^{\infty} C_n(k) \times_{\Sigma_k} X^k / \approx$$

[J. P. May, *The geometry of iterated loop spaces*, LNM, Vol. 271 (1972)], and recall the "approximation theorem":

$$C_nX \simeq \Omega^n S^n X \quad (X \text{ connected})$$

Problem. *Let G be compact Lie, X a pointed G-space, and V a representation of G. Develop an approximation theorem for $\Omega^V S^V X$.*

There are two "recognition principles" for infinite loop spaces, namely Γ-spaces and C_∞-algebras [G. Segal,*Categories and cohomology theories*, Topology 13, 293-312 (1974)] and [op cit].

For $|G| < \infty$, there is a Γ_G-version of the recognition principle, [K. Shimakawa, Infinite loop G-spaces associated to monoidal G-graded categories, Publ. RIMS 25 (1989), 239-262]. A G-operad version for $|G| < \infty$ exists, but was never published.

Problem. *Develop a G recognition theorem for dimG > 0.*

Problem. *Develop a recognition principle for module spectra over a given E_∞ ring spectrum R and for E_∞ R-algebras.*

8. Group completions and spectra

Ralph Cohen

Let C_n be the little cubes operad, E a spectrum and let C_nE be the half smash product

$$C_nE = C_n(k)_+ \wedge_{\Sigma_k} E^{\wedge k}.$$

The spectrum E is a C_n-algebra if there is a map $C_nE \to E$ with obvious extra associative conditions.

Problem. *Suppose the spectrum E is a C_n-algebra. Define the notion of "group completion" E^+ and calculate its homotopy.*

9. The stable mapping class group.

Ib Madsen

Let $\Sigma_{g,1}$ be a genus g surface with one boundary circle, and $\mathrm{Diff}(\Sigma_{g,1})$ the group of oriented diffeomorphisms that keep the boundary fixed. The components of $\mathrm{Diff}(\Sigma_{g,1})$ are contractible, so

$$\mathrm{Diff}(\Sigma_{g,1}) \xrightarrow{\simeq} \pi_0\mathrm{Diff}(\Sigma_{g,1}) = \Gamma_{g,1}$$

is an equivalence. Adding a torus with two discs removed gives inclusions $\Gamma_{g,1} \subset \Gamma_{g+1,1}$ with limit $\Gamma_{\infty,1}$. The plus construction $B\Gamma^+_{\infty,1}$ is an infinite loop space. [U. Tillmann, *On the homotopy of the stable mapping class group*, Invent. Math 130 (1997), 257-275].

Let $L_s^\perp$ be the s-dimensional complex vector bundle over $\mathbb{C}P^s$, complementary to the canonical line bundle, and $\mathrm{Th}(L_s^\perp)$ its Thom space. Define an infinite loop space

$$\Omega^\infty\mathbb{C}P^\infty_{-1} = hocolim\,\Omega^{2s+2}\mathrm{Th}(L_s^\perp).$$

Conjecture. $\mathbb{Z} \times B\Gamma^+_{\infty,1} \simeq \Omega^\infty\mathbb{C}P^\infty_{-1}$.

A recent paper [I. Madsen, U. Tillmann, *The stable mapping class group and* $Q(\mathbb{C}P^\infty)$, Inventiones Math., (to appear)] gives supporting evidence for the conjecture. It constructs an infinite loop map from $\mathbb{Z} \times B\Gamma^+_{\infty,1}$ to $\Omega^\infty(\mathbb{C}P^\infty_{-1})$, shows that it is 2-connected and shows that all but one of the "Adams components" of $(\Omega^\infty\mathbb{C}P^\infty_{-1})^\wedge_p$ sits also in $(\mathbb{Z} \times B\Gamma^+_{\infty,1})^\wedge_p$, where p is odd.

10. A topological Rees algebra.

Gunnar Carlsson

For a $k[x]$-algebra A, equipped with a family of ideals $I_n \subset A$ with

$$I_n \cdot I_m \subseteq I_{n+m}$$

one can form the Rees subalgebra of the localization $A\left[\frac{1}{x}\right]$, where $\operatorname{Rees}(A, I_\bullet) = \sum_{k \geqslant 0} I_n/x^n \subseteq A\left[\frac{1}{x}\right]$.

Problem. *Is there an analogue of the Rees algebra in the category of ring spectra?*

The "group ring" $ku[G]$ and the fixed set ku^G of the G equivariant ku, with "powers" of the augmentation ideal as the ideals I_n should yield interesting examples.

11. Atiyah-Segal completion theorem for profinite groups.

Gunnar Carlsson

Let $G = \lim_{\leftarrow} G_k$ be a profinite group. Define its complex K-theory to be

$$K^*(BG) = \lim_{\substack{\longleftarrow \\ n}} \lim_{\substack{\longrightarrow \\ k}} K^*(BG_k^{(n)}),$$

where (n) indicates the n-skeleton. There is a spectral sequence

$$E_2^{*,*} = H^*_{\text{cont}}(G, K^*) \Rightarrow K^*(BG).$$

On the representation side, let $\tilde{R}_G$ be the representation ring Green functor, [A. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Annals of Math 102 (1975) 291-325],

$$\widetilde{R}_G(G/H) = R(H) \quad , \quad |G : H| < \infty,$$

where $R(H)$ is the usual complex representation ring. The augmentation $R(H) \to \mathbb{Z}$ defines an augmentation $\varepsilon : \widetilde{R}_G \to \widetilde{\mathbb{Z}}$ into the constant Green functor, $\widetilde{\mathbb{Z}}(G/H) = \mathbb{Z}$.

Define the "derived Mackey functor completion" as the total space of the cosimplicial Mackey functor:

$$\widetilde{\mathbb{Z}}_G^\bullet : \widetilde{\mathbb{Z}} \rightrightarrows \widetilde{\mathbb{Z}} \Box_{\widetilde{R}_G} \widetilde{\mathbb{Z}} \rightrightarrows , \ \widetilde{\mathbb{Z}} \Box_{\widetilde{R}_G} \widetilde{\mathbb{Z}} \Box_{\widetilde{R}_G} \widetilde{\mathbb{Z}} \ldots$$

Here the boxes denote derived versions of the box product of modules over Green functors, first defined by G. Lewis. [see S. Bouc, *Green Functors and G-sets*, LNM 1671, 1997]. We now define the derived completion to be

$$(\widetilde{R_G})^\wedge_\varepsilon = \operatorname{Tot}(\widetilde{\mathbb{Z}}_G^\bullet).$$

This is a Mackey functor, and it can be evaluated at the one point G-space. Define

$$\widetilde{R}(G)^\wedge_\varepsilon = (\widetilde{R}_G)^\wedge_\varepsilon(G/G).$$

This is a simplicial abelian group which is not in general homotopy discrete.

Conjecture.

(i) $K^0(BG) = \bigoplus_{k \geqslant 0} \pi_{2k} \widetilde{R}(G)^\wedge_\varepsilon$,

(ii) $K^1(BG) = \bigoplus_{k \geqslant 0} \pi_{2k+1} \widetilde{R}(G)^\wedge_\varepsilon$.

12. Equivariant cohomology

Gaunce Lewis

For a G-space X (G finite), and an abelian valued Mackey functor M, we let $H^G_*(X;M)$ and $H^*_G(X,M)$ be the Bredon homology and cohomology groups, [G. E. Bredon, *Equivariant cohomology theories*, LNM 34 (1967)] and [L.G. Lewis, J. P. May, J.E. McClure,*Ordinary $RO[G]$-graded cohomology*, Bull. AMS (1981), 128-130]. We consider both $H^*_G(-;M)$ and $H^G_*(-;M)$ to be $RO(G)$-graded. Let $\widetilde{A}_G$ be the Burnside ring Green functor,

$$\widetilde{A}_G(G/H) = A(H),$$

where $A(G)$ is the isomorphism classes of virtual finite G-sets, the Burnside ring.

For a finite dimensional G-representation V, $Gr_n(V)$ denotes the Grassmann manifold of n-dimensional subspaces of V with its induced action of G.

Problem. *Is $H^G_*(Gr_n(V);\ \widetilde{A}_G)$ finitely generated over $H^G_*(X, \widetilde{A}_G)$?*

13. Equivariant Moore spaces

John Greenlees

Let L be a Mackey functor, and suppose ML is a Moore space for it in the sense that

$$H^*_K(ML,\ \widetilde{A}_G) = L(G/K),$$

where $\widetilde{A}_G$ is the Burnside ring Mackey functor.

Problem. *Is L necessarily of finite cohomological dimension over $\widetilde{A}_G$?*

14. Continuity of (topological) Hochschild homology

Lars Hesselholt

Let S be a commutative ring and J an ideal in S. Let $\mathrm{HH}_*(S)$ and $\mathrm{THH}_*(S)$ be the Hochschild and topological Hochschild homology groups of S (with coefficient in S). There are maps

$$f^s_*\colon S/J^s \otimes_S \mathrm{HH}_*(S) \to \mathrm{HH}_*(S/J^s),$$
$$g^s_*\colon S/J^s \otimes_S \mathrm{THH}_*(S) \to \mathrm{THH}_*(S/J^s)$$

that defines a pro-system for varying s.

Problem. *Are $f_*^\bullet$ and $g_*^\bullet$ pro-isomorphisms?*

It follows from the isomorphism $\mathrm{HH}_1(S) \cong \Omega^1_{S/\mathbb{Z}}$ that the map $f_1^\bullet$ is a pro-isomorphism.

Let $\mathrm{TR}^n(S;p) = \mathrm{THH}(S)^{C_{p^{n-1}}}$ be the fixed set under the cyclic group action on $\mathrm{THH}(S)$ that exists by Connes' theory of cyclic sets, [L. Hesselholt, I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology 36 (1997), 29-101], and let $\mathrm{TR}^n_*(S;p) = \pi_* \mathrm{TR}^n(S;p)$. The components are the ring of Witt vectors of length n in S, $\mathrm{TR}^n_0(S;p) \cong W_n(S)$. Then, more generally, there are maps

$$g_*^{n,s} \colon W_n(S/J^s) \otimes_{W_n(S)} \mathrm{TR}^n_*(S;p) \to \mathrm{TR}^n_*(S/J^s;p).$$

If the map $g_*^\bullet = g_*^{1,\bullet}$ is a pro-isomorphism then the same holds for the maps $g_*^{n,\bullet}$, for all $n \geqslant 1$.

15. Calculations in connective K-theory

Ib Madsen

At the conference Bob Bruner talked about some calculations of connective K-theory of finite groups. For permutation groups it is sometimes better to take them all under one and use Quillen's result that

$$\Omega B\left(\coprod_{n=0}^{\infty} B\Sigma_n\right) = Q(S^0).$$

The mod p homology $H_*(Q(S^0);\ \mathbb{Z}/p)$ was calculated by Kudo and Araki for $p = 2$ and Dyer-Lashof for odd p. The mod p periodic K-homology was calculated by L. Hodgkin, and more systematically by McClure in [R. R. Bruner, J. P. May, J. E. McClure, M. Steinberger, *H_∞ ring spectra and their applications*, LNM 1176, Springer (1980)].

Problem. *Calculate mod p connective K-homology of $Q(S^0)$ or more generally of $Q(X)$.*

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