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EQUIVARIANT FORMAL GROUP LAWS AND COMPLEX ORIENTED COHOMOLOGY THEORIES.

J.P.C.GREENLEES

(*communicated by Gunnar Carlsson*)

Abstract

The article gives an introduction to equivariant formal group laws, and explains its relevance to complex oriented cohomology theories in general and to complex cobordism in particular.

Contents

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1. Introduction

1.A. Purpose.

This article is designed to give an introduction to the notion of an equivariant formal group law, and to explain its relevance to complex oriented cohomology theories in general and to complex cobordism in particular. Historically, the definition came directly from topology, but only as the algebraic picture emerged did the notion seem natural. The extraordinary effectiveness of formal groups in non-equivariant topology has become so familiar that we barely need to explain the purpose of the enterprise. The idea is to construct algebraic models of certain basic structures in topology, and to exploit the rigidity of the algebra, both as a skeleton and as a tool in calculation. In particular we hope to have accurate algebraic models of K theory and cobordism. Since K theory encodes representation theory, a full realization of this dream would include a higher representation theory.

The article summarizes results of [**4, 14, 5, 16, 15, 21**] and other unpublished preprints, but with growing hindsight some of this appears in rather a new light. In any case, the account is arranged rather differently from [**4**], and it contains a number of new results and examples. Wherever it makes statements more digestible we restrict to the case where the group is of prime order. Quite in accordance with the philosophy of algebraic topology, we have found that the topological constructions are clarified once the relevant algebra is distilled. In practice the distillation was greatly assisted by the topology, but exposition need not be historical, and we have arranged the article as if the algebra had already existed. We resisted the temptation to follow Adams's alternation of algebra and topology in [**1**]: in a written account we hope to achieve a similar effect with effective cross-referencing.

1.B. Organization.

Part 1 gives a purely algebraic account of A-equivariant formal group laws and Part 2 discusses complex oriented equivariant cohomology theories, identifying topological counterparts of various structures already seen in algebra. Finally, Part 3 gives a very brief discussion of the non-abelian case: N.P.Strickland and the author plan a more thorough and systematic account.

1.C. Notation

We are concerned with A-equivariant cohomology theories and A-equivariant formal groups, where A is an abelian compact Lie group.

First we take the dual group $A^* = \mathrm{Hom}(A, S^1)$, which is a finitely generated abelian group; we write $\alpha, \beta, \gamma, \ldots$ for its elements, and ϵ is the trivial representation, its identity.

We are also concerned with algebraic groups of various sorts. In Part 1 we write G for an algebraic group (a group object in a category of schemes), and $\mathbb{G}$ for a formal group (a group object in a category of formal schemes). In Section 12 and Part 3 we use G for a compact Lie group.

Part 1. Equivariant formal group laws.

In Part 1 we give an algebraic introduction to equivariant formal group laws, postponing application to topology to Parts 2 and 3. We have used the language of algebraic geometry for motivation, but most of the development is in terms of representing rings of functions as this is more familiar (to topologists) from the non-equivariant case. Section 2 introduces equivariant formal group laws from several points of view. Sections 3 and 4 discuss the underlying ring structure of an equivariant formal group law and Euler clases. Sections 6, 7 and 8 describe examples: Euler-local, Euler-complete, multiplicative and rational.

2. Algebraic introduction to equivariant formal groups.

We begin with an informal motivation of the definition, before formalizing it as Definition 2.1. We then return to discuss the additive structure, the group scheme point of view and the universal ring.

2.A. Multiplication in a neighbourhood of a discrete subgroup.

This subsection is intended to motivate the formal definition of an equivariant formal group (Definition 2.1): it is informal, and some may prefer to skip to Subsection 2.B

In all algebraic discussion we find it helpful to state things first in geometric terms, and then make it more explicit in terms of functions. The scheme-theoretic language is very suggestive, but some of the issues only become apparent when it is unpacked.

We confine ourselves throughout to formal groups which are one dimensional and commutative. Before giving the equivariant case we briefly recall the familiar, classical, non-equivariant case. The idea is that a formal group encodes the behaviour of an algebraic group in an infinitesimal neighbourhood of the identity. Suppose then that G is a group with multiplication $\mu : G \times G \longrightarrow G$. We can look at μ in a formal neighbourhood of the identity e, thought of as a tiny patch around e. Since G is smooth, the formal neighbourhood of e is isomorphic to a formal neighbourhood $\hat{\mathbb{A}}^1$ of 0 in affine space $\mathbb{A}^1$, but it inherits a multiplication from G which gives it the structure of a group object $\mathbb{G}$ in the category of formal schemes: this is a *formal group*. There are many examples of formal groups $\mathbb{G}$ which do not arise as the formal completion of an algebraic group G.

To rigidify the structure and to make calculations, we choose a coordinate y around the identity e. This is a function $y : G \longrightarrow \mathbb{A}^1$, only defined near e, with the properties (i) that $y(e) = 0$ and (ii) that it gives an identification of a neighbourhood of e in G with a neighbourhood of 0 in the affine line $\mathbb{A}^1$. If $g \in G$ is a point near e with $y(g) = s$, we use the identification to write the more suggestive equation $g = e + s$ (the reason for making the identity explicit on the right hand side will emerge in the equivariant case). Now consider $\mu(e + s, e + t)$ for small s and t and write

$$\mu(e + s, e + t) = e + F(s, t).$$

We then view $F(s, t)$ as a power series in s and t. This power series will inherit properties corresponding to commutativity, associativity, identity and inverse from G. An arbitrary power series satisfying these conditions is called *a formal group law.* A formal group law is equivalent to a formal group with a specified choice of coordinate.

We can also define these notions explicitly in terms of the ring R of functions on $\hat{\mathbb{A}}^1$. A coordinate function y generates the functions on $\hat{\mathbb{A}}^1$ in the sense that $R = k[[y]]$, and F may be viewed as defining a map

$$\Delta : R = k[[y]] \longrightarrow k[[y \otimes 1, 1 \otimes y]] = R \hat{\otimes} R$$

by $\Delta(y) = F(y \otimes 1, 1 \otimes y)$. The conditions on F show that Δ makes R into a commutative and cocommutative complete topological Hopf algebra. In these terms, a formal group corresponds to a bicommutative complete topological Hopf algebra R so that the underlying ring is isomorphic to $k[[y]]$, but with no isomorphism specified. The coordinate is a chosen generator y of the augmentation ideal $\ker(R \longrightarrow k)$, which specifies an isomorphism $R \cong k[[y]]$; a formal group law corresponds to R with a chosen coordinate.

The idea of an A-equivariant formal group may be obtained analogously. We suppose given a homomorphism $\zeta : A^* \longrightarrow G$, and then look at the multiplication μ of G in a formal neighbourhood of the image of ζ. We think of the formal neighbourhood as a collection of little patches around the elements $\zeta(\alpha)$, giving a formal scheme $\mathbb{G}$; it is a group object because the image of A^* is a group. An *A-equivariant formal group* is a formal group $\mathbb{G}$ together with a homomorphism $A^* \longrightarrow \mathbb{G}$ with properties modelled on this one. As in the non-equivariant case, there are equivariant formal groups $\mathbb{G}$ which cannot be obtained by completing a group scheme G.

Again we may choose a coordinate $y(\epsilon)$ around $e = \zeta(\epsilon)$, and by translation we obtain a coordinate $y(\alpha)$ around $\zeta(\alpha)$ for each $\alpha \in A^*$. Now we can consider $\mu(\zeta(\alpha) + s, \zeta(\beta) + t)$ for small s and t and write

$$\mu(\zeta(\alpha) + s, \zeta(\beta) + t) = \zeta(\alpha\beta) + F_{\alpha,\beta}(s, t)$$

for a power series $F_{\alpha,\beta}(s,t)$ in s and t. Again the collection of power series $F_{\alpha,\beta}(s,t)$ will have formal properties inherited from those of μ. However the completion of G around $\zeta(A^*)$ is not the formal neighbourhood of a single point, so its ring of functions R is not a power series ring. If the coordinate patches around the different points $\zeta(\alpha)$ do not interact then R will be a product of power series rings, but in general this need not happen. Accordingly, some care is necessary in interpreting the axioms in terms of the expressions $F_{\alpha,\beta}(s,t)$. If they are assumed to take values in separate power series rings we obtain the notion of an Okonek equivariant formal group law [**26**]. In general the appropriate packaging of the functions $F_{\alpha,\beta}(s,t)$ is in terms of the coproduct

$$\Delta : R \longrightarrow R \hat{\otimes} R,$$

which encodes how functions on the formal group behave under multiplication: $\Delta = \mu^*$. This gives all the ingredients for an A-equivariant formal group law: the most subtle new question in the equivariant case is how to axiomatize the properties of the coordinate $y(\epsilon)$ at the identity $\zeta(\epsilon)$.

2.B. The definition.

We now begin the formal development: first we need the ring theoretic counterpart of the discrete group A^*. We let k^{A^*} denote the ring of k-valued functions on A^*. If A is finite, this is a Hopf algebra over k using the group multiplication of A^* to give the coproduct, and the inclusion of the identity element in A^* to give the counit. If A is not finite, k^{A^*} is topologized as a product of copies of k and becomes a complete topological Hopf algebra.

We shall give the axioms for an equivariant formal group law may in terms of the coproduct Δ and the map

$$\theta : k^{A^*} \longrightarrow R$$

corresponding to ζ, and a coordinate $y(\epsilon)$.

Definition 2.1. ([**4**]) If A is an abelian compact Lie group, an *A-equivariant formal group law* over a commutative ring k is

(Afgl1) a complete topological Hopf k-algebra R with

(Afgl2) a homomorphism $\theta : R \longrightarrow k^{A^*}$ of topological Hopf k-algebras so that the topology on R is defined by the finite intersections of kernels of its components $\theta_\alpha : R \longrightarrow k$ for $\alpha \in A^*$.

(Afgl3) an element $y(\epsilon) \in R$ which is (i) regular and (ii) generates the kernel of the ϵth component, θ_ϵ of θ; equivalently, $y(\epsilon)$ gives an exact sequence

$$0 \longrightarrow R \xrightarrow{y(\epsilon)} R \longrightarrow k \longrightarrow 0. \quad \square$$

Remark 2.2. (i) If A is finite, (Afgl2) shows that the topology on R is defined by the single ideal $\ker(\theta)$.
(ii) Since θ is a map of Hopf algebras it follows that θ_ϵ is the counit of R.
(iii) The element $y(\epsilon)$ is called the *coordinate* of the formal group law, since in geometric terms it is a function whose vanishing defines the identity of the group. If the coordinate is not specified, the resulting structure represents an *equivariant formal group*. Indeed, by (Afgl1), R may be viewed as the ring of functions on a group object $\mathbb{G}$ in the category of formal schemes over k. In these terms, (Afgl2) states that we are given a homomorphism $\zeta : A^* \longrightarrow \mathbb{G}$, so that $\mathbb{G}$ is a formal neighbourhood of the image, and (Afgl3) states that $y(\epsilon)$ is a good coordinate at $\zeta(\epsilon)$. The geometry of an equivariant formal group is best understood using Strickland's notion of a multicurve [**31**].

2.C. Additive structure of an equivariant formal group law.

The k-module structure of every equivariant formal group law is topologically free, and we may therefore express the structure maps of R in terms of the basis. To describe the basis, we note that we may define an action of A^* on R via $l_\alpha r = (\theta_{\alpha^{-1}} \otimes 1)\Delta(r)$. Thus the element $y(\epsilon)$ determines elements $y(\alpha)$ for $\alpha \in A^*$ by the formula $y(\alpha) = l_\alpha y(\epsilon)$. The completeness is thus equivalent to completeness with respect to the system of principal ideals generated by all finite products $\prod_\alpha y(\alpha)$. In the following statement, a complex complete A-universe is a countably infinite dimensional complex representation of A in which every simple representation occurs infinitely often.

Theorem 2.3. *[4, 13.2] If we choose a complete A-invariant flag $F = (V^1 \subset V^2 \subset \cdots)$ in a complex complete A-universe, then an equivariant formal group law R has an additive topological k-basis $1, y(V^1), y(V^2), \ldots$ where $y(V^n) = y(\alpha_1)y(\alpha_2) \cdots y(\alpha_n)$ if $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$.* $\square$

Remark 2.4. Note that if A is the trivial group, Theorem 2.3 shows that Definition 2.1 reduces to the usual concept of a (non-equivariant, commutative, one dimensional) formal group law.

2.D. Formal group schemes.

We return to make explicit how the topological k-algebra R represents a formal group $\mathbb{G}$, restricting attention to the case that A is finite. By definition, $\mathbb{G}$ is the set valued functor on k-algebras whose l-valued points are the continuous k-algebra homomorphisms into l:

$$\mathbb{G}(l) = k\text{-Alg}_{cts}(R, l).$$

The coproduct on R gives $\mathbb{G}(l)$ the structure of an abelian group. Furthermore the homomorphism θ defines a group homomorphism

$$\zeta : A^* \longrightarrow \mathbb{G}(l)$$

by the formula $\zeta(\alpha)(r) = \theta_\alpha(r)$.

Lemma 2.5. *Evaluation at the elements $y(\alpha)$ gives an identification*

$$\mathbb{G}(l) \subseteq A\text{-}nil(l)$$

where A-nil$(l) \subseteq l^{A^*}$ is defined by

$$A\text{-}nil(l) = \{(y_\alpha) \mid \Pi_\alpha y_\alpha \textit{ is topologically nilpotent}\}.$$

Under this identification, the group operation is given by

$$[(y'_\alpha) \odot (y''_\beta)]_\gamma = \Delta(y(\gamma))((y'_\alpha \otimes 1), (1 \otimes y''_\beta)) :$$

the element $\Delta(y(\gamma))$ may be expressed in terms of any flag basis, and then evaluated by putting $y(\alpha) \otimes 1 = y'_\alpha$ and $1 \otimes y(\beta) = y''_\beta$. □

This shows that $\mathbb{G}(l)$ can be viewed as a subset of the affine space l^{A^*}: we will identify the defining equations for this algebraic set when A is of order 2 in Section 3 below. We should draw attention to several differences from a classical formal group law. Firstly, the element $y(\epsilon)$ is not usually topologically nilpotent, and secondly, it is not a free generator. From the geometric point of view, we may ask how to think of a point $(y_\alpha) \in \mathbb{G}(l)$. Because $y(\epsilon)$ does not generate R, the coordinate y_ϵ does not generally determine all the others, so that the projection onto the ϵth factor of l need not be injective. Nonetheless, we consider the ϵth coordinate. Classically, only topologically nilpotent elements of l qualify: these are points infinitesimally close to the identity $0 = \zeta(\epsilon)$. In the equivariant case, a point of l infinitesimally close to any of the points $\zeta(\alpha)$ in the image of ζ qualifies as a candidate for y_ϵ. Thus the projection of $\mathbb{G}(l)$ onto the ϵth factor is an infinitesimally thickened copy of A^* in l.

2.E. The universal ring for A-equivariant formal group laws.

First we note that the set A-fgl(k) of A-equivariant formal group laws over k is a functor of the ring k. Indeed, if $f : k \longrightarrow l$ is a ring homomorphism and R is an A-equivariant formal group law over k then we may define an A-equivariant formal group law f_*R over l by applying $\hat{\otimes} l$. The result is again an A-equivariant formal group, since by 2.3 R is a topologically free k-module. In other words, we use the fact that the structure of R may be described by certain structure constants in k, and let f_*R be described by their images in l.

It follows quite easily [**4**, 14.3] that the functor A-fgl$(\cdot)$ is represented by a ring L_A in the sense that

$$A\text{-fgl}(k) = \mathrm{Ring}(L_A, k).$$

Indeed, L_A may be constructed by giving generators for each of the structure constants, and imposing relations to ensure that the axioms of Definition 2.1 hold. The A-equivariant formal group law over k corresponding to a ring homomorphism $f : L_A \longrightarrow k$ is the one with structure constants given by the image of the corresponding generators of L_A.

3. The underlying ring of an equivariant formal group law.

We want to describe the ring R (or equivalently the underlying formal scheme) of an equivariant formal group. We will give a satisfactory answer when A is of order 2, but not in general. If A is finite it is natural to choose an ordering of the simple representations $\alpha_1 = \epsilon, \alpha_2, \ldots, \alpha_n$ of A and use the flag obtained by repeatedly adding these representations in order. Thus $V^n, V^{2n}, \ldots$ are multiples of the regular representation and the topology is defined by the powers of $x = y(V^n) = \Pi_\alpha y(\alpha)$. It is thus natural to view R as an extension of $k[[x]]$, but we warn that $k[[x]]$ is not usually closed under the coproduct.

Example 3.1. (*A* of order 2.) Suppose A is of order 2, with non-trivial simple representation α and let $y = y(\epsilon), y' = y(\alpha)$. Using the periodic flag, the basis is $1, y, x, yx, x^2, yx^2, x^3, \ldots$ so we may write

$$y' = p(x) + yq(x).$$

Rotating by α we find

$$y = p(x) + y'q(x),$$

so that

$$y^2 = yp(x) + xq(x).$$

Furthermore, substituting the expression for y' into that for y, we find

$$y = p(x) + [p(x) + yq(x)]q(x) = p(x)(1 + q(x)) + yq(x)^2$$

and therefore

$$p(x)(1 + q(x)) = 0 \text{ and } q(x)^2 = 1.$$

In particular we note that $q(x)$ is a unit, so that working modulo any power of x we can recursively express every element as a polynomial in y: this is Cole's theorem [**3**] that the theorem on projective bundles holds for all complex oriented cohomology theories provided A is of order 2 (it does not hold in general if A is of any larger order). We also conclude

$$R = k[[x]][y]/(y^2 = yp(x) + xq(x)),$$

telling us that the formal group of R is a ramified double cover of the formal affine line $\hat{\mathbb{A}}^1$.

Example 3.2. **(A of order 3.)** Suppose A is of order 3 with $A^* = \langle\alpha\rangle$, and let $y = y(\epsilon), y' = y(\alpha)$ and $y'' = y(\alpha^2)$. Using the periodic flag, the basis is $1, y, yy', x, yx, yy'x, x^2, \ldots$ so we may write

$$y' = p'(x) + yq'(x) + yy'r'(x) \text{ and } y'' = p''(x) + yq''(x) + yy'r''(x).$$

One may then deduce in turn expressions for $yy'', y^2, (y'')^2, yy', y'y'', (y')^2$ in terms of the standard bases. Of course the element yy' is already a basis element, so the expression gives relations amongst the power series. The coefficient of yy' shows that $q''(x) + r''(x)$ is a unit, and then the three coefficients together allow one to express $p'(x), q'(x)$ and $r'(x)$ in terms of $p''(x), q''(x)$ and $r''(x)$. This gives

$$R = k[[x]][y, y']/(y' = p' + yq' + yy'r', y^2 = a + yb + yy'c, (y')^2 = a' + yb' + yy'c')$$

where $p', q', r', a, b, c, a', b', c'$ are polynomials in $p''(x), q''(x), r''(x)$ and $(q''(x) + r''(x))^{-1}$. It is straightforward to make these polynomials explicit, or to rewrite the above as a quotient of $k[[x]][y, (yy')]$.

One point to note is that $q'(x)$ turns out to be a unit, but if we write $y' = e(\alpha) + yw$ for a scalar $e(\alpha)$, it need not be true that w is a unit (the analogue of Example 6.2 below for the group of order 3 provides an example).

4. Euler classes.

One view is that an equivariant formal group law is a structure precisely designed to encode the formal properties of Euler classes.

Definition 4.1. Given an A-equivariant formal group law we may define the *Euler class* of a one dimensional representation α by

$$e(\alpha) = \theta_\epsilon(y(\alpha)).$$

Remark 4.2. (i) In geometric terms, $e(\alpha)$ is the value of the coordinate $y(\alpha)$ at the identity, or equivalently, the value of $y(\epsilon)$ at the point $\zeta(\alpha^{-1})$.
(ii) When the formal group law arises from a complex oriented cohomology theory as described in Section 9 below, these coincide with Euler classes in the topological sense [**4**].
(iii) Note that $e(\epsilon) = 0$ by (Afgl3) (ii).

Lemma 4.3. *The Euler classes determine the structure map θ.*

Proof: Since θ is a continuous ring homomorphism it suffices to identify $\theta(y(\beta))$ for all $\beta \in A^*$. Indeed, we calculate

$$\theta(y(\beta))(\alpha^{-1}) = \theta(y(\epsilon))(\alpha^{-1}\beta^{-1}) = e(\alpha\beta),$$

where the first equality follows by applying l_β, since θ is a Hopf map. □

To see that an equivariant formal group law encodes the formulae for the Euler class of a tensor product, we consider the diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d] & k^{A^*} \arrow[d] \\ R\hat{\otimes}R \arrow[r, "\theta\hat{\otimes}\theta"] & k^{A^*\times A^*} \end{tikzcd}$$

encoding the fact that θ respects coproducts. Choose a complete flag $V^1 \subseteq V^2 \subseteq V^3 \subseteq \cdots$ with $V^i = \alpha_1 \oplus \cdots \oplus \alpha_i$ and $\alpha_1 = \epsilon$. We may then write the coproduct in terms of this basis

$$\Delta(y(\epsilon)) = \sum_{i,j} f_{i,j} y(V^i) \otimes y(V^j);$$

the coefficients $f_{i,j}$ depend on the flag, although this is not displayed in the notation. Now consider the effect of Δ on the coordinate $y(\epsilon)$ and evaluate at $(\beta^{-1}, \gamma^{-1})$. We see

$$\begin{array}{rcl} e(\beta\gamma) & = & (\mu^* \circ \theta(y(\epsilon)))(\beta^{-1}, \gamma^{-1}) \\ & = & ((\theta \otimes \theta) \circ \Delta y(\epsilon))(\beta^{-1}, \gamma^{-1}) \\ & = & ((\theta \otimes \theta)(\sum_{i,j} f_{i,j} y(V^i) \otimes y(V^j))(\beta^{-1}, \gamma^{-1}) \\ & = & e(\beta) + e(\gamma) + \\ & & e(\beta)e(\gamma)\left[\sum_{i,j\geqslant 1} f_{i,j} e(\beta\alpha_2)e(\beta\alpha_3)\cdots e(\beta\alpha_i)e(\gamma\alpha_2)e(\gamma\alpha_3)\cdots e(\gamma\alpha_j)\right] \end{array}$$

The usefulness of this is most obvious when $f_{i,j} = 0$ for $i \geqslant 2$ or $j \geqslant 2$: it then says $e(\beta\gamma) = e(\beta) + e(\gamma) + f_{1,1}e(\beta)e(\gamma)$. In general higher terms are zero modulo progressively higher powers of the ideal of Euler classes. In any case, since both β^{-1} and γ^{-1} occur in any complete flag, the sum is finite.

Remark 4.4. It is shown in [**4**, 16.1] that, for any choice of flag, the universal ring L_A is generated by the Euler classes $e(\alpha)$ and the coefficients $f_{i,j}$ in the coproduct.

5. The identity component of an equivariant formal group.

We are used to the idea that when dealing with a compact Lie group G it may be useful to consider the short exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

where the normal subgroup G_0 is its identity component and $\pi_0(G) = G/G_0$ is its group of components. The purpose of this section is to explain the analogue for equivariant formal groups: its usefulness is that the identity component and the component group are more familiar objects. In fact, an equivariant formal group may be thought of as an extension of a non-equivariant (one dimensional, commutative) formal group, with a "discrete group" as the quotient.

Theorem 5.1. *If $\mathbb{G}$ is an A-equivariant formal group over k, there is a short exact sequence*

$$0 \longrightarrow \mathbb{G}_0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{\acute{e}t} \longrightarrow 0$$

of formal groups, where

- $\mathbb{G}_0$ *is a classical non-equivariant formal group and*
- *the composite* $A^* \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{\acute{e}t}$ *is surjective; indeed, over a field* $\mathbb{G}_{\acute{e}t}$ *is a discrete group and a quotient of* A^**. More precisely, if the* k*-algebra* l *is a field,*

$$\mathbb{G}_{\acute{e}t}(l) = A^*/B_l^*$$

where

$$B_l^* = \{\beta \mid e(\beta) = 0 \text{ in } l\}$$

is the subgroup of representations whose Euler classes are trivial in l*.*

Proof: The group $\mathbb{G}_0$ is constructed as a formal neighbourhood of the identity in $\mathbb{G}$ so that we have a homomorphism $\mathbb{G}_0 \longrightarrow \mathbb{G}$: this corresponds to completing the ring of functions R with respect to the ideal of functions vanishing at the identity, i.e., to the map

$$R \longrightarrow R^{\wedge}_{(y(\epsilon))},$$

of topological Hopf algebras. It is easy to check $R^{\wedge}_{(y(\epsilon))} = k[[y(\epsilon)]]$, so that $\mathbb{G}_0$ is a non-equivariant formal group. The quotient group $\mathbb{G}_{\acute{e}t}$ is represented by the Hopf kernel $R \square_{R^{\wedge}_{(y(\epsilon))}} k$.

The idea for proving the statement about $\mathbb{G}_{\acute{e}t}$ is as follows. If $e(\beta) = 0$ it follows from [**4**, 16.7] that $y(\alpha) = y(\alpha\beta)$, so that, over l, the distinct coordinates are parametrized by A^*/B_l^*. Corresponding to the inclusion of the coordinate neighbourhoods we consider the map

$$R \longrightarrow \prod_{\gamma \in A^*/B^*} R^{\wedge}_{(y(\gamma))}$$

whose components are completions. Over a field, the other Euler classes are units, and so one may check the the coordinates $y(\gamma)$ are coprime. The Chinese Remainder Theorem shows the map is an isomorphism, so that $\mathbb{G}_{\acute{e}t}(l) = A^*/B_l^*$ as required. □

Remark 5.2. (i) We refer to $\mathbb{G}_0$ as the "identity component" of $\mathbb{G}$.
(ii) A coordinate $y(\epsilon)$ on the group $\mathbb{G}$ giving rise to an equivariant formal group law also provides a coordinate on $\mathbb{G}_0$ so that it gives rise to a non-equivariant formal group law.
(iii) We refer to $\mathbb{G}_{\acute{e}t}$ as the étale quotient, although the terminology is really only appropriate if A is finite. We will see in Section 6 that the $\mathbb{G}_{\acute{e}t}$ is usually not a constant group.

6. Complete and local examples.

We describe two examples representing the two extremes in the behaviour of an equivariant formal group. A general equivariant formal group can be constructed by amalgamating them, at least over a Noetherian ring [**17**]. The corresponding topological constructions are described in Section 11.

We start with a non-equivariant formal group $\overline{\mathbb{G}}$ over $\overline{k}$, and form an A-equivariant formal group in two ways; if a coordinate is specified on $\overline{\mathbb{G}}$ we describe a coordinate on the equivariant formal group. In both cases the minimal additional data we need is something to specify the Euler classes, but the data required is very different in the two cases.

Non-equivariant ingredients 6.1. We assume given a non-equivariant formal group: notation related to this is indicated by a bar.

- $\overline{\mathbb{G}}$, the (non-equivariant, commutative, one dimensional) formal group
- $\overline{R}$, its ring of functions
- $\overline{k}$, its ground ring
- $\overline{\Delta} : \overline{R} \longrightarrow \overline{R}\hat{\otimes}\overline{R}$, its coproduct
- $\overline{\theta} : \overline{R} \longrightarrow \overline{k}$, its counit
- $\overline{y}$, a coordinate on $\overline{\mathbb{G}}$

We construct two quite different equivariant formal groups $\mathbb{G}$ with identity component $\mathbb{G}_0 = \overline{\mathbb{G}}$.

6.A. Euler-invertible extensions.

In this example the ground ring $\overline{k}$ is unaltered (so that $k = \overline{k}$) but the formal group $\overline{\mathbb{G}}$ is replaced by its product with A^*: the A-equivariant formal group is

$$A^* = 1 \times A^* \longrightarrow \overline{\mathbb{G}} \times A^* =: \mathbb{G}.$$

This is essentially an equivariant formal group law in the sense of Okonek [**26**].

Translating this description into statements about functions, we take

$$k = \overline{k}, R = \overline{R}^{A^*}, \text{ and } \theta = \overline{\theta}^{A^*}\!: R = \overline{R}^{A^*} \longrightarrow \overline{k}^{A^*} = k^{A^*}.$$

Given $\overline{y}$ and the Euler classes $e(\alpha)$ there are many ways to specify the coordinate, but the one needing no further choice is

$$y(\epsilon) = (\overline{y}, e(\alpha^{-1}), e(\beta^{-1}), \ldots),$$

where we have recorded the ϵth, αth and βth components.

There is one constraint: the Euler classes must be units. This is necessary to ensure that a sum such as $\sum_i (a_i \overline{y}^i, 0, 0, \ldots)$ converges in R. Indeed, since l_β simply permutes the factors,

$$y(\beta) = (e(\beta), e(\alpha^{-1}\beta), \overline{y}, \ldots)$$

(in general $y(\beta)(\gamma) = y(\epsilon)(\gamma^{-1}\beta)$). Now, to see that $e(\alpha)$ must be a unit we attempt to express $y(\alpha)$ in terms of the basis $1, y(\epsilon), y(\epsilon)y(\alpha), y(\epsilon)y(\epsilon)y(\alpha), \ldots$. In the αth component, this basis has $1, e(\alpha), \overline{y}e(\alpha)$, followed by terms divisible by $\overline{y}^2$ whereas $y(\alpha)$ has $\overline{y}$.

Example 6.2. It is perhaps worth a short calculation with A of order 2 to show how great the difference between equivariant and non-equivariant formal groups can be. Let α denote the nontrivial one dimensional representation, and we suppose $e = e(\alpha)$ is invertible. Thus

$$R = k[[\overline{y}]]^{A^*} = k[[\overline{y}]] \times k[[\overline{y}]],$$

and we may take

$$y(\epsilon) = (\overline{y}, e),$$

so that

$$y(\alpha) = (e, \overline{y}).$$

We can thus write $y(\alpha)$ in terms of the $\epsilon, \alpha, \epsilon, \alpha, \epsilon, \ldots$ basis

$$y(\alpha) = (0, e) \cdot 1 + (1, -1) \cdot y(\epsilon) + (0, e^{-1}) \cdot y(\epsilon) y(\alpha). \quad \square$$

Other possible choices of the coordinate $y(\epsilon)$ must have a non-equivariant coordinate in the ϵth factor and the Euler classes must still be the constant terms in other factors, but these are the only constraints. Thus the ϵth factor is a unit multiple of $\overline{y}$ and any power series $\overline{y} f_\alpha(\overline{y})$ may be added in the αth factor where $\alpha \neq \epsilon$. Accordingly the universal ring for A-equivariant formal group laws of this sort is

$$L[e(\alpha), e(\alpha)^{-1}, \gamma_1(\alpha), \gamma_2(\alpha), \ldots \mid \alpha \neq \epsilon],$$

where L is Lazard's universal ring for non-equivariant formal group laws.

For any given equivariant formal group law we can discuss Euler classes. We consider the case when $e(\alpha)$ is invertible in the ground ring k for all $\alpha \neq \epsilon$.

Theorem 6.3. (Localization theorem.) *If R is an A-equivariant formal group law with all Euler classes invertible, then $R = \overline{R}^{A^*}$ for some non-equivariant formal group law $\overline{R}$ as above.*

The idea is the same as in the proof of 5.1: when Euler classes are invertible, the elements $y(\alpha)$ are coprime, and we can use the Chinese remainder theorem to say that R splits. Details are given in [**16**].

The localization theorem immediately gives us the universal ring for Euler-invertible equivariant formal group laws.

Corollary 6.4. *If $\mathcal{E}$ denotes the multiplicatively closed subset generated by the Euler classes $e(\alpha)$ for $\alpha \neq \epsilon$, we have*

$$\mathcal{E}^{-1} L_A = L[e(\alpha), e(\alpha)^{-1}, \gamma_1(\alpha), \gamma_2(\alpha), \ldots \mid \alpha \neq \epsilon]. \quad \square$$

The Localization Theorem suggests alternative nomenclature. After all, $e(\alpha)$ is the value of $y(\epsilon)$ at $\zeta(\alpha^{-1})$, whereas $y(\epsilon)$ takes the value 0 at ϵ: thus if $e(\alpha)$ is a unit, patches around the different points $\zeta(\alpha)$ are completely independent. Pictorially, the map $\mathbb{G} = A^* \times \overline{\mathbb{G}} \longrightarrow \overline{\mathbb{G}}$ is projection. In this case $\mathbb{G}_0 = \overline{\mathbb{G}}$ and $\mathbb{G}_{\acute{e}t} = A^*$, so it would also be reasonable to call $\mathbb{G}$ the unramified extension of $\overline{\mathbb{G}}$.

6.B. Euler-complete extensions.

This time we adjoin new Euler classes: the ground ring has various roots of n-series adjoined, but the group itself is unchanged. We want the equivariant formal group to be given by a homomorphism

$$\zeta : A^* \longrightarrow \overline{\mathbb{G}},$$

but it may well happen that A^* is finite and $\overline{\mathbb{G}}$ has no points of finite order over $\overline{k}$, so we must adjoin some. This is closely related to constructions of Hopkins-Kuhn-Ravenel [**22**].

We need to use the formal group law F associated to $\overline{\mathbb{G}}$, and the associated n-series $[n](z) = [n]_F(z)$ defined recursively by $[1](z) = z$ and $[n+1](z) = F(z, [n](z))$ as usual; equivalently $[n](z)$ is the pullback of the function z along $n : \overline{\mathbb{G}} \longrightarrow \overline{\mathbb{G}}$. For notational simplicity we restrict attention to the case of a cyclic group of order n, and suppose $A^* = \langle \alpha \rangle$. The general case behaves just as this suggests. Thus the homomorphism ζ is to be specified by giving a point $\zeta(\alpha)$ of order n, and we take

$$k = \overline{k}[[z]]/([n](z)), R = \overline{R}, \text{ and } \theta : R \longrightarrow k^{A^*}.$$

As our coordinate we may use the non-equivariant coordinate $\overline{y}$ of $\overline{\mathbb{G}}$:

$$y(\epsilon) = \overline{y}.$$

We then have Euler classes

$$e(\alpha^s) = \theta_{\alpha^{-s}}(\overline{y}) = [s](z).$$

It is immediate from the definition that this gives an A-equivariant formal group law. It is clear by construction that the universal ring for A-equivariant formal group laws of this form is

$$L[[z]]/([n](z)).$$

For any given equivariant formal group law we can discuss Euler classes. We consider the case when k is complete for the ideal

$$I\mathcal{E} = (e(\alpha) \mid \alpha \in A^*).$$

Theorem 6.5. (Completion Theorem.) *If R is an A-equivariant formal group law complete for the ideal $I\mathcal{E}$ generated by Euler classes, then R is constructed in this way from a non-equivariant formal group law $\overline{R}$.*

The idea of proof is as follows. Quite generally, the coordinates $y(\alpha)$ are all congruent modulo Euler classes, and hence in the Euler-complete case $y(\epsilon)$ itself is topologically nilpotent and its powers are dense. Details are given in [**16**].

This immediately gives us the universal ring for Euler-complete equivariant formal group laws.

Corollary 6.6. *If A is cyclic of order n and $I\mathcal{E}$ denotes the ideal generated by the Euler classes $e(\alpha)$ for $\alpha \neq \epsilon$, we have*

$$(L_A)^{\wedge}_I = L[[z]]/([n](z)).$$

The Completion Theorem suggests alternative nomenclature. After all, $e(\alpha)$ is the value of $y(\epsilon)$ at $\zeta(\alpha^{-1})$: if k is complete for the ideal $I\mathcal{E}$ of Euler classes, this states that $y(\epsilon)$ is topologically nilpotent in the patches around the different points $\zeta(\alpha)$. In this case $\mathbb{G}_0 = \overline{\mathbb{G}} \otimes_{\overline{k}} k$, and $\mathbb{G}_{\acute{e}t} = 0$ so an alternative terminology would state that $\mathbb{G}$ is the totally ramified extension of $\overline{\mathbb{G}}$.

7. Multiplicative equivariant formal group laws.

In this section we consider equivariant formal group laws of a very simple form, summarizing results from [**14**], to which we refer for all proofs. The topological counterparts of these results are discussed in Section 12.

7.A. The definition.

We first give the definition apparently by simple transposition from the non-equivariant case. In the final subsection we give a more helpful geometric explanation.

Definition 7.1. (i) An equivariant formal group law R is *multiplicative* if its coproduct has the property

$$\Delta y(\epsilon) = 1 \otimes y(\epsilon) + y(\epsilon) \otimes 1 - vy(\epsilon) \otimes y(\epsilon)$$

for some $v \in k$.
(ii) Given a multiplicative equivariant formal group law over k, we define a binary operation on k-algebras by $x \odot y = x + y - vxy$.

Remark 7.2. (i) From 2.3 the products $y(V^i) \otimes y(V^j)$ give a basis of $R\hat{\otimes}R$ for any flag, so the coproduct determines the element v. The letter v is chosen to correspond to the Bott element in topological K-theory.
(ii) Note that v is not required to be a unit. In particular, we allow the degenerate case $v = 0$, which is usually referred to as an *additive* law. If v is a unit we say the formal group law is *strictly* multiplicative.
(iii) The notion depends heavily on the coordinate: it is a property of the formal group *law* and not of its associated formal group.

For such a simple coproduct it is easy to make the structure quite explicit. For example the equivariance equation $\Delta \circ l_\alpha = (1 \otimes l_\alpha) \circ \Delta$ may be applied to $y(\epsilon)$ to give an expression for $y(\alpha)$.

Lemma 7.3.

$$y(\alpha) = e(\alpha) + (1 - ve(\alpha))y(\epsilon). \quad \square$$

7.B. Euler classes.

First we define the polynomial $[n](x)$ inductively by $[0](x) = 0$ and $[n](x) = ([n-1](x)) \odot x$. Thus

$$[n](x) = (1 - (1 - vx)^n)/v.$$

The notation is consistent with the usage in Subsection 6.B above because this is the n-series $[n]_m(x)$ for the non-equivariant multiplicative formal group.

Because of the very simple form of the coproduct, the formula in Section 4 for Euler classes of a tensor product is considerably simplified.

Lemma 7.4. *The Euler class of a tensor product is described by the formula*

$$e(\alpha\beta) = e(\alpha) \odot e(\beta).$$

Furthermore,

$$e(\alpha^n) = [n](e(\alpha)),$$

so that if α is of order n, we have $[n](e(\alpha)) = 0$, and

$$e(\alpha^n) = e(\alpha^{n-1}) + e(\alpha)(1 - ve(\alpha))^{n-1}. \quad \square$$

Corollary 7.5. *For any one dimensional representation α, the element $1 - ve(\alpha)$ is a unit with inverse $1 - ve(\alpha^{-1})$.*

Proof: We have

$$(1 - ve(\alpha))(1 - ve(\beta)) = 1 - ve(\alpha) \odot e(\beta) = 1 - ve(\alpha\beta),$$

so that $1 - ve(\alpha^{-1})$ is inverse to $1 - ve(\alpha)$. $\square$

7.C. Universal rings.

Since multiplicative formal group laws are defined by the vanishing of most terms in the coproduct, the set $A\text{-fgl}_m(k)$ of multiplicative equivariant formal group laws is also a functor of k. It follows from the existence of L_A that there is also a representing ring L_A^m for the multiplicative A-equivariant formal group law functor:

$$A\text{-fgl}_m(k) = \mathrm{Ring}(L_A^m, k).$$

Similarly, there are representing rings L_A^a and L_A^{sm} for additive and strictly multiplicative A-equivariant formal group laws. The results in this subsection give explicit presentations of these representing rings.

We have seen that the entire structure of a multiplicative A-equivariant formal group law over k is determined by polynomials in the element v and the Euler classes $e(\alpha)$ of a set of characters α generating A^*: this shows that L_A^m is generated by elements v and $e(\alpha)$ corresponding to these structure constants. The main content in the descriptions of the universal rings is in showing that all relations follow from relations on the Euler classes that we have already met.

Theorem 7.6. *For any compact abelian Lie group A there is a representing ring L_A^m for multiplicative equivariant formal group laws. The ring L_A^m is a $\mathbb{Z}[v]$-algebra and may be described as follows.*
(i) If $A = B \times C$ then

$$L_A^m = L_B^m \otimes_{\mathbb{Z}[v]} L_C^m$$

(ii) If A is a finite cyclic group of order n with dual group $A^* = \langle\alpha\rangle$ then

$$L_A^m = \mathbb{Z}[v,e]/([n](e)),$$

where $e = e(\alpha)$. This becomes a graded ring if v has degree 2 and e is of degree -2.
(iii) If A is a circle group and $A^* = \langle z\rangle$ then

$$L_A^m = \mathbb{Z}[v,f,f']/(vff' - f - f'),$$

where $f = e(z)$ and $f' = e(z^{-1})$. This becomes a graded ring if v has degree 2 and both f and f' have degree -2.

The resulting description of L_A^m depends strongly on the chosen presentation of the group A. In some cases this can be avoided by using a standard construction in commutative algebra. The Rees ring $\mathrm{Rees}(R,J)$ associated to an ideal J of a ring R is the graded subring of the graded ring $R[v,v^{-1}]$ generated by R, v and $v^{-1}\cdot J$, where v has degree 2. It is thus R in degree 0 and each positive even degree, and it is J^n in degree $-2n$. Since the Euler classes in representation theory satisfy the universal relations, and the Rees ring is generated by v together with e's, f's and f''s we find a useful connection.

Corollary 7.7. *There is a natural map*

$$\ell : L_A^m \longrightarrow R(A)[v,v^{-1}],$$

with image equal to the Rees ring. □

If A is topologically cyclic we have a more satisfactory description of L_A^m.

Proposition 7.8. *(i) If A is topologically cyclic then the map ℓ of 7.7 induces an isomorphism*

$$L_A^m \cong Rees(R(A),J).$$

(ii) For any abelian group A, the map ℓ is the localization away from v:

$$L_A^m[v^{-1}] \cong R(A)[v,v^{-1}].$$

(iii) If A is not topologically cyclic then L_A^m contains $\mathbb{Z}$-torsion and v-torsion. □

Corollary 7.9. *If A is topologically cyclic, the representing ring L_A^m for multiplicative A-equivariant formal group laws, may be identified with the Rees ring*

$$L_A^m = \mathrm{Rees}(R(A),J).$$

For an arbitrary abelian compact Lie group A, the representing ring for strictly multiplicative A-equivariant formal group laws is given by

$$L_A^{sm} = R(A)[v,v^{-1}]. \quad \square$$

Finally, we record the corresponding results for additive formal group laws, which follow by setting $v = 0$.

Corollary 7.10. *There is a universal ring L_A^a for additive A-equivariant formal group laws. It is the free commutative ring on the abelian group A^*, and if $A^* = \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_r) \oplus \mathbb{Z}^d$, it has the presentation*

$$L_A^a = \mathbb{Z}[e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_d]/(n_1 e_1, n_2 e_2, \ldots, n_r e_r),$$

where $e_1, \ldots, e_r, f_1, \ldots, f_d$ are the Euler classes of the cyclic generators of A^*. $\square$

7.D. Outline via group schemes.

First recall the multiplicative group scheme G_m over k, defined for k-algebras l by

$$G_m(l) = k\text{-Alg}(k[z, z^{-1}], l) = \text{Units}(l)$$

where the group multiplication is induced by the coproduct $\Delta(z) = z \otimes z$ and the inverse by $\iota(z) = z^{-1}$. The universal example is the case with $k = \mathbb{Z}$. Now take $x = 1 - z$ as a coordinate and note that the coproduct takes the form $\Delta(x) = x \otimes 1 + 1 \otimes x - x \otimes x$. If v is a unit we may replace x by $y = v^{-1}x$ and obtain the coproduct $\Delta(y) = y \otimes 1 + 1 \otimes y - vy \otimes y$ in the form we have been discussing. But note that to define the inverse it is essential that $z = 1 - vy$ is invertible. Of course, this holds for the rings $k[y, (1 - vy)^{-1}] = k[z, z^{-1}]$ and $k[[y]]$, which define the multiplicative group G_m and the multiplicative formal group $\mathbb{G}_m$ over k, at least provided v is invertible in k.

By contrast, we need to discuss the multiplicative monoid scheme $\mathbb{M}_{m,v}$ with parameter v. This is defined by

$$\mathbb{M}_{m,v}(l) = k\text{-Alg}(k[y], l)$$

with monoid structure defined by the coproduct $\Delta(y) = y \otimes 1 + 1 \otimes y - vy \otimes y$. We choose y as a coordinate at the identity to obtain a monoid law. By construction, for any multiplicative equivariant formal group law $(\mathbb{G}, y(\epsilon), v)$ there is a canonical homomorphism

$$\mathbb{G} \longrightarrow \mathbb{M}_{m,v}$$

of monoid schemes determined by the coordinate $y(\epsilon)$. Furthermore, behaviour of Euler classes shows that any monoid homomorphism $\zeta' : A^* \longrightarrow \mathbb{M}_{m,v}$ factors uniquely through a group homomorphism $\zeta : A^* \longrightarrow \mathbb{G}$. Thus A-equivariant formal group laws with parameter v correspond to homomorphisms ζ', and

$$A\text{-fgl}_{m,v}(l) = \text{Monoid}(A^*, \mathbb{M}_{m,v}(l)).$$

The universal case has $k = \mathbb{Z}[v]$, and we conclude

$$A\text{-fgl}_m = \text{Monoid}(A^*, \mathbb{M}_m)$$

where $\mathbb{M}_m$ is the universal multiplicative nonequivariant formal group law $\mathbb{M}_{m,v}$ over $\mathbb{Z}[v]$. Thus the representing ring L_A^m is the ring of functions on $\text{Monoid}(A^*, \mathbb{M}_m)$.

To calculate the ring of functions, we remark that the fact the coproduct lies in the polynomial ring in the coordinate shows that if $A = B \times C$ then

$$\text{Monoid}(A^*, \mathbb{M}_{m,v}) = \text{Monoid}(B^*, \mathbb{M}_{m,v}) \times \text{Monoid}(C^*, \mathbb{M}_{m,v}),$$

so that the ring of functions is the tensor product of the rings of functions of the cyclic factors. Finally, when A^* is cyclic of order n

$$\mathrm{Monoid}(A^*, \mathbb{M}_{m,v}) = \mathbb{M}_{m,v}[n]$$

is the *group* scheme of points of order dividing n, and this is represented by the ring $k[y]/([n](y))$. However, the infinite cyclic group $A^* = \langle z \rangle$ is generated as a monoid by the elements z and z^{-1}, subject to the relation $zz^{-1} = \epsilon$, so that $\mathrm{Monoid}(A^*, \mathbb{M}_{m,v})$ is represented by the ring $k[y, y']/(y \odot y')$. The case when $k = \mathbb{Z}[v]$ gives the calculation of L_A^m, and hence Theorem 7.6.

8. The easy life over the rationals.

In this section we assume that A is finite and its order is invertible in k: this drastically simplifies the situation. In fact, idempotents show that any example is essentially a product of the Euler-local and Euler-complete examples. As might be expected, for a general finite group A there is one factor for each (conjugacy class of) subgroups of A, but for simplicity we restrict attention to the case that A is cyclic of prime order p and $A^* = \langle \alpha \rangle$.

Proposition 8.1. *If A is of prime order p and p is invertible in k, then if R is an A-equivariant formal group law over k there is an idempotent $f \in k$ giving rise to splittings*

$$k = k' \times k'', R = R' \times R'' \text{ and } y(\epsilon) = (y(\epsilon)', y(\epsilon)'')$$

so that

- *R' is an Euler-local A-equivariant formal group law over k' and*
- *R'' is an A-equivariant formal group law over k'' with trivial Euler classes.*

Accordingly the universal ring for A-equivariant formal group laws over $\mathbb{Z}[1/p]$ is

$$L_A[1/p] = L[1/p] \times L[1/p][e(\alpha^i), e(\alpha^i)^{-1}, \gamma_1(\alpha^i), \gamma_2(\alpha^i), \ldots \mid i = 1, 2, \ldots, p-1].$$

Proof: The formula for the Euler class of a product gives

$$0 = e(\alpha^p) = pe(\alpha) + e(\alpha)r$$

for some r in k. Equivalently, if we take $f = -e(\alpha)r/p$, we have $f^2 = f$. Inverting f inverts $e(\alpha)$, and inverting $(1-f)$ kills $e(\alpha)$.

If $e(\alpha)$ is inverted, then expressing $e(\alpha)$ in terms of $e(\alpha^i)$ for $i \neq 0 \bmod p$ we see that we invert $e(\alpha^i)$ as well, so that inverting e gives an Euler-invertible A-equivariant formal group law. Similarly, if $e(\alpha) = 0$ then all Euler classes are zero and we have a trivial example of an Euler-complete formal group law. The result now follows from Section 6. □

Part 2. Complex oriented equivariant cohomology theories.

Complex oriented cohomology theories are more complicated in the equivariant setting, even when the ambient group of equivariance is abelian. The main source of the complexity is that topologically trivial line bundles need not be trivial: they are classified by one dimensional representations. A secondary problem, which won't appear explicitly here, is that to exploit the good behaviour of complex oriented theories one must decompose spaces into cells based on representations rather than on homogeneous spaces.

By design, the language of equivariant formal group laws is ideally suited to explain this more complicated structure. In Part 2 we re-examine topological examples in this framework, and give a number of examples.

9. Complex oriented cohomology theories.

Equivariant formal group laws were introduced to study complex oriented cohomology theories in general and equivariant bordism in particular. These are theories well behaved on line bundles, so we begin by summarizing properties of relevant spaces.

9.A. The classifying space for line bundles.

For each complex representation V we may form the A-space $\mathbb{C}P(V)$ of complex lines in V. It is sometimes useful to view projective space as the quotient $\mathbb{C}P(V) = S(V \otimes z)/\mathbb{T}$, where z is the natural representation of the circle group $\mathbb{T}$, $V \otimes z$ is the resulting representation of $A \times \mathbb{T}$, and $S(V \otimes z)$ is its unit sphere.

Note that if W is one dimensional, $\mathbb{C}P(W)$ is a point, so that a one dimensional subspace of V specifies a basepoint of $\mathbb{C}P(V)$: this is significant because basepoints may lie in different components of the fixed point set. At the other extreme, if α is one dimensional one may verify there is a cofibre sequence

9.1.

$$\mathbb{C}P(V) \longrightarrow \mathbb{C}P(V \oplus \alpha) \longrightarrow S^{V \otimes \alpha^{-1}}.$$

The A-invariant complex lines are exactly the subrepresentations of V, so it is easy to see that

9.2.

$$\mathbb{C}P(V)^A = \coprod_{\alpha} \mathbb{C}P(V_\alpha)$$

where $V_\alpha = \mathrm{Hom}_A(\alpha, V)$ is the α-isotypical part of V.

For convenience we take $\mathcal{U} = \bigoplus_{k \geqslant 0} \bigoplus_{\alpha \in A^*} \alpha$ as our complete A-universe and consider $\mathbb{C}P(\mathcal{U})$, with its topology as a colimit of its subspaces $\mathbb{C}P(V)$ with V finite dimensional.

The importance of projective spaces is the following standard fact.

Lemma 9.3. *The A-space $\mathbb{C}P(\mathcal{U})$ classifies line bundles.* □

The tensor product of line bundles is commutative and associative up to coherent isomorphism, and has ϵ as a unit: we constantly use the represented counterpart.

Corollary 9.4. *The A-space $\mathbb{C}P(\mathcal{U})$ is an abelian group object up to homotopy, and the inclusion of fixed points is a group homomorphism.* □

We have seen

$$\mathbb{C}P(\mathcal{U})^A = \coprod_\alpha \mathbb{C}P(\mathcal{U}_\alpha) \cong A^* \times \mathbb{C}P(\mathcal{U}_\epsilon).$$

Two maps arising from this are important. Firstly, note that since $\mathbb{C}P(\mathcal{U}_\alpha)$ is connected, there is a unique homotopy class

$$\zeta' : A^* \longrightarrow (\mathbb{C}P(\mathcal{U}))^A$$

splitting the natural augmentation $(\mathbb{C}P(\mathcal{U}))^A \longrightarrow A^*$, and it is a group homomorphism up to homotopy. In particular A^* acts on $\mathbb{C}P(\mathcal{U})$ through A-maps, by $\alpha \cdot L = \alpha \otimes L$. Secondly, there is the group homomorphism

$$j : \mathbb{C}P(\mathcal{U}_\epsilon) \longrightarrow \mathbb{C}P(\mathcal{U}),$$

which is a non-equivariant equivalence.

9.B. Complex stability and Euler classes.

We have already seen that the fixed point spaces of interesting A-spaces are disconnected, so it is rare for there to be a preferred basepoint. This is one reason it is convenient to work throughout in the *unbased* context.

A genuine equivariant cohomology theory $E_A^*(\cdot)$ is an exact contravariant functor on A-spaces, which admits an $RO(G)$-graded extension so that we have coherent suspension isomorphisms

$$\tilde{E}_A^{V+n}(S^V \wedge X) \cong \tilde{E}_A^n(X)$$

for all real representations V. Amongst these, many familiar ones have a stronger stability property

$$\tilde{E}_A^{|V|+n}(S^V \wedge X) \cong \tilde{E}_A^n(X)$$

when V is a complex representation, where $|V|$ denotes the space V with trivial action. This is very convenient: for most purposes we only need to look at the theory in integer gradings. Following tom Dieck we call these theories *complex stable.* As examples, we have the cohomology theory of the Borel construction, defined in terms of a complex orientable nonequivariant cohomology theory by $X \longmapsto E^*(EA \times_A X)$. A Serre spectral sequence argument shows this is complex stable, since A acts trivially on $H^{|V|}(S^V)$ when V is complex. The other examples we discuss below include complex equivariant K-theory and complex cobordism MU.

Now let us suppose given a multiplicative, complex stable equivariant cohomology theory $E_A^*(\cdot)$. For any complex representation V, complex stability provides an element

$$\lambda(V) \in \tilde{E}_A^{|V|}(S^V)$$

corresponding to the unit in E_A^0, and the E_A^*-module $\tilde{E}^*(S^V)$ is free of rank 1 on this generator. All complex stability isomorphisms are given by multiplication by $\lambda(V)$, and we have $\lambda(V \oplus W) = \lambda(V)\lambda(W)$. We then define the Euler class $\chi(V) = e_V^*(\lambda(V)) \in E_A^{|V|}$, where $e_V : S^0 \longrightarrow S^V$ is the inclusion. Thus we have $\chi(V \oplus W) = \chi(V)\chi(W)$.

9.C. Orientations and the cohomology of $\mathbb{C}P(\mathcal{U})$.

Notice that for any α we have containments $\epsilon \subseteq \epsilon \oplus \alpha \subseteq \mathcal{U}$, so that

$$*_\epsilon = \mathbb{C}P(\epsilon) \subseteq \mathbb{C}P(\epsilon \oplus \alpha) = S^{\alpha^{-1}} \subseteq \mathbb{C}P(\mathcal{U}).$$

Definition 9.5. [**3**] We say that $x(\epsilon) \in E_A^*(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\epsilon))$ is an *orientation* if for all one dimensional representations $\alpha \in A^*$,

$$\mathrm{res}^{\mathcal{U}}_{\epsilon\oplus\alpha} x(\epsilon) \in E_A^*(\mathbb{C}P(\epsilon \oplus \alpha), \mathbb{C}P(\epsilon)) \cong \tilde{E}_A^*(S^{\alpha^{-1}})$$

is a generator. The Euler class arising from this particular generator is denoted $e(\alpha^{-1})$.

We may generate many other elements from an orientation. Firstly, pulling back along the action

$$\alpha^{-1} : (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\alpha)) \longrightarrow (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\epsilon))$$

we have $x(\alpha) \in E_A^*(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\alpha))$. Writing $l_\alpha = (\alpha^{-1})^*$, we have $x(\alpha) = l_\alpha x(\epsilon)$. Taking external products, if $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ we obtain $x(V) \in E_A^*(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V))$. Forgetting the subspace, $x(V)$ defines an element $y(V) \in E_A^*(\mathbb{C}P(\mathcal{U}))$ which restricts to zero on $\mathbb{C}P(V)$. It turns out that the pair $(\mathbb{C}P(V \oplus W), \mathbb{C}P(V))$ defines a short exact sequence

$$0 \longleftarrow E_A^*(\mathbb{C}P(V)) \longleftarrow E_A^*(\mathbb{C}P(V \oplus W)) \longleftarrow E_A^*(\mathbb{C}P(V \oplus W), \mathbb{C}P(V)) \longleftarrow 0.$$

In particular $y(\epsilon)$ is the image of $x(\epsilon)$, and, since the restriction is injective, either one determines the other. It is clear that $y(0) = 1$, that $y(V \oplus W) = y(V)y(W)$, and that $(\alpha^{-1})^* y(V) = y(V \otimes \alpha)$. Thus all the elements $y(V)$ can be obtained from $y(\epsilon)$ using the action of A^* and the multiplication.

To obtain a topological additive basis of $E_A^*(\mathbb{C}P(\mathcal{U}))$ we choose a complete flag

$$F = (V^0 \subset V^1 \subset V^2 \subset \cdots),$$

so that $\dim_{\mathbb{C}}(V^i) = i$, and $\bigcup_{i \geqslant 0} V^i = \mathcal{U}$. Associated to any such complete flag F we have the sequence $\alpha_1, \alpha_2, \ldots$ of subquotients $\alpha_i = V^i/V^{i-1}$, so that $V^n \cong \alpha_1 \oplus \cdots \oplus \alpha_n$, and $y(V^n) = y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n)$. The basis only depends on the isomorphism classes of the sequence V^n, so the important structure is represented by the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$. We use the notation $k\{\{y_i | i \in I\}\}$ to denote the product $\prod_{i \in I} k$ where y_i is the characteristic function of the ith factor.

The Splitting Theorem of [**3**] is the appropriate substitute for the collapse of the Atiyah-Hirzebruch spectral sequence in the non-equivariant case.

Theorem 9.6. *(Cole [3]) A complete flag $F = (V^0 \subset V^1 \subset V^2 \cdots)$ specifies a basis of $E_A^*(\mathbb{C}P(\mathcal{U}))$ as follows:*

$$E_A^*(\mathbb{C}P(\mathcal{U})) = E_A^*\{\{y(V^0) = 1, y(V^1), y(V^2), \dots\}\}.$$

Similar results hold for products of copies of $\mathbb{C}P(\mathcal{U})$, in the sense that the Künneth theorem holds with completed tensor products.

Proof: The cofibre sequence

$$(S^{\alpha_{n+1}^{-1}}, *) = (\mathbb{C}P(V^{n+1}), \mathbb{C}P(V^n)) \longrightarrow (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V^{n+1})) \longrightarrow (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V^n))$$

is split in cohomology by $x(V^{n+1})$. □

For our purposes the main consequence it that we obtain an equivariant formal group law from a complex oriented cohomology theory.

Corollary 9.7. *A complex oriented cohomology theory $E_A^*(\cdot)$ gives rise to an A-equivariant formal group law*

- $k = E_A^*$
- $R = E_A^*(\mathbb{C}P(\mathcal{U}))$
- *the coproduct $\Delta : R \longrightarrow R\hat{\otimes}R$ is induced by $\otimes : \mathbb{C}P(\mathcal{U}) \times \mathbb{C}P(\mathcal{U}) \longrightarrow \mathbb{C}P(\mathcal{U})$*
- *the map $\theta : R \longrightarrow k^{A^*}$ is induced by $\zeta' : A^* \longrightarrow \mathbb{C}P(\mathcal{U})$*
- *the coordinate $y(\epsilon)$ is obtained from the orientation $x(\epsilon)$*
- *the inclusion of the identity component $\mathbb{G}_0 \longrightarrow \mathbb{G}$ is induced by $j : \mathbb{C}P(\mathcal{U}_\epsilon) \longrightarrow \mathbb{C}P(\mathcal{U})$*

Proof: The Künneth theorem, together with the properties of $\mathbb{C}P(\mathcal{U})$, shows that $R = E_A^*(\mathbb{C}P(\mathcal{U}))$ satisfies (Afgl1) and (Afgl2) with $k = E_A^*$, except perhaps the statement about topologies. The particular basis shows that $y(\epsilon)$ is a coordinate in the sense of (Afgl3), and that the topology on R is as required. □

10. Equivariant K-theory.

We include this section on K theory partly because it is one of the few cases where calculations are easy, but also because it is a template for an illuminating general approach to global equivariant formal group laws (see Remark 10.1). In fact, Bott periodicity shows K-theory is complex stable, and that we may work entirely in degree 0. Thus the coefficient ring $K_A = R(A)$ is the complex representation ring. It is important not to confuse the action of $R(A)$ on $K_A(\mathbb{C}P(\mathcal{U}))$ with action of A^* through ring homomorphisms induced from its action on $\mathbb{C}P(\mathcal{U})$.

The key observation is the change of group isomorphism

$$K_{A\times\mathbb{T}}(S(V \otimes z)) = K_A(\mathbb{C}P(V)),$$

where z is the natural representation of $\mathbb{T}$. This fundamental observation is a process of "uncompleting" the associated formal group (although the uncompleted object need no longer be a group).

Now we may exploit the based cofibre sequence $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$ and its associated Gysin sequence. By Bott periodicity, it takes the form

$$0 \longleftarrow K_{A\times\mathbb{T}}(S(V\otimes z)) \longleftarrow R(A)[z,z^{-1}] \overset{\chi(V\otimes z)}{\longleftarrow} R(A)[z,z^{-1}] \longleftarrow \cdots .$$

Furthermore, if $V = \alpha_1 \oplus \cdots \oplus \alpha_n$, then

$$\chi(V\otimes z) = \chi(\alpha_1 z)\chi(\alpha_2 z)\cdots\chi(\alpha_n z) = (1-\alpha_1 z)(1-\alpha_2 z)\cdots(1-\alpha_n z),$$

which is a regular element. Thus

$$K_A(\mathbb{C}P(V)) = K_{A\times\mathbb{T}}(S(V\otimes z)) = R(A)[z,z^{-1}]/\chi(V\otimes z).$$

Next, note that z is already invertible in $R(A)[z]/\chi(V\otimes z)$; indeed

$$1-\chi(V\otimes z) = z\cdot(V + \text{higher terms}).$$

Either by the completion theorem, or simply by passage to inverse limits, we see that

$$K_A(\mathbb{C}P(\infty V)) = R(A)[z]^{\wedge}_{\chi(V\otimes z)}.$$

Now observe that $y = 1-z$ is an orientation; K-theory is unusual in that this has finite degree in z. To verify it is indeed an orientation, we note that $1-z$ makes sense as an element of $K_A(\mathbb{C}P(V))$ for any V, and that $1-z$ visibly generates the kernel of

$$K_A(\mathbb{C}P(\epsilon\oplus\alpha)) = R(A)[z]/(1-z)(1-\alpha z) \longrightarrow R(A)[z]/(1-z) = K_A(\mathbb{C}P(\epsilon)).$$

The element z, regarded as an element of $K_A(\mathbb{C}P(V))$, is the canonical line bundle over $\mathbb{C}P(V)$, so it is easy to identify the A^* action: $l_\alpha z = \alpha z$. Since the action is through ring homomorphisms, $y(\alpha) = l_\alpha(1-z) = 1-\alpha z$.

Next, we specialize to case A is finite and V is the regular representation, and let $\Pi = \chi(\mathbb{C}A\otimes z) = \prod_\alpha(1-z\alpha)$. There is a straightforward and standard way to adapt the discussion to an arbitrary abelian compact Lie group A. The inclusion $i: A^*\times\mathbb{C}P(\mathcal{U}_\epsilon) \longrightarrow \mathbb{C}P(\mathcal{U})$ induces a map

$$i^*: R(A)[z]^{\wedge}_{\Pi} \longrightarrow \prod_\alpha R(A)[z]^{\wedge}_{(1-\alpha z)}.$$

The αth component is induced by completing the identity map of $R(A)[z]$ with respect to Π in the domain and $(1-\alpha z)$ in the codomain, as is legitimate since $(1-\alpha z)$ divides Π. Note in particular that i^* is injective, since the same primes contain the product and the intersection of the ideals $(1-\alpha z)$. It is also not hard to see that if we invert all the Euler classes $\chi(\alpha) = 1-\alpha$ then the ideals become coprime. Thus if we invert the Euler classes *before* completion, we obtain an isomorphism by the Chinese Remainder Theorem.

Remark 10.1. The discussion in this section applies to more general complex oriented equivariant cohomology theories $E^*_A(\cdot)$ defined for all abelian compact Lie groups. The main requirement is that the theory should be split in the sense that (i) $E^*_{A\times\mathbb{T}}(X) = E^*_A(X/\mathbb{T})$ when X is $\mathbb{T}$-free, and (ii) the coefficient rings are related by ring maps $E^*_A \longrightarrow E^*_{A\times\mathbb{T}}$ compatibly with the isomorphism.

The representation ring $R(A) = K_A = K^0_A$ must be replaced by E^*_A, and the Euler classes adapted accordingly. The most significant difference is that $E^*_{A\times\mathbb{T}} \neq E^*_A \otimes_{E^*} E^*_{\mathbb{T}}$ in general, and Euler classes $\chi(V \otimes z)$ need not be regular in $E^*_{A\times\mathbb{T}}$.

11. Completion and localization in topology.

We briefly discuss the topological counterparts of the Euler-local and Euler complete equivariant formal group laws described in Section 6. Historically, these well known topological versions of the localization and completion theorems suggested their algebraic counterparts.

Throughout this section we assume that E is a split, complex oriented cohomology theory, so that it has Euler classes.

11.A. Euler-local cohomology theories.

If we assume all Euler classes are invertible we have a reduction to fixed point spaces.

Lemma 11.1. *If all Euler classes are invertible in E^*_A then for any G-space X,*

$$E^*_A(X) = E^*_A(X^A).$$

Proof: Under the stated hypotheses, $F(S^{\infty\overline{\rho}}, E) = E$, where

$$S^{\infty\overline{\rho}} := \lim_{\rightarrow\, W^G=0} S^W.$$

Hence

$$[X, E]^*_A = [X, F(S^{\infty\overline{\rho}}, E)]^*_A = [X \wedge S^{\infty\overline{\rho}}, E]^*_A.$$

The map

$$X^A \wedge S^{\infty\overline{\rho}} \longrightarrow X \wedge S^{\infty\overline{\rho}}$$

is an A-equivalence: indeed, it is an equivalence in B-fixed points for all proper subgroups B since there is a representation W with $W^A = 0$ and $W^B \neq 0$, and it is obviously an equivalence in A-fixed points. □

Now take $X = \mathbb{C}P(\mathcal{U})$ and note that

$$E^*_A(\mathbb{C}P(\mathcal{U})) = E^*_A(\mathbb{C}P(\mathcal{U})^A) = E^*_A(\mathbb{C}P(\mathcal{U}_\epsilon) \times A^*) = E^*_A(\mathbb{C}P(\mathcal{U}_\epsilon)))^{A^*},$$

showing immediately that the associated equivariant formal group law is of the Euler-local form discussed in Subsection 6.A.

11.B. Euler-complete cohomology theories.

If we assume the cohomology theory is complete for the ideal $I\mathcal{E}$ generated by Euler classes we have a reduction to non-equivariant topology.

Lemma 11.2. *If $E^*_A(\cdot)$ is complete for the ideal $I\mathcal{E}$ generated by Euler classes then any A-map $X' \longrightarrow X$ which is a non-equivariant equivalence induces an $E^*_A(\cdot)$ isomorphism, and hence we have the Borel theory*

$$E^*_A(X) = E^*(EA \times_A X).$$

Proof: Since A is abelian we may construct EA as a product of spaces $S(\infty\alpha)$ with $\alpha \neq \epsilon$, and hence $I\mathcal{E}$ has the same radical as the augmentation ideal $I = \ker(E_A^* \longrightarrow E^*)$ (see [**12**]). Hence the completeness hypothesis gives $F(EA_+, E) = E$ and therefore

$$[X, E]_A^* = [X, F(EA_+, E)]_A^* = [X \wedge EA_+, E]_A^*.$$

□

Now take $X = \mathbb{C}P(\mathcal{U})$ and note that the inclusion

$$j : \mathbb{C}P(\mathcal{U}_\epsilon) \longrightarrow \mathbb{C}P(\mathcal{U})$$

is an A-map which is a non-equivariant isomorphism. Hence

$$E_A^*(\mathbb{C}P(\mathcal{U})) = E_A^*(EA \times \mathbb{C}P(\mathcal{U}_\epsilon)) = E^*(BA)[[z]],$$

showing immediately that the associated equivariant formal group law is of the Euler-complete form discussed in Subsection 6.B, giving the topological side of the connection with the work of Hopkins-Kuhn-Ravenel [**22**].

12. Equivariant connective K theory.

The results of this section give the topological counterpart of Section 7. It was shown in 7.9 that the universal ring $L_A^{sm} = L_A^m[v^{-1}]$ for strictly multiplicative equivariant formal group laws is the coefficient ring $K_A^* = R(A)[v, v^{-1}]$ of Atiyah-Segal equivariant periodic complex K-theory. One may do a little better. In fact there is a complex oriented, equivariant version $ku_A^*(\cdot)$ of connective K theory and, for simple enough groups A, its coefficient ring is L_A^m. Note immediately that this means ku_A^* is not connective (unless A is trivial), so that one must refer to "equivariant connective K theory", not to "connective equivariant K theory": this is forced by complex orientability and the relationship with periodic K theory.

This section summarizes results from [**15**], and in the topological context, the results are available for all compact Lie groups G. The previous paragraph summarized the results when $G = A$ is abelian, and when G is non-abelian this section can be viewed as motivation for Part 3, where equivariant formal group laws are discussed for non-abelian groups.

Theorem 12.1. *The G spectrum ku representing equivariant connective K theory has the following properties.*

1. *It is a commutative ring spectrum up to homotopy, and a highly structured commutative ring spectrum if G is finite.*
2. *If H is any subgroup of G then if we view the G-equivariant spectrum ku as an H-spectrum we obtain the H-equivariant construction. In particular, ku is non-equivariantly the connective cover of the periodic K-theory spectrum K.*
3. *ku is a split ring spectrum, and hence ku_G^* is an algebra over $ku^* = \mathbb{Z}[v]$.*
4. *There is a ring map $ku \longrightarrow K$ of G-spectra which is localization to invert v.*
5. *ku_G^* is a Noetherian ring.*

6. *There is a ring map $MU \longrightarrow ku$ of G-spectra, so that ku is complex orientable (if G is abelian this agrees with the notion of orientability in Definition 9.5 by [**5**]).*

For the purposes of the present article, the main reason for interest in the theory is its relation with multiplicative equivariant formal group laws.

Proposition 12.2. *If G is a product of two topologically cyclic groups then ku^*_G is the representing ring for multiplicative equivariant formal group laws as in Section 7:*

$$ku^*_G = L^m_G. \quad \square$$

Remark 12.3. Note that it follows from 7.9 that if G is topologically cyclic, ku^*_G is the Rees ring

$$ku^*_G = \mathrm{Rees}(R(G), J).$$

For other groups the connection is not so tight. In fact ku^*_G is non-zero in odd degrees for elementary abelian groups of rank 3 or more, and its even degree part can have $\mathbb{Z}$-torsion. In the rest of the secion we explain this in more detail. We also discuss non-abelian groups G, thereby providing some useful preparation for the general G-equivariant formal group laws of Part 3.

The connection with the non-equivariant theory is as good as possible in that the completion theorem and the local cohomology theorem hold. Let $I = \ker(ku^*_G \longrightarrow ku^*)$ be the augmentation ideal.

Proposition 12.4. *For any finite group G*

1. *The completion theorem holds for ku in the sense that*
$$ku^*(BG) = ku^*_G(EG) = (ku^*_G)^\wedge_I.$$
2. *The local cohomology theorem holds for ku in the sense that there is a spectral sequence*
$$H^{*,*}_I(ku^G_*) \Rightarrow ku^G_*(EG). \quad \square$$

We have partial results about the coefficient ring. The most satisfactory of these are in terms of the ring homomorphism

$$\ell : ku^*_G \longrightarrow K^*_G = R(G)[v, v^{-1}]$$

comparing connective and periodic K-theory, where $R(G)$ is the complex representation ring of G. We have already used the Rees ring when G is abelian in Section 7. One may think of the Rees ring as a ring in which elements of J^n become divisible by v^n. In topology, the Euler class $e(V)$ of an n-dimensional complex representation V naturally lies in degree $-2n$, and $v^n e(V) = e^R(V)$. Here

$$e^R(V) = \lambda_{-1}(V) = \sum_i (-1)^i \lambda^i(V) \in R(G)$$

is the representation theory Euler class. Thus we would expect the representation theory Euler class $e^R(V)$ of an n-dimensional representation to be divisible by v^n

in ku_G^*, even if it is not in J^n. Similarly the ith representation theory Chern class $c_i^R(V)$ should be divisible by v^i. Accordingly we define an algebraic model for ku_G^* with this property.

Definition 12.5. [**2**] The modified Rees ring ModRees(G) is the subring of $R(G)[v, v^{-1}]$ generated by $R(G)$, v and $v^{-i}c_i^R(V)$ for all representations V.

Remark 12.6. (i) If G is abelian, then ModRees(G) = Rees$(R(G), J)$, but in general the inclusion

$$\mathrm{Rees}(R(G), J) \subseteq \mathrm{ModRees}(G)$$

is proper.
(ii) The Rees ring only depends on $R(G)$ as an augmented ring. However, the modified Rees ring also depends on the exterior powers, so we write it as a functor of G rather than the ring $R(G)$.

Example 12.7. The groups Q_8 and D_8 have isomorphic augmented representation rings, and hence also isomorphic Rees rings. Their modified Rees rings are not only different from the Rees ring, but also different from each other. Indeed, if V is the 2-dimensional simple representation of Q_8 then $c_2(V) = 2 - V$, whilst if W is the 2-dimensional simple representation of D_8 then $c_2(V) = 1 - V + r$ where r is the one dimensional representation with kernel the rotation subgroup. □

We are now ready to state results about the coefficient rings. First we have a general statement if we invert v or tensor with the rational numbers.

Proposition 12.8. *1. In positive degrees, connective and periodic K theory agree:*

$$ku_i^G = K_i^G \text{ if } i \geqslant 0.$$

2. Above degree -4, the coefficients are as follows

$$ku_i^G = \begin{cases} 0 & \text{if } i \text{ is odd and } \geqslant 0 \\ R(G) & \text{if } i \text{ is even and } \geqslant 0 \\ 0 & \text{if } i = -1 \\ J & \text{if } i = -2 \\ 0 & \text{if } i = -3 \end{cases}$$

3. Localized away from v, connective and periodic K theory agree:

$$ku_G^*[1/v] \cong K_G^* = R(G)[v, v^{-1}].$$

4. If G is finite, the map $ku_G^* \longrightarrow K_G^*$ is a rational monomorphism and the image is the rationalized modified Rees ring. □

The lower coefficients behave in a more complicated way: it follows from calculations for elementary abelian groups in [**2**] that ku_{-6}^G can contain torsion (and is therefore not equal to the modified Rees ring in this degree) and ku_{-7}^G can be non-zero. One may show that $ku^G_{-5} = 0$, and it is natural to conjecture that ku^G_{-4} agrees with the modified Rees ring in that degree.

One special case is worth special mention. The calculations of [**2**] give the coefficient ring for a number of other groups.

Proposition 12.9. *If $G = U(n)$ then ku^*_G is the modified Rees ring*

$$ku^*_{U(n)} = \mathrm{ModRees}(U(n)).$$

We may therefore define ku-theory Chern classes for arbitrary representations by naturality. □

13. Complex cobordism and the universal equivariant formal group law.

One of the purposes of introducing equivariant formal group laws was the hope that they might provide an algebraic description of tom Dieck's homotopical complex cobordism ring MU^*_A. It is shown in [**5**] that complex orientable theories have Thom classes for all complex bundles, and hence that the representing spectrum MU is topologically universal for complex oriented theories, in the sense that an orientation in cohomological degree 2 gives rise to a ring map $MU \longrightarrow E$.

We are concerned here with algebraic universality, and illustrate the results of [**16**] in special cases. The construction of MU gives a canonical complex orientation, and therefore an equivariant formal group law over MU^*_A. There is thus a canonical map

$$\nu : L_A \longrightarrow MU^*_A$$

classifying its equivariant formal group law.

I conjecture that it is an isomorphism, but the best available result at present is a little weaker.

Theorem 13.1. *If A is a finite group, the ring MU^*_A represents A-equivariant formal group laws over Noetherian rings. More precisely, the map ν is surjective and its kernel is Euler torsion and Euler divisible.*

The reason for the unsatisfactory statement in the theorem is that the proof relies on a Hasse square which is only known to be a pullback for Noetherian rings or rings with bounded Euler torsion. It is known to be a pullback for MU^*_A since the boundedness of torsion is proved by topological means. However it is not known to be a pullback for L_A. Another approach via tori gives hope of a proof that ν is actually an isomorphism. Strickland has a more precise approach, and when A is of order 2 he has already given a presentation of MU^*_A and shown that it is a retract of L_A [**32**].

13.A. Outline of the proof.

We may illustrate the argument given in general in [**16**] by describing the case when A is of prime order p. The geometrical pullback square

$$\begin{array}{ccc} MU_A^* & \longrightarrow & \mathcal{E}^{-1}MU_A^* \\ \downarrow & & \downarrow \\ MU^*(BA) & \longrightarrow & \mathcal{E}^{-1}MU^*(BA) \end{array}$$

was first used by tom Dieck [**8**], but has recently been analyzed in more detail by Kriz [**23**], and Strickland has used it to give an exact presentation of MU_A^* in terms of formal group data, at least when A is of order 2 [**32**]. In view of the completion theorem [**24, 6, 19**],

$$MU^*(BA) = H_0^I(MU_A^*) = (MU_A^*)_I^\wedge,$$

where H_0^I denotes local homology in the sense of [**18**], and hence the corresponding algebraic square is

$$\begin{array}{ccc} L_A & \longrightarrow & \mathcal{E}^{-1}L_A \\ \downarrow & & \downarrow \\ H_0^I(L_A) & \longrightarrow & \mathcal{E}^{-1}H_0^I(L_A). \end{array}$$

Unfortunately this is not known to be a pullback unless $H_1^I(L_A)$ can be shown to be zero, but L_A does map onto the pullback. Evidently there is a comparison map from the algebraic to the topological square. The pullbacks will be isomorphic provided the maps

$$\mathcal{E}^{-1}\nu : \mathcal{E}^{-1}L_A \longrightarrow \mathcal{E}^{-1}MU_A^*$$

and

$$H_0(\nu) : H_0^I(L_A) \longrightarrow H_0^I(MU_A^*)$$

are shown to be isomorphic.

The topological pullback square for more general groups is of interest even without the comparison with the algebraic square.

13.B. The Euler-local isomorphism.

Given an equivariant manifold M, its fixed point set M^A will be a manifold equipped with an equivariant bundle, which can be split up into α-isotypical pieces for the non-trivial simple representations α. In tom Dieck's stabilized bordism certain Euler classes are inverted: this is the geometric content of tom Dieck's calculation [**8**]

$$\mathcal{E}^{-1}MU_A^* = (\Phi^A MU)_* = MU_*(BU[z, z^{-1}]^{\wedge(A^*\setminus\{\epsilon\})}).$$

Combining this with our algebraic calculation of $\mathcal{E}^{-1}L_A$ in 6.4 we obtain a local version of the equivariant Quillen theorem.

Proposition 13.2. *The universal map*

$$\nu : L_A \longrightarrow MU_A^*$$

induces an isomorphism

$$\mathcal{E}^{-1}L_A \xrightarrow{\cong} \mathcal{E}^{-1}MU_A^*. \quad \square$$

13.C. The Euler-complete isomorphism.

Combining the calculation 6.6 of $(L_A)_I^\wedge$ with the fact that MU is non-equivariantly complex cobordism, we obtain a complete version of the equivariant Quillen theorem.

Proposition 13.3. *The universal map*

$$\nu : L_A \longrightarrow MU_A^*$$

induces an isomorphism

$$(L_A)_I^\wedge \xrightarrow{\cong} (MU_A^*)_I^\wedge = MU^*(BA). \quad \square$$

Part 3. Towards equivariant formal groups for the non-abelian case.

In the final part we replace the abelian compact Lie group A by a non-abelian group G and discuss features of a possible definition of a G-equivariant formal group law. As in the abelian case our motivation is to understand complex oriented cohomology theories. We will give the definition in outline; a full presentation and proper development is planned by N.P.Strickland and the author [**21**]. The theory is not fully tested from the topological point of view because at present we have only shown that K theory provides an example, but the structure is certainly all relevant.

14. Ingredients.

When $G = A$ is abelian, all simple representations are one dimensional, so it is reasonable to expect some sort of splitting principle by which behaviour for arbitrary complex vector bundles can be reduced to that of line bundles. That is why it is reasonable to encode all the behaviour in terms of $R = E_A^*(BU(1))$, and all relevant structure comes from the map $\otimes : BU(1) \times BU(1) \longrightarrow BU(1)$ classifying tensor product of line bundles. The different equivariant types of a line bundle are described by vector bundles over a point hence by $A^* \longrightarrow BU(1)$. It is convenient to have the particular model $\mathbb{C}P(\mathcal{U})$ for the classifying space $BU(1)$.

In the general case we need to discuss the G-equivariant classifying spaces $BU(n)$ for all $n \geqslant 0$, and it is again convenient to have the Grassmannian model $\mathrm{Gr}_n(\mathcal{U})$ for $BU(n)$. This time, the relevant structure includes direct sum

$$\oplus_{m,n} : BU(m) \times BU(n) \longrightarrow BU(m+n),$$

tensor product

$$\otimes_{m,n} : BU(m) \times BU(n) \longrightarrow BU(mn),$$

and exterior power maps

$$\lambda^i_m : BU(m) \longrightarrow BU(\binom{m}{i}).$$

We also need a point in $BU(1)$ which is the unit of $\otimes$, which then gives a map

$$BU(m) \longrightarrow BU(m+1)$$

adding a trivial representation. The formal properties of these maps are known. This is perhaps best encoded in terms of the category $\mathbb{N}$ of natural numbers equipped with the two monoidal structures direct sum $(m \oplus n := m+n)$ and tensor product $(m \otimes n := mn)$, together with exterior power operations and a map $0 \longrightarrow 1$ giving the unit of $\otimes$. We say that a functor

$$F : (\mathbb{N}, \oplus, \otimes, \lambda^i) \longrightarrow \mathcal{C}$$

to a symmetric monoidal category $\mathcal{C}$ is an $\mathbb{N}$-*algebraic object of* $\mathcal{C}$ if it is a lax symmetric monoidal functor with tensor product distributing over sum, and exterior powers behaving in the usual way. Similarly an $\mathbb{N}$-*coalgebraic object of* $\mathcal{C}$ is an $\mathbb{N}$-algebra in $\mathcal{C}^{op}$.

Remark 14.1. Strickland considers the same structure in [**30**] where he refers to $\mathbb{N}$-algebras as Λ-semirings, and the reader will find it useful to refer to his more precise and thorough treatment. We have chosen different terminology to emphasize the view that an $\mathbb{N}$-algebra is an $\mathbb{N}$-indexed collection of objects connected with addition, multiplication and additional structure.

We shall be guided by a simple example.

Example 14.2. The unitary group $U(\cdot)$ functor defines an $\mathbb{N}$-algebraic group, and $R(U(\cdot))$ defines an $\mathbb{N}$-coalgebraic ring.

Slightly more generally, for any compact Lie group G, the functor $G \times U(\cdot)$ defines an $\mathbb{N}$-algebraic group, and $R(G \times U(\cdot))$ defines an $\mathbb{N}$-coalgebraic ring.

We record the motivating example in these terms.

Example 14.3. The G-equivariant classifying space functor $BU(\cdot)$ is an $\mathbb{N}$-algebraic G-space up to homotopy, and if $E^*_G(\cdot)$ is a multiplicative theory with a Künneth theorem for the spaces $BU(n)$, $E^*_G(BU(\cdot))$ is an $\mathbb{N}$-coalgebraic E^*_G-algebra.

The final example we need is the simplest.

Example 14.4. For any compact Lie group G, the functor $\mathrm{Rep}_G(\cdot)$ with $\mathrm{Rep}_G(n)$ the set of isomorphism classes of n-dimensional complex representations of G is an $\mathbb{N}$-algebraic set with the usual operations

$$\oplus_{m,n} : \mathrm{Rep}_G(m) \times \mathrm{Rep}_G(n) \longrightarrow \mathrm{Rep}_G(m+n),$$

$$\otimes_{m,n} : \mathrm{Rep}_G(m) \times \mathrm{Rep}_G(n) \longrightarrow \mathrm{Rep}_G(mn),$$

and

$$\lambda^i_m : \mathrm{Rep}_G(m) \longrightarrow \mathrm{Rep}_G(\binom{m}{i}).$$

Taking rings of k-valued functions $k^{\mathrm{Rep}_G(\cdot)}$ gives an $\mathbb{N}$-coalgebraic k-algebra.

Notice that choosing a point in each component of the G-fixed point set gives a comparison G-map

$$\mathrm{Rep}_G(n) \longrightarrow BU(n).$$

Since operations on bundles reduce to the corresponding operation on each fibre, this is compatible up to homotopy with all structure maps.

15. The definition of G-equivariant formal group laws.

Combining Examples 14.3 and 14.4 we obtain the basis of the definition of a G-equivariant formal group. Take

$$k = E_G^* \text{ and } R_n = E_G^*(BU(n)),$$

and note that $E_G^*(\mathrm{Rep}_G(n)) = k^{\mathrm{Rep}_G(n)}$. Thus we have two $\mathbb{N}$-coalgebraic k-algebras and a homomorphism

$$\theta : R \longrightarrow k^{\mathrm{Rep}_G}$$

of $\mathbb{N}$-coalgebraic k-algebras.

We now give an outline definition, referring to [**21**] for details. We will comment on the imprecision below.

Outline Definition 15.1. If G is a finite group, a *G-equivariant formal group law* over a commutative ring k is

(Gfgl1) an $\mathbb{N}$-coalgebra

$$R = (R_n) : (\mathbb{N}, \oplus, \otimes, \lambda^i) \longrightarrow k\text{-alg}$$

in complete topological k-algebras.

(Gfgl2) a homomorphism $\theta : R \longrightarrow k^{\mathrm{Rep}_G}$ of $\mathbb{N}$-coalgebras in complete topological k-algebras in terms of which the topology on R is defined

(Gfgl3) an element $y_n(n\epsilon) \in R_n$ for each n so that there is an exact sequence

$$0 \longrightarrow R_n \xrightarrow{y_n(n\epsilon)} R_n \longrightarrow R_{n-1} \longrightarrow 0,$$

where the map $R_n \longrightarrow R_{n-1}$ is the restriction structure map.

Remark 15.2. (i) We can use the structure to define Chern classes. Indeed, we may think of $y_n(n\epsilon)$ as the nth Chern class c_n of the natural representation of $U(n)$, and from the exact sequence of (Gfgl3), we may lift $y_{n-1}((n-1)\epsilon), \ldots, y_1(\epsilon)$ to give a regular sequence $c_n, c_{n-1}, \ldots, c_1$ in R_n. The choice of lifts is not part of the structure.

It then follows that

$$(c_n, c_{n-1}, \ldots, c_1) = \ker(\theta_{n\epsilon} : R_n \longrightarrow k),$$

showing that the kernel of $\theta_{n\epsilon}$ is a regular ideal, exactly as in the abelian case.
(ii) The imprecision in the outline is that we have not described how the topology

on R is defined using θ. In particular, the topology on R_1 is finer than that defined by the kernel of $R_1 \longrightarrow k^{\mathrm{Rep}_G(1)}$, because we need to take account of representations of higher dimension. The idea for a finite group of order n is to use the kernel $I_n(\rho)$ of

$$R_n \xrightarrow{\theta_\rho} k,$$

where ρ is the regular representation. Extending the ideal $I_n(\rho)$ along the map $R_n \longrightarrow R_1$ induced by $\oplus^n : 1 \longrightarrow n$ we define an ideal in R_1 which gives the topology.

Example 15.3. If $G = A$ is abelian then Definition 2.1 is consistent with the new definition. First note that any n-dimensional representation is the sums of n one dimensional representations, so that $\mathrm{Rep}_G(n) = \mathrm{Rep}_G(1)^n/\Sigma_n$ and therefore

$$k^{\mathrm{Rep}_G(n)} = ([k^{\mathrm{Rep}_G(1)}]^{\hat{\otimes} n})^{\Sigma_n}.$$

Thus if R_1 is an equivariant formal group law in the sense of Definition 2.1 we may define an equivariant formal group law in the sense of the Definition 15.1 as follows. First, we take

$$R_n = (R_1^{\hat{\otimes} n})^{\Sigma_n},$$

and similarly for θ. Now take

$$y_n(n\epsilon) = y(\epsilon)^{\otimes n}.$$

The map $R_n \longrightarrow R_{n-1}$ is given by setting $1 \otimes \cdots \otimes 1 \otimes y(\epsilon)$ to be zero, and it is not hard to check that $y_n(n\epsilon)$ is regular and generates the kernel of this map. In this case the topology on R_n is defined by the kernel of $\theta_n : R_n \longrightarrow k^{\mathrm{Rep}_G(n)}$.

Note also that in this case we may give canonical lifts of the elements $y_n(n\epsilon)$ using the symmetric functions in $y(\epsilon)$.

Remark 15.4. In general, direct sum gives a map

$$R_n \longrightarrow (R_1^{\hat{\otimes} n})^{\Sigma_n},$$

and we call an equivariant formal group law R *symmetric* if this is an isomorphism. For these equivariant formal groups, the comments in Remark 15.2 completely describe the topology so 15.1 is complete.

For these symmetric laws, the connection with the geometry of divisors as in Strickland's work is close and illuminating [**28, 29**]. However it is not clear that all topological examples of equivariant formal groups are symmetric.

16. Complex oriented cohomology theories.

The intention is that complex oriented cohomology theories are those that behave in the best possible way for complex vector bundles. At present we do not have a proper codification of this best possible behaviour, but whatever it means it should cover tom Dieck's homotopical complex cobordism MU, so it is natural to make the following definition.

Definition 16.1. We say that a G-equivariant cohomology theory E is *complex oriented* if it is a ring theory equipped with a ring map $y : MU \longrightarrow E$ of G-spectra.

Results of [**5**] show that this is consistent with Definition 9.5.

Construction 16.2. If E is a complex oriented theory in this sense, we hope to define a G-equivariant law by taking

$$k = E_G^* \text{ and } R_n = E_G^*(BU(n)).$$

The structure maps are immediate from those of the unitary groups provided E_G^* has a Künneth theorem for products of $BU(n)$. We may define the classes $y_n(n\epsilon)$ by pullback from MU provided they may be defined in that case.

Remark 16.3. If A is abelian, this is consistent with the construction of Corollary 9.7. Indeed, results of [**5**] show that in the topological case

$$R_n = E_A^*(BU(n)) = (E_A^*(BU(1))^{\hat{\otimes} n})^{\Sigma_n} = (R_1^{\hat{\otimes} n})^{\Sigma_n}.$$

We may therefore define $y_n(n\epsilon)$ as the nth symmetric function in $y(\epsilon)$. As in Example 15.3, the regularity assertion (Gfgl3) follows.

We may then ask when Construction 16.2 actually does give a G-equivariant formal group law.

Lemma 16.4. *(i) If G is abelian, any complex oriented theory in the sense of Definition 2.1 gives a G-equivariant formal group law in the sense of 15.1 by the construction of 16.3.*
(ii) Equivariant K theory gives a G-equivariant formal group laws for all compact Lie groups G.
(iii) Any global equivariant cohomology theory with Künneth theorems for products of $BU(n)$'s and a completion theorem will provide a structure satisfying (Gfgl1) and (Gfgl2).
(iv) Any complex oriented equivariant cohomology theory is equipped with classes $y_n(n\epsilon)$.

Proof: We have already outlined the proof of Part (i). For Part (iv) we need only observe that there such a class for MU, but this is given by the canonical map

$$BU(n) \longrightarrow MU(n) \longrightarrow \Sigma^{2n} MU.$$

We will prove Part (ii) in detail for equivariant K theory, and the proof of Part (iii) will be clear.

Now we turn to K theory and show it does give a G-equivariant formal group law. The proof really takes the $\mathbb{N}$-coalgebraic k-algebra $R(G \times U(\cdot))$ representing an $\mathbb{N}$-algebraic group scheme over k and makes an appropriate completion.

First, the split condition allows us to make calculations using the completion theorem, because

$$K_G(BU(m)) = K_{G \times U(m)}(EU(m)),$$

and the coefficients are given by

$$K_{G\times U(m)} = R(G)\otimes R(U(m)) =$$
$$R(G)[\lambda^1,\dots\lambda^{m-1},\lambda^m,(\lambda^m)^{-1}] = R(G)[c_1,\dots c_{m-1},c_m,(\lambda^m)^{-1}].$$

The $G\times U(m)$-space $EU(m)$ is the universal space for the family $[\cap U(m)=1] = \{H \mid H\cap U(m)=1\}$. Accordingly the completion theorem states that

$$K_{G\times U(m)}(EU(m)) = R(G\times U(m))^{\wedge}_{I[\cap U(m)=1]},$$

where

$$I[\cap U(m)=1] = \bigcap_{H\in[\cap U(m)=1]} \ker\left[R(G\times U(m)) \longrightarrow R(H)\right].$$

Lemma 16.5. *If G is finite*

$$I[\cap U(m)=1] = (\chi(\rho\lambda^1),\dots\chi(\rho\lambda^{m-1}),\chi(\rho\lambda^m))$$

where ρ is the regular representation of G. □

In view of the presence of $\chi(\rho\lambda^m)$ in the ideal, we again see that we may omit $(\lambda^m)^{-1}$ from the list of generators.

Summary 16.6.

$$K_G(BU(m)) = R(G)[c_1,\dots,c_m]^{\wedge}_{(\chi(\rho\lambda^1),\dots,\chi(\rho\lambda^m))}.$$

For an m-dimensional representation V

$$\theta_V : R(G)[c_1,\dots,c_m]^{\wedge}_{(\chi(\rho\lambda^1),\dots,\chi(\rho\lambda^m))} \longrightarrow R(G)$$

is reduction modulo $(c_1(V),\dots,c_m(V))$. The special element $y_n(n\epsilon)=\lambda^1$.

It is now clear that $K_G(BU(\cdot))$ is a G-equivariant formal group law. □

Remark 16.7. (i) Note that the example of K theory is symmetric in the sense that $R_n = (R_1^{\hat{\otimes} n})^{\Sigma_n}$.
(ii) The example of the symmetric group Σ_3 of degree 3 shows that the topology of $K_G(BU(1))$ is not defined by the map

$$\theta_1 : K_G(BU(1)) \longrightarrow R(G)^{\mathrm{Rep}_G(1)},$$

so that the further discussion of the topology in Remark 15.2 (ii) is necessary.

The general question of when $E^*(BG)$ is generated by Chern classes may be viewed as a problem about Euler-complete G-equivariant formal group laws. It has been much studied for ordinary cohomology [**33, 11**] and there are some calculations in [**2**]. However Strickland's study [**30**] is exactly in the present spirit and gives extensive general information.

A number of obvious questions remain.

Questions 16.8. 1. For which groups G does equivariant bordism provide a G-equivariant formal group law? (Known for abelian groups [**4, 5**]. The general case will be discussed further in [**21**]; the remaining obstacle is to show that a suitable exact Künneth theorem holds.)

2. Does every complex oriented cohomology theory give an equivariant formal group law? (Known for abelian groups [**4, 5**].)
3. For which groups G is there a universal ring for G-equivariant formal group laws? (Known for abelian groups [**4, 5**].)
4. For which groups G is MU_G^* in even degrees? (Known for abelian groups [**24, 6**].)
5. For which groups G is MU_G^* the universal ring for G-equivariant formal group laws? (Known for the trivial group. Nearly known for abelian groups [**16**]; more nearly known for the group of order 2 [**32**].)

References

[**1**] J.F.Adams "Stable homotopy and generalized homology." University of Chicago Press (1974)

[**2**] R.R.Bruner and J.P.C.Greenlees "The connective K-theory of finite groups." Submitted for publication 131pp.

[**3**] M.M.Cole "Complex oriented $RO(G)$-graded equivariant cohomology theories and their formal group laws." Thesis, University of Chicago (1996)

[**4**] M.M.Cole, J.P.C.Greenlees and I. Kriz "Equivariant formal group laws." Proc. London Math. Soc **81** (2000) 355-386.

[**5**] M.M.Cole, J.P.C.Greenlees and I. Kriz "Universality of equivariant bordism." Math. Z. (to appear) 16pp

[**6**] G.Comezana "Calculations in complex equivariant bordism." CBMS Regional conference series in math. **91** AMS (1996) 333-352

[**7**] G.Comezana and J.P.May "A completion theorem in complex cobordism." CBMS Regional conference series in math. **91** AMS (1996) 327-332

[**8**] T.tom Dieck "Bordism of G-manifolds and integrality theorems." *Topology* **9** (1970) 345-358

[**9**] T. tom Dieck "Lokalisierung äquivarianter Kohomologie-Theorien." *Math Z.*, 121, 253-262 (1971).

[**10**] T.tom Dieck "Characteristic numbers of G-manifolds II" *J. Pure and Applied Algebra* **4** (1974) 31-39

[**11**] D.J.Green and I.J.Leary "The spectrum of the Chern subring." Comm. Math. Helv. **73** (1998) 406-426

[12] J.P.C.Greenlees "Augmentation ideals of equivariant cohomology rings." Topology **37** (1998) 1313-1323

[13] J.P.C.Greenlees "Equivariant forms of connective K-theory." Topology **38** (1999) 1075-1092.

[14] J.P.C.Greenlees "Multiplicative equivariant formal group laws." J. Pure and Applied Algebra (to appear) 20pp.

[15] J.P.C.Greenlees "Equivariant connective K theory for compact Lie groups" (Submitted for publication) 17pp

[16] J.P.C.Greenlees "The coefficient ring of equivariant homotopical bordism classifies equivariant formal group laws over Noetherian rings." (Submitted for publication) 16pp

[17] J.P.C.Greenlees "Equivariant formal groups over tame rings" Preprint 16pp

[18] J.P.C.Greenlees and J.P. May "Derived functors of I-adic completion and local homology" Journal of Algebra **149** (1992) 438-453.

[19] J.P.C.Greenlees and J.P.May "Localization and completion theorems for MU-module spectra." Annals of Maths. **146** (1997) 509-544

[20] J.P.C.Greenlees and N.P.Strickland "Varieties and local cohomology for chromatic group cohomology rings." Topology **38** (1999) 1093-1139.

[21] J.P.C.Greenlees and N.P.Strickland "The geometry of equivariant formal group laws." (In preparation)

[22] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. "Generalised group characters and complex oriented cohomology theories." J. American Math. Soc. **13** (2000) 553-594

[23] I. Kriz "The $\mathbb{Z}/p$-equivariant stable complex cobordism ring". Contemporary Math. **239** (1999) 217-223, American Maths. Society.

[24] P. Löffler "Equivariant unitary bordism and classifying spaces. " Proc. International Symposium on Topology and its Applications, Budva, Yugoslavia (1973) 158-160

[25] C.Okonek "Der Conner-Floyd-Isomorphismus für Abelsche Gruppen" *Math. Z.* **179** (1982) 201-212.

[26] C.Okonek "Die äquivariante unitäre Kobordimentheorie und ihre formale Gruppe." *Arch. Math.* **40** (1983) 208-220.

[27] D.P. Sinha "Computations in complex equivariant bordism theory." Preprint (1998) 20pp. (available from http://hopf.math.purdue.edu/pub/hopf.html)

[28] N.P.Strickland "Finite subgroups of formal groups." J. Pure and Applied Algebra **121** (1997) 161-208

[29] N.P.Strickland "Formal schemes and formal groups." Cont. Math. **239** (1999) 263-352

[30] N.P.Strickland "Chern approximations" Topology (to appear)

[31] N.P.Strickland "Multicurves." Preprint (2000)

[32] N.P.Strickland "Complex cobordism of involutions." Geometry and Topology **5** (2001) 335-345

[33] C.B.Thomas "Characteristic classes and the cohomology of finite groups." Cambridge University Press (1986) xii+129 pp

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J.P.C.Greenlees `j.greenlees@sheffield.ac.uk`

Department of Pure Mathematics,
Hicks Building,
Sheffield S3 7RH.
UK.