Homology, Homotopy and Applications, vol.3(2), 2001, pp.265–339

EQUIVARIANT ORIENTATION THEORY

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(*communicated by Gunnar Carlsson*)

Abstract

We give a long overdue theory of orientations of G-vector bundles, topological G-bundles, and spherical G-fibrations, where G is a compact Lie group. The notion of equivariant orientability is clear and unambiguous, but it is surprisingly difficult to obtain a satisfactory notion of an equivariant orientation such that every orientable G-vector bundle admits an orientation. Our focus here is on the geometric and homotopical aspects, rather than the cohomological aspects, of orientation theory. Orientations are described in terms of functors defined on equivariant fundamental groupoids of base G-spaces, and the essence of the theory is to construct an appropriate universal target category of G-vector bundles over orbit spaces G/H. The theory requires new categorical concepts and constructions that should be of interest in other subjects, such as algebraic geometry.

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The second author was partially supported by the NSF
Received March 8, 2001, revised July 7, 2001; published on September 13, 2001.
2000 Mathematics Subject Classification: Primary 55P91; Secondary 18B40, 20L15, 55N25, 55N91, 55P20, 55R91, 57Q91, 57R91.
Key words and phrases: Equivariant classifying space, equivariant bundle, equivariant fibration, equivariant fundamental groupoid, equivariant orientation, groupoid.

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Introduction

When is a smooth G-manifold orientable, where G is a compact Lie group? Accepting the obvious answer, "when its tangent bundle is orientable," what does it mean to say that a G-vector bundle is orientable? There is a good, straightforward, answer to this question. Suppose that $p\colon E \to B$ is an orthogonal G-vector bundle. Recall that, if H is a subgroup of G, then the fiber $F = p^{-1}(x)$ over an H-fixed point $x \in B$ is a representation of H. We say that p is equivariantly orientable if, whenever we transport such a fiber F around a loop in the H-fixed set B^H, the resulting self-map of F is homotopic to the identity map through H-linear isometries. Already we can see one complication that does not arise nonequivariantly: If we ask that the self-map of F be homotopic to the identity through equivariant PL maps, or homeomorphisms, or homotopy equivalences, rather than linear isometries, we may get different notions of orientability. One purpose of the categorical framework we develop here is to allow us to handle all of these cases with the same machinery.

The next obvious question is, what do we mean by an orientation of an orientable G-manifold or G-vector bundle? Surprisingly, there is no satisfactory answer in the literature except under rather restrictive hypotheses. One of us began work on this question in a 1986 preprint [**31**], and the three of us took up the problem soon after. This paper is a revision of an undistributed 1989 preprint, and in the meantime a number of papers have appeared that are explicitly or implicitly based on that preprint [**1, 3, 4, 5, 6, 7, 25**]. The answer to the question is necessarily complicated, and our present categorical framework is a significant improvement on our original one. We give the idea by reviewing one approach to classical orientation theory.

Nonequivariantly, an elaborately pedantic way of defining an orientation of an n-plane bundle $p\colon E \longrightarrow B$ runs as follows. We consider the category $\mathscr{V}(n)$ with one object $\mathbb{R}^n$ and the two morphisms given by the two homotopy classes of linear isometries $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. We have the inclusion S of the discrete subcategory $\mathscr{S}\mathscr{V}(n)$ with just the identity morphism. We may think of $S\colon \mathscr{S}\mathscr{V}(n) \longrightarrow \mathscr{V}(n)$ as obtained from $SO(n) \longrightarrow O(n)$ by passing to components. We choose and fix an isomorphism from each n-dimensional vector space V to $\mathbb{R}^n$, thereby obtaining an equivalence of categories from the category $\overline{\mathscr{V}}(n)$ of all n-dimensional vector spaces to $\mathscr{V}(n)$. Using the bundle covering homotopy property (CHP), we see that p induces a functor p^* from the fundamental groupoid ΠB to $\overline{\mathscr{V}}(n)$ that sends a point b to the fiber V_b over b. Using our fixed equivalence of categories, this gives a functor $p^*\colon \Pi B \longrightarrow \mathscr{V}(n)$. The functor p^* fixes a choice of orientation of each fiber and describes how the orientations of fibers change as one traverses paths in the base space. The bundle p is orientable if and only if there is a lift of this functor to $\mathscr{S}\mathscr{V}(n)$, by which we mean a functor $F\colon \Pi B \longrightarrow \mathscr{S}\mathscr{V}(n)$ together with a natural isomorphism $\phi\colon S \circ F \longrightarrow p^*$. A choice of such a lift is an orientation of p. Here the functor F is obviously unique if it exists, but there are then two choices of ϕ if B is path connected.

We shall mimic this procedure equivariantly. We define the equivariant fundamental groupoid $\Pi_G X$ and the equivariant analogue $\mathscr{V}_G(n)$ of $\mathscr{V}(n)$ in §1 and §2, where we also define $p^*\colon \Pi_G B \longrightarrow \mathscr{V}_G(n)$ for a G-vector bundle $p\colon E \longrightarrow B$ and explain what it means for p to be orientable. There are two main variants of the relevant categories, which coincide when G is finite. In §3 and §4, we show how to topologize $\Pi_G X$ and $\mathscr{V}_G(n)$ so that the more commonly used variants are the respective homotopy categories of the variants most appropriate to our theory.

The definition of the equivariant analogue $\mathscr{S}\mathscr{V}_G(n)$ of $\mathscr{S}\mathscr{V}(n)$ turns out to be quite subtle. The idea is to find a functor $S\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{V}_G(n)$ such that p is orientable if and only if p^* factors through $\mathscr{S}\mathscr{V}_G(n)$ and such that $\mathscr{S}\mathscr{V}_G(n)$ is the "smallest" category with this property. To carry out this idea, we need the categorical notion of a bundle of groupoids over the orbit category $\mathscr{O}_G$, or over any category $\mathscr{B}$ with similar structure. We call these objects groupoids over $\mathscr{B}$ for short. This notion is defined in §5; $\Pi_G X$ and $\mathscr{V}_G(n)$ are examples of groupoids over $\mathscr{O}_G$. To construct $\mathscr{S}\mathscr{V}_G(n)$, we need the restricted types of groupoids over $\mathscr{B}$ that are described in §6. These arise as quotients of $\Pi_G B$ through which p^* factors when p is orientable. We introduce and explain a kind of representation theory of bundles of groupoids that allows us to define orientations of orientable G-bundles in §7. The construction and characterization of the "universal orientable representation" $S\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{V}_G(n)$ used in the definition is carried out in §8 and §9. We obtain the following theorem.

Theorem 0.1. *A G-vector bundle $p\colon E \longrightarrow B$ of dimension n is orientable if and only if $p^*\colon \Pi_G B \longrightarrow \mathscr{V}_G(n)$ can be lifted to a functor $F\colon \Pi_G B \longrightarrow \mathscr{S}\mathscr{V}_G(n)$ together with a natural isomorphism $\phi\colon S \circ F \longrightarrow p^*$. A choice of such a lift (F, ϕ) is an orientation of p.*

This notion correctly encodes the intuitive idea that an orientation should be a consistent set of orientations of the restricted bundles over orbits of B. Here consistency entails consistency with all paths in all fixed point spaces in B. Since $\mathscr{S}\mathscr{V}_G(n)$ must allow for all possibilities, its construction is intrinsically complicated. The categorical representation theory that is involved may well have applications in other fields.

The very abstract definitions and constructions in Part II (§§5 – 9) are illustrated by concrete examples in §10. Specifically, we trace through the steps of the construction and give an explicit description of the universal orientable representation for a cyclic group of prime order. The reader may find it helpful to refer to this section while reading Part II.

We discuss G-bundles "of dimension V" for a representation V of G and illustrate the need for our theory with a simple example in §11. For G-bundles over a general compact Lie group, or even over a general finite group G, there seems to be no precursor to our theory in the literature. There is a naive notion of an orientation of a V-dimensional G-bundle that is sometimes used, but we show that this notion is insufficient to give a satisfactory theory. An obvious desideratum of a satisfactory theory is that every orientable G-bundle must admit an orientation, but this fails with the naive notion. In fact, the 2-sphere S^2 with the circle group S^1 or any of its cyclic subgroups acting by rotation around the polar axis (say) gives an elementary example of an orientable G-manifold that admits no naive orientation. For cyclic groups of prime order, we display the orientations of S^2 explicitly. We urge the reader who has not thought about equivariant orientation theory to consider that example first, since it well illustrates both the problem and our solution of it.

We describe the universal orientable representation explicitly for any odd order finite group G in §12. Here it turns out that a G-vector bundle is orientable in the equivariant sense if and only if it is orientable in the nonequivariant sense, and then equivariant orientations are uniquely determined by their underlying nonequivariant orientations. The equivariant orientation describes additional fixed point space information that is implicit in the nonequivariant orientation. The nature of this information is not obvious. In fact, there is a naive notion of an equivariant orientation of any G-vector bundle for a group of odd order. For V-dimensional bundles it coincides with the naive notion of §11, so the example there shows that not every oriented G-vector bundle can be naively oriented. We also describe the essentially trivial complex analogue of our theory in §12.

We give a conjectural description of the universal orientable representation for an elementary Abelian 2-group in §13. We doubt that the conjecture is right, but with more work the ideas presented should lead to a correct description of the universal orientable representation for any Abelian compact Lie group G. As a first non-Abelian example, we display the universal orientable representation for 2-dimensional G-vector bundles for the dihedral group $G = D_6$ in §14.

We return to the general categorical theory in §15 and §16, first showing how the theory of categorical fibrations gives an alternative way of thinking about orientations in §15 and then discussing the functoriality with respect to changes of the reference groupoid over $\mathscr{B}$ into which representations map in §16. In §17, we use this discussion to show that an orientation of a G-bundle $p\colon E \longrightarrow B$ induces orientations of the H-fixed point bundle over B^H and of its complementary bundle over B^H for all subgroups H of G. For an oriented smooth G-manifold M, this means that the fixed point manifolds M^H and the normal bundles of the inclusions $M^H \subset M$ inherit appropriate orientations.

While our main focus is on G-vector bundles, the theory also applies to topological and PL G-bundles, to spherical G-fibrations, and to stable and virtual variants of each of these. We explain this in §18 and §19. The discussion of functoriality in §16 allows comparisons among these versions of orientation theory.

In §20 – §23, we describe classifying G-spaces and prove classification theorems for oriented G-bundles and oriented spherical G-fibrations. We prove a related classification theorem for representations of fundamental groupoids in §24.

Despite the length of this paper, we have by no means obtained a complete theory. Nonequivariantly, there are geometric and cohomological notions of orientation, and the geometric theory coincides with the cohomological theory when we take ordinary cohomology with integral coefficients. That is a calculational fact that does not carry over to the equivariant context. While ideas here have been used successfully in work towards the cohomological theory in [**1, 3, 4, 5, 6, 7, 25**], there remains much work to be done, particularly in unifying and systematizing the several different approaches that are taken in the cited papers. We plan to return to this matter elsewhere.

Part I. Fundamental groupoids and categories of bundles

1. The equivariant fundamental groupoid

We recall the definition and properties of the fundamental groupoid of a G-space X. We understand spaces to be compactly generated (= weak Hausdorff k-spaces), and we let $\mathscr{U}$ denote the category of (unbased) spaces. A topological category is a category enriched over $\mathscr{U}$, so that its hom sets are spaces and composition is continuous. A functor between topological categories is continuous if it is continuous on hom sets. Recall that a category is a *groupoid* if all of its morphisms are isomorphisms. We shall later be interested in topological groupoids, but we focus on the underlying untopologized categories in this section and the next.

Our ambient group G is a compact Lie group, and subgroups are understood to be closed. The *orbit category* $\mathscr{O}_G$ is the topological category whose objects are the orbit G-spaces G/H and whose morphisms are the G-maps between orbits. The morphism set $\mathscr{O}_G(G/H, G/K)$ is topologized as the subspace $(G/K)^H$ of G/K.

The following definition is given by tom Dieck [**10**, 10.7]. We regard an element $x \in X^H$ as the G-map $G/H \longrightarrow X$ that sends eH to x, going back and forth at will between the two interpretations, and similarly for paths, etc, in X^H.

Definition 1.1. Let X be a G-space. The *(equivariant) fundamental groupoid* $\Pi_G X$ of X is the category whose objects are the G-maps $x\colon G/H \longrightarrow X$ and whose morphisms $x \longrightarrow y$, $y : G/K \longrightarrow X$, are the pairs (ω, α), where $\alpha\colon G/H \longrightarrow G/K$ is a G-map and ω is an equivalence class of paths $x \longrightarrow y \circ \alpha$ in X^H. As usual, two paths are equivalent if they are homotopic rel endpoints. Composition is induced by composition of maps of orbits and the usual product on path classes. Let $\pi\colon \Pi_G X \longrightarrow \mathscr{O}_G$ be the functor given by $\pi(x\colon G/H \longrightarrow X) = G/H$ and $\pi(\omega, \alpha) = \alpha$.

Lemma 1.2. *A G-map $f\colon X \longrightarrow Y$ induces a functor $f_*\colon \Pi_G X \longrightarrow \Pi_G Y$. A G-homotopy $h\colon f \simeq f'$ induces a natural isomorphism $h_*\colon f_* \longrightarrow f'_*$.*

We write ΠX for the nonequivariant fundamental groupoid of a space X.

Remark 1.3. For a category $\mathscr{B}$ and an object b, we have the category $\mathscr{B}/b$ of objects $a \longrightarrow b$ over b. Taking $X = G/H$, the functor $\pi\colon \Pi_G(G/H) \longrightarrow \mathscr{O}_G$ factors through a functor $\Pi_G(G/H) \longrightarrow \mathscr{O}_G/(G/H)$ that is surjective on objects and morphisms and is an isomorphism if G is finite.

We record some properties of the fundamental groupoid that will later be abstracted to give the notion of a bundle of groupoids. For a functor $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$, the *fiber* $\mathscr{E}_b$ over an object $b \in \mathscr{B}$ is the subcategory of objects and morphisms of $\mathscr{E}$ that map to b and its identity morphism.

Remarks 1.4. Let X be a G-space.

(i) The fiber $(\Pi_G X)_{G/H}$ is the nonequivariant fundamental groupoid ΠX^H.

(ii) For an object $y\colon G/K \longrightarrow X$ in $\Pi_G X$ and a map $\alpha\colon G/H \longrightarrow G/K$ in $\mathscr{O}_G$, there is an object $x\colon G/H \longrightarrow X$ and a morphism $(\omega, \alpha)\colon x \longrightarrow y$. In fact, $x = y \circ \alpha$ and the constant path ω give canonical choices for x and (ω, α).

(iii) Let $x\colon G/H \longrightarrow X$, $y\colon G/J \longrightarrow X$, and $z\colon G/K \longrightarrow X$ be objects in $\Pi_G X$. Suppose that we have maps $(\nu,\gamma)\colon x \longrightarrow z$ and $(\mu,\beta)\colon y \longrightarrow z$ in $\Pi_G X$ and a map $\alpha\colon G/H \longrightarrow G/J$ such that $\beta\alpha = \gamma$:

$$\begin{tikzcd} x \arrow[rr, dashed, "{(\omega,\alpha)}"] \arrow[dr, "{(\nu,\gamma)}"'] & & y \arrow[dl, "{(\mu,\beta)}"] \\ & z & \end{tikzcd} \quad \xrightarrow{\ \pi\ } \quad \begin{tikzcd} G/H \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & G/J \arrow[dl, "\beta"] \\ & G/K. & \end{tikzcd}$$

There is a unique map $(\omega,\alpha)\colon x \longrightarrow y$ in $\Pi_G X$ such that $(\mu,\beta)(\omega,\alpha) = (\nu,\gamma)$, namely the one given by $\omega = (\mu\alpha)^{-1}\nu$. The existence and uniqueness of (ω,α) are encoded in the statement that the following diagram is a pullback:

$$\begin{tikzcd} \Pi_G X(y,z) \times \Pi_G X(x,y) \arrow[r, "\circ"] \arrow[d, "\mathrm{id}\times\pi"'] & \Pi_G X(x,z) \arrow[d, "\pi"] \\ \Pi_G X(y,z) \times \mathscr{O}_G(G/H,G/J) \arrow[r, "\circ(\pi\times\mathrm{id})"'] & \mathscr{O}_G(G/H,G/K). \end{tikzcd}$$

2. Categories of G-vector bundles and orientability

We need reference categories of G-vector bundles over orbits. By a G-bundle, we will understand a real G-vector bundle with orthogonal structure group.

Definition 2.1. Let $\overline{\mathscr{V}}_G$ be the category whose objects are the G-bundles over orbits of G and whose morphisms are the equivalence classes of G-bundle maps between them. Here two maps are equivalent if they are G-bundle homotopic, with the homotopy inducing the constant homotopy on base spaces. Let $\pi\colon \overline{\mathscr{V}}_G \longrightarrow \mathscr{O}_G$ be the functor that sends a G-bundle to its base space and sends an equivalence class of bundle maps to its map of base spaces. Let $\overline{\mathscr{V}}_G(n)$ be the full subcategory of $\overline{\mathscr{V}}_G$ consisting of the n-dimensional bundles.

These categories are not small, but they have small skeleta.

Definition 2.2. Let $\mathscr{V}_G(n)$ be the full subcategory of $\overline{\mathscr{V}}_G$ whose objects are the n-plane G-bundles of the form $G \times_H \mathbb{R}^n \longrightarrow G/H$, where H acts on $\mathbb{R}^n$ through some representation $\lambda\colon H \longrightarrow O(n)$ and we choose one such λ in each $O(n)$-conjugacy class. We obtain a retraction equivalence $\overline{\mathscr{V}}_G(n) \longrightarrow \mathscr{V}_G(n)$ by choosing a fixed isomorphism from each object in $\overline{\mathscr{V}}_G(n)$ to an object of $\mathscr{V}_G(n)$, choosing the identity map if the object is in $\mathscr{V}_G(n)$. Note that we still have functors $\pi\colon \mathscr{V}_G(n) \longrightarrow \mathscr{O}_G$. Let $\mathscr{V}_G$ be the disjoint union of the categories $\mathscr{V}_G(n)$; it is equivalent to $\overline{\mathscr{V}}_G$.

We continue to write V for representations, even when we are thinking in terms of objects of $\mathscr{V}_G$. The following observations give a description of this category.

Lemma 2.3. *Up to equivalence, a G-bundle over the orbit G-space G/H has the form $G \times_H V \longrightarrow G/H$ for some real representation V of H. A map*

$$\tilde{\alpha}\colon G \times_H V \longrightarrow G \times_K W$$

of G-bundles over a map $\alpha\colon G/H \longrightarrow G/K$ has the form $\tilde{\alpha}(g,v) = (gg_0, \tau(v))$, where $\alpha(eH) = g_0K$ (hence $g_0^{-1}Hg_0 \subset K$) and $\tau\colon V \longrightarrow W$ is a linear isometry which is H-linear in the sense that $\tau(hv) = (g_0^{-1}hg_0)\tau(v)$. Two maps $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ over α so determined by τ_0 and τ_1 are G-bundle homotopic over α if and only if there is a path τ_t connecting τ_0 to τ_1 in the space of H-linear isometries $V \longrightarrow W$.

The skeletal nature of $\mathscr{V}_G(n)$ implies the following useful observation.

Lemma 2.4. *If there is a map $G \times_H V \longrightarrow G \times_H W$ in $\mathscr{V}_G(n)$, then $V = W$.*

Remark 2.5. The fiber $\mathscr{V}_G(n)_{G/H}$ is a groupoid that has one object V in each isomorphism class of representations of H in $O(n)$ and has morphisms $V \longrightarrow V$ the homotopy classes of H-linear isometries; it has no morphisms $V \longrightarrow V'$ if $V \neq V'$. By inspection of pullbacks, the evident analogues of Remarks 1.4(ii) and (iii) hold for the functor $\pi\colon \mathscr{V}_G(n) \longrightarrow \mathscr{O}_G$.

The following well known fact clarifies the structure of $\mathscr{V}_G(n)$. Let $O_G(V)$ be the group of G-linear isometries of a representation V of G and let $\pi_0(O_G(V))$ be its group of components.

Lemma 2.6. *The group $\pi_0(O_G(V))$ is an elementary Abelian 2-group.*

Proof. Write $V = \oplus V_i$, where the V_i are the isotypical components of V, so that $V_i \cong W_i \otimes \mathbb{R}^{q_i}$ for some irreducible representation W_i. Then $O_G(V) \cong \prod_i O_G(V_i)$. Let $\mathbb{K}_i = \mathrm{Hom}_G(W_i, W_i)$. Each $\mathbb{K}_i$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and $\mathrm{Hom}_G(V_i, V_i) \cong M_{q_i}(\mathbb{K}_i)$. The corresponding subgroup of underlying real linear isometries is connected when $\mathbb{K}_i = \mathbb{C}$ or $\mathbb{H}$ and has two components when $\mathbb{K}_i = \mathbb{R}$. □

The following basic construction is central to our work. Recall Lemma 1.2.

Proposition 2.7. *A G-bundle $p\colon E \longrightarrow B$ determines a functor $p^*\colon \Pi_G B \longrightarrow \mathscr{V}_G$ over $\mathscr{O}_G$. A G-bundle map $(\tilde{f}, f)\colon p \longrightarrow q$, with $\tilde{f}\colon E \longrightarrow E'$ the map of total spaces and $f\colon B \longrightarrow B'$ the map of base spaces, determines a natural isomorphism $\tilde{f}_*\colon p^* \longrightarrow q^* \circ f_*$ over the identity functor of $\mathscr{O}_G$. If $(\tilde{h}, h)\colon (\tilde{f}, f) \simeq (\tilde{f}', f')$ is a G-bundle homotopy, then the following diagram commutes:*

$$\begin{tikzcd} & p^* \arrow[dl, "\tilde{f}_*"'] \arrow[dr, "\tilde{f}'_*"] & \\ q^* \circ f_* \arrow[rr, "q^* \circ h_*"'] & & q^* \circ f'_*. \end{tikzcd}$$

In the last statement and later, we compose a functor (in this case q^*) with a natural transformation (in this case h_*) by applying the functor to the maps that define the natural transformation; we often omit $\circ$ in writing such composites.

Proof. It suffices to work in $\overline{\mathscr{V}}_G$, since we can then transfer information to the equivalent category $\mathscr{V}_G$. Pulling p back along G-maps $x\colon G/H \longrightarrow B$, we obtain a system of G-bundles $p^*(x) \longrightarrow G/H$ and bundle maps $\tilde{x}\colon p^*(x) \longrightarrow E$. For a G-map $\alpha\colon G/H \longrightarrow G/K$ and a path $w\colon x \longrightarrow y \circ \alpha$, the G-bundle covering homotopy property (G-bundle CHP) gives a homotopy $\tilde{w}: p^*(x) \times I \longrightarrow E$ of $\tilde{x}$ that covers w. The map $\tilde{w}_1$ covers $y \circ \alpha$ and factors through a G-bundle map $p^*(w, \alpha): p^*(x) \longrightarrow p^*(y)$ whose equivalence class depends only on the equivalence class ω of w. This constructs p^*, and the remaining verifications are similar. □

With these definitions in place, we can define orientability precisely.

Definition 2.8. The G-bundle $p\colon E \longrightarrow B$ is *orientable* if the functor $p^* : \Pi_G B \longrightarrow \mathscr{V}_G$ satisfies $p^*(\omega, \alpha) = p^*(\omega', \alpha)$ for every pair of morphisms (ω, α) and (ω', α) with the same source and target and the same image in $\mathscr{O}_G$. That is, $p^*(\omega, \alpha)$ is independent of the choice of the path class ω. For example, for a representation V of G, the projection $B \times V \longrightarrow B$ is orientable.

Remark 2.9. A G-bundle p is orientable if the defining condition holds when $x = y$ and $\alpha = \mathrm{id}$. Indeed, if (ω, α) and (ω', α) are maps $x \longrightarrow y$, then, by Remark 1.4(iii), there is a map $(\xi, \mathrm{id}) : x \longrightarrow x$ such that $(\omega', \alpha) = (\omega, \alpha)(\xi, \mathrm{id})$. If $p^*(\xi, \mathrm{id}) = \mathrm{id}$, then $p^*(\omega, \alpha) = p^*(\omega', \alpha)$. This gives the claimed implication. Thus orientability is a property of the restrictions of p over fixed point spaces B^H.

The following observation is immediate from Remark 1.3.

Proposition 2.10. *If G is finite, any G-bundle over an orbit G/H is orientable.*

Example 2.11. This result fails for general compact Lie groups. For example, let $\mathbb{L}$ be the sign representation of the cyclic group H of order 2. Regarding H as a subgroup of S^1, we can identify the open Möbius strip and its retraction to the circle as the S^1-bundle $S^1 \times_H \mathbb{L} \longrightarrow S^1/H$. Clearly this is non-orientable.

3. The topologized fundamental groupoid

When G is a general compact Lie group, we shall need topologies on the categories $\Pi_G X$ and $\mathscr{V}_G$ in order to define orientations of vector bundles. This section and the next deal with this issue and may be skipped by the reader who wishes to focus on finite groups. However, this material illuminates the structure of all of the categories that we have defined and should be of independent interest. The following easy, but basic, observation appears to be new. Here and later, we use the term "bundle with discrete fibers" instead of "covering space" to emphasize that we are not assuming that the base spaces or total spaces are connected. In particular, we allow some fibers to be empty.

Proposition 3.1. *The category $\Pi_G X$ is a topological category such that, for objects $x\colon G/H \longrightarrow X$ and $y\colon G/K \longrightarrow X$,*

$$\pi\colon \Pi_G X(x, y) \longrightarrow \mathscr{O}_G(G/H, G/K)$$

is a bundle with discrete fibers.

Proof. For a simply connected open neighborhood U of a point $\alpha \in G/K^H$ and a point $\beta \in U$, there is a unique path class $\nu_{\alpha,\beta}$ connecting α to β in U. Composing with y gives a path class $\tilde{\nu}_{\alpha,\beta}$ connecting $y \circ \alpha$ to $y \circ \beta$. For $\omega\colon x \longrightarrow y \circ \alpha$, let

$$U(\omega, \alpha) = \{(\tilde{\nu}_{\alpha,\beta}\omega, \beta) | \beta \in U\} \subset \Pi_G X(x, y).$$

The $U(\omega,\alpha)$ are the open sets of a basis for a topology on $\Pi_G X(x,y)$ such that $\pi\colon \Pi_G X(x,y) \longrightarrow \pi(\Pi_G X(x,y))$ is a bundle with discrete fiber $\Pi X^H(x, y\circ\alpha)$ over α. Indeed, if α is in the image of π, then $\pi^{-1}(U)$ is the disjoint union of the $U(\omega,\alpha)$ as ω ranges over the inequivalent classes of paths $x \longrightarrow y\circ\alpha$. □

Corollary 3.2. *Maps $(\omega,\alpha),(\xi,\beta)\colon x \longrightarrow y$, $x\colon G/H \longrightarrow X$ and $y\colon G/K \longrightarrow X$, are homotopic if and only if there is a homotopy $j\colon G/H\times I \longrightarrow G/K$ from α to β and a homotopy $k\colon G/H\times I\times I \longrightarrow X$ from a path $w\colon G/H\times I \longrightarrow X$ in the path class ω to a path $z\colon G/H\times I \longrightarrow X$ in the path class ξ such that $k(a,0,t) = x(a)$ and $k(a,1,t) = yj(a,t)$ for $a\in G/H$ and $t\in I$.*

Remark 3.3. Identifying homotopic maps, we obtain the homotopy category $h\Pi_G X$ and a functor $\pi\colon h\Pi_G X \longrightarrow h\mathscr{O}_G$. By the corollary, $h\Pi_G X$ is tom Dieck's "discrete fundamental groupoid" [**10**, 10.9]. When G is finite, there is no distinction. Much of our theory can be carried out in terms of homotopy categories, that being the approach taken in the original version (circa 1989) of this work. However, use of $\Pi_G X$ turns out to be preferable since it gives a closer relationship between $\Pi_G X$ and the ΠX^H and allows a more natural variant of representation theory.

So far, G could have been any (locally simply connected) topological group. However, since G is a compact Lie group, we can give an explicit description of the quotient functor $\Pi_G X \longrightarrow h\Pi_G X$. This depends on the following description of homotopies in $\mathscr{O}_G$ [**25**, 1.1].

Lemma 3.4. *Let $j\colon \alpha \longrightarrow \beta$ be a G-homotopy between G-maps $G/H \longrightarrow G/K$. Then j is the composite of α with a homotopy $c\colon G/H\times I \longrightarrow G/H$ such that $c(eH,t) = c(t)H$, where $c(0) = e$ and the $c(t)$ specify a path in the identity component of the centralizer $C_G H$ of H in G. In particular $\beta = \alpha\circ c(1)\colon G/H \longrightarrow G/K$.*

For a path c in $C_G H$ and a path w in X^H, we obtain a path $c\cdot w$ in X^H by setting $(c\cdot w)(s) = c(s)w(s)$. If ω is the path class of w, we write $c\cdot\omega$ for the path class of $c\cdot w$. Combining the notations and hypotheses of Corollary 3.2 and Lemma 3.4, we obtain the following description of homotopies in $\Pi_G X$.

Proposition 3.5. *Consider objects $x\colon G/H \longrightarrow X$ and $y\colon G/K \longrightarrow X$ of $\Pi_G X$. Let (k,j) give a homotopy between maps $(\omega,\alpha),(\xi,\beta)\colon x \longrightarrow y$ in $\Pi_G X$, where $j = \alpha\circ c$ and thus $\beta = \alpha\circ c(1)$. Then (ω,α) is homotopic to $(c\cdot\omega, \alpha\circ c(1))$ (independent of ξ), and $(c\cdot\omega,\beta)$ is equal to (ξ,β) in $\Pi_G X$ (independent of the homotopy j). Therefore $h\Pi_G X(x,y)$ is the quotient of $\Pi_G X(x,y)$ obtained by identifying (ω,α) with $(c\cdot\omega,\alpha\circ c(1))$ for all paths c in $C_G H$ such that $c(0) = e$.*

Proof. Define $h = h(w,c)\colon I\times I \longrightarrow X^H$ by $h(s,t) = c(st)w(s)$, where w represents ω. Then $h\colon w \simeq c\cdot w$, $h(0,t) = w(0)$, and $h(1,t) = c(t)w(1)$. Interpreting in terms of equivariant maps on orbits, this means that (h,j) gives a homotopy $(\omega,\alpha) \simeq (c\cdot\omega,\beta)$. Regarding k as a map $I\times I \longrightarrow X^H$, define a new map $\ell\colon I\times I \longrightarrow X^H$ by $\ell(s,t) = c(s)c(st)^{-1}k(s,t)$. Then ℓ is a homotopy rel endpoints between $c\cdot w$ and a representative z of ξ. □

Remark 3.6. The functor $\pi\colon h\Pi_G X \longrightarrow h\mathscr{O}_G$ has properties similar to but less convenient than those of Remarks 1.4. The fiber $(h\Pi_G X)_{G/H}$ is a quotient of ΠX^H, there is a noncanonical solution x and (ω, α) to the "source lifting" question in Remarks 1.4(ii), and there is a non-unique solution (ω, α) to the "divisibility" question in Remarks 1.4(iii).

4. The topologized category of G-vector bundles over orbits

We have a topologization of the category $\mathscr{V}_G$ that is precisely analogous to the topologization of $\Pi_G X$ in Proposition 3.1. We work with $\mathscr{V}_G$ for simplicity of notation, but it will be the topological disjoint union of the $\mathscr{V}_G(n)$. To emphasize the analogy with $\Pi_G X$, we write ω or (ω, α) for a morphism $p \longrightarrow q$ over α, so that ω is an equivalence class of G-bundle maps $\tilde{\alpha}$ over α.

Proposition 4.1. *The category $\mathscr{V}_G$ is a topological category such that, for n-plane G-bundles $p\colon D \longrightarrow G/H$ and $q\colon E \longrightarrow G/K$,*

$$\pi\colon \mathscr{V}_G(p, q) \longrightarrow \mathscr{O}_G(G/H, G/K)$$

is a bundle with discrete fibers.

Proof. As in the proof of Proposition 3.1, consider a simply connected open neighborhood U of a point $\alpha \in G/K^H$ and the path classes $\nu_{\alpha,\beta}$ of paths $v_{\alpha,\beta}$ connecting α to points β in U. Let $\tilde{\alpha}$ be a G-bundle map over α. Applying the G-bundle CHP, we obtain a G-bundle homotopy $\tilde{v}_{\alpha,\beta}(\tilde{\alpha})\colon D \times I \longrightarrow E$ of $\tilde{\alpha}$ that covers $v_{\alpha,\beta}$. Write $\tilde{\beta}_{\alpha,\beta}(\tilde{\alpha})\colon p \longrightarrow q$ for the G-bundle map over β obtained at the end of the homotopy. Further application of the G-bundle CHP shows that the equivalence class of $\tilde{\beta}_{\alpha,\beta}(\tilde{\alpha})$ depends only on the equivalence class ω of $\tilde{\alpha}$ and the path class $\nu_{\alpha,\beta}$. We write $\zeta_{\alpha,\beta}(\omega)$ for the equivalence class of $\tilde{\beta}_{\alpha,\beta}(\tilde{\alpha})$, and we define

$$U(\omega, \alpha) = \{\zeta_{\alpha,\beta}(\omega) | \beta \in U\} \subset \mathscr{V}_G(p, q).$$

The $U(\omega, \alpha)$ are the open sets of a basis for a topology on $\mathscr{V}_G(p, q)$, such that $\pi\colon \mathscr{V}_G(p, q) \longrightarrow \pi(\mathscr{V}_G(p, q))$ is a bundle with discrete fibers. Indeed, if α is in the image of π, then $\pi^{-1}(U)$ is the disjoint union of the sets $U(\omega, \alpha)$ as ω ranges over the equivalence classes of bundle maps $p \longrightarrow q$ over α. □

Remark 4.2. We have a quotient category $h\mathscr{V}_G$ obtained by identifying bundle maps $p \longrightarrow q$ over α and β if they are bundle homotopic over a homotopy $\alpha \simeq \beta$. If G is finite, then $\mathscr{V}_G = h\mathscr{V}_G$. Passage to base spaces gives a functor $\pi\colon h\mathscr{V}_G \longrightarrow h\mathscr{O}_G$.

The precise relationship between $\mathscr{V}_G$ and $h\mathscr{V}_G$ is analogous to that between $\Pi_G X$ and $h\Pi_G X$ described in Proposition 3.5. We see this by extending the last sentence of Lemma 2.3 to allow for homotopies on the base space level. Consider a path c in $C_G H$ and a bundle map $\tilde{\alpha}\colon G \times_H V \longrightarrow G \times_K W$ over α specified by $\tilde{\alpha}(g, v) = (gg_0, \tau_0(v))$, where $\alpha(eH) = g_0 K$ and $\tau_0\colon V \longrightarrow W$ is an H-linear isometry. We obtain a bundle homotopy

$$c \cdot \tilde{\alpha}\colon (G \times_H V) \times I \longrightarrow G \times_K W \tag{4.3}$$

by setting $(c \cdot \tilde{\alpha})(g, v, t) = (gc(t)g_0, \tau_0(v))$. If ω denotes the equivalence class of $\tilde{\alpha}$, we let $c(1) \cdot \omega$ denote the equivalence class of the map over β given by setting $t = 1$.

Proposition 4.4. *Let $p\colon G \times_H V \longrightarrow G/H$ and $q\colon G \times_K W \longrightarrow G/K$ be G-bundles. Let $(\tilde{j}, j)$ give a homotopy between maps (ω, α), $(\xi, \beta)\colon p \longrightarrow q$ in $\mathscr{V}_G$, where $j = \alpha \circ c$. Then ω is homotopic to $c(1) \cdot \omega$ (independent of ξ) and $c(1) \cdot \omega$ is equal to ξ in $\mathscr{V}_G(p, q)$ (independent of the homotopy j). Therefore $h\mathscr{V}_G(p, q)$ is the quotient of $\mathscr{V}_G(p, q)$ obtained by identifying (ω, α) with $(c(1) \cdot \omega, \alpha \circ c(1))$ for all paths c in $C_G H$ such that $c(0) = e$.*

Proof. The bundle homotopy (4.3) gives $\omega \simeq c(1) \cdot \omega$. We must show that $c(1) \cdot \omega = \xi$. We are given a bundle homotopy $\tilde{j}\colon (G \times_H V) \times I \longrightarrow G \times_K W$ over j from $\tilde{\alpha}$ to $\tilde{\beta}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are in the homotopy classes ω and ξ. We must define a bundle homotopy $k\colon (G \times_H V) \times I \longrightarrow G \times_K W$ over the constant homotopy at β from $c(1) \cdot \tilde{\alpha}$ to $\tilde{\beta}$. Observe that the H-action on W defined by $hw = g_0^{-1} c(t)^{-1} h c(t) g_0$ is independent of t. We may write $\tilde{j}(g, v, t) = (gc(t)g_0, \tau_t(v))$, where the τ_t specify a path in the space of H-linear isometries $V \longrightarrow W$. We define

$$k(g, v, t) = (gc(1)g_0, \tau_t(v)). \qquad \square$$

We have a perhaps more natural alternative definition of orientability. We define it, but we then show that it agrees with the definition already given.

Remark 4.5. Let $p\colon E \longrightarrow B$ be a G-bundle. Since the full strength of the G-bundle CHP allows us to vary maps on base spaces by homotopies, Proposition 2.7 remains valid if we replace the categories $\Pi_G B$ and $\mathscr{V}_G$ over $\mathscr{O}_G$ with the categories $h\Pi_G B$ and $h\mathscr{V}_G$ over $h\mathscr{O}_G$. Thus we have a functor $p^*\colon h\Pi_G B \longrightarrow h\mathscr{V}_G$ over $h\mathscr{O}_G$.

Definition 4.6. Thinking in terms of $p^*\colon h\Pi_G B \longrightarrow h\mathscr{V}_G$, we say that a G-bundle p is *h-orientable* if $p^*(\omega, \alpha) = p^*(\zeta, \beta)$ for every pair of morphisms (ω, α) and (ξ, β) of $h\Pi_G B$ with the same source and target and the same image in $h\mathscr{O}_G$.

Proposition 4.7. *A G-bundle p is orientable if and only if it is h-orientable.*

Proof. The construction of both functors p^* is by use of the G-bundle CHP, starting from homotopies on the base space level. We see that p is orientable if it is h-orientable by starting with constant homotopies. Conversely, suppose that p is orientable. Then Proposition 3.5 implies that p is h-orientable. Indeed, with the notation there, the fact that p^* in Remark 4.5 is well-defined implies that $p^*(\omega, \alpha) = p^*(\alpha \circ c(1), c \cdot \omega)$ in $h\overline{\mathscr{V}}_G$. This can also be seen by direct verification using an evident cover of the homotopy h used in the proof of Proposition 3.5. $\square$

There is thus no a priori reason to prefer orientability to h-orientability. However, as we mentioned in Remark 3.3, we prefer to use $\Pi_G B$ rather than $h\Pi_G B$. In particular, we want the uniqueness specified in Definition 5.1(iv) below, and this would fail for $h\Pi_G B$.

Part II. Categorical representation theory and orientations

5. Bundles of Groupoids

We abstract the properties of the equivariant fundamental groupoid to obtain the notion of a bundle of groupoids. We fix a topological category $\mathscr{B}$, which the reader should think of as $\mathscr{O}_G$. We assume that $\mathscr{B}$ is small, its morphism spaces are locally path connected, and every endomorphism of an object of $\mathscr{B}$ is an isomorphism. We call such a category a *base category.* If G is finite, we give all categories in sight the discrete topology. It is helpful to think in terms of 2-categories [**19**, XII§2], which have objects (the "0-cells"), morphisms between objects (the "1-cells"), and morphisms between morphisms or "homotopies" (the "2-cells").

Definition 5.1. A *bundle of groupoids over* $\mathscr{B}$, or a *groupoid over* $\mathscr{B}$ for short, is a small topological category $\mathscr{E}$ together with a continuous functor $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ that satisfy the following properties.

(i) Each map $\pi\colon \mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y))$ is a fiber bundle with discrete fibers (possibly empty and varying over different components of the target).

(ii) For each object b of $\mathscr{B}$, the fiber $\mathscr{E}_b$ (the subcategory of objects and morphisms of $\mathscr{E}$ that map to b and its identity) is a groupoid (possibly empty).

(iii) (Source Lifting) For each object y of $\mathscr{E}$ and morphism $\alpha\colon a \longrightarrow \pi(y)$ of $\mathscr{B}$, there is a morphism $\omega\colon x \longrightarrow y$ of $\mathscr{E}$ such that $\pi(x) = a$ and $\pi(\omega) = \alpha$.

(iv) (Divisibility) For objects x, y, z and morphisms $\nu\colon x \longrightarrow z$, $\mu\colon y \longrightarrow z$ of $\mathscr{E}$ and a morphism $\alpha\colon \pi(x) \longrightarrow \pi(y)$ of $\mathscr{B}$ such that $\pi(\mu)\alpha = \pi(\nu)$, there is a unique morphism $\omega\colon x \longrightarrow y$ of $\mathscr{E}$ such that $\pi(\omega) = \alpha$ and $\mu \circ \omega = \nu$:

$$\begin{tikzcd} x \arrow[rr, dashed, "\omega"] \arrow[dr, "\nu"'] & & y \arrow[dl, "\mu"] \\ & z & \end{tikzcd} \xrightarrow{\ \pi\ } \begin{tikzcd} \pi(x) \arrow[rr, "\alpha"] \arrow[dr, "\pi(\nu)"'] & & \pi(y) \arrow[dl, "\pi(\mu)"] \\ & \pi(z) & \end{tikzcd}$$

Moreover, ω varies continuously with the data. The existence, uniqueness, and continuity are encoded by requiring the following diagram to be a pullback:

$$\begin{tikzcd} \mathscr{E}(y,z) \times \mathscr{E}(x,y) \arrow[r, "\circ"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathscr{E}(x,z) \arrow[d, "\pi"] \\ \mathscr{E}(y,z) \times \mathscr{B}(\pi(x),\pi(y)) \arrow[r, "\circ(\pi\times\mathrm{id})"'] & \mathscr{B}(\pi(x),\pi(z)). \end{tikzcd}$$

We write π generically for the projections of groupoids over $\mathscr{B}$, and we often write $\mathscr{E}$ for $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$. The groupoids over $\mathscr{B}$ are the 0-cells of a 2-category. The 1-cells are the continuous functors $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathscr{F} \arrow[dl, "\pi"] \\ & \mathscr{B}. & \end{tikzcd}$$

We refer to these as *functors over* $\mathscr{B}$. The 2-cells $\eta\colon F \longrightarrow F'$ are the natural transformations $\eta\colon F \longrightarrow F'$ such that $\pi(\eta(x)) = \mathrm{id}_{\pi(x)}$ for all objects x of $\mathscr{E}$. Since $\eta(x)$ is then a morphism of the groupoid $\mathscr{F}_{\pi(x)}$, η must be a natural isomorphism. We refer to these as *isomorphisms over* $\mathscr{B}$. Let $\mathscr{B}(\mathscr{E},\mathscr{F})$ denote the groupoid whose objects are the functors $\mathscr{E} \longrightarrow \mathscr{F}$ over $\mathscr{B}$ and whose morphisms are the isomorphisms over $\mathscr{B}$. Two groupoids $\mathscr{E}$ and $\mathscr{F}$ over $\mathscr{B}$ are *equivalent* if there are functors $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ and $F^{-1}\colon \mathscr{F} \longrightarrow \mathscr{E}$ over $\mathscr{B}$ whose composites are isomorphic over $\mathscr{B}$ to the respective identity functors; $\mathscr{E}$ and $\mathscr{F}$ are *isomorphic* if there are functors F and F^{-1} over $\mathscr{B}$ whose composites are equal to the respective identity functors.

Condition (ii) of the definition is redundant, being implied by unique divisibility (see Remark 6.3 below). It is stated for emphasis. If we ignore the topology, then a groupoid over $\mathscr{B}$ is exactly a "catégorie fibrée en groupoides" over $\mathscr{B}$, as defined by Grothendieck [**15**, pp. 165–166]; see also [**8**, p. 96]. Nevertheless, what we have defined is not some kind of stack. It must be kept in mind that the convenient abbreviation "groupoid $\mathscr{E}$ over $\mathscr{B}$" is an abuse of language, since $\mathscr{E}$ is not a groupoid. Remarks 1.4 and 2.5, Lemma 1.2, and Propositions 2.7, 3.1, and 4.1 are summarized in the following motivating examples.

Proposition 5.2. *For a G-space X, $\pi\colon \Pi_G X \longrightarrow \mathscr{O}_G$ is a groupoid over $\mathscr{O}_G$. For a G-map $f\colon X \longrightarrow Y$, $f_*\colon \Pi_G X \longrightarrow \Pi_G Y$ is a functor over $\mathscr{O}_G$. A G-homotopy $h\colon f \simeq f'$ induces an isomorphism $h_*\colon f_* \longrightarrow f'_*$ over $\mathscr{O}_G$.*

Proposition 5.3. *The functor $\pi\colon \mathscr{V}_G(n) \longrightarrow \mathscr{O}_G$ is a groupoid over $\mathscr{O}_G$. For a G-bundle $p\colon E \longrightarrow B$, $p^*\colon \Pi_G B \longrightarrow \mathscr{V}_G$ is a functor over $\mathscr{O}_G$. For a G-bundle map $(\tilde{f}, f)\colon p \longrightarrow q$, $\tilde{f}_*\colon p^* \longrightarrow q^* \circ f_*$ is an isomorphism over $\mathscr{O}_G$. The following diagram commutes for a G-bundle homotopy $(\tilde{h}, h)\colon (\tilde{f}, f) \simeq (\tilde{f}', f')$:*

$$\begin{tikzcd} & p^* \arrow[dl, "\tilde{f}_*"'] \arrow[dr, "\tilde{f}'_*"] & \\ q^* \circ f_* \arrow[rr, "q^* \circ h_*"'] & & q^* \circ f'_*. \end{tikzcd}$$

Divisibility implies the following result about the bundles of morphisms of π. For an object x of $\mathscr{E}$, let $\mathrm{Aut}(x)$ denote the (discrete) group of self-maps of x in the fiber $\mathscr{E}_{\pi(x)}$.

Proposition 5.4. *The map $\mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x), \pi(y))$ is a principal $\mathrm{Aut}(x)$-bundle onto its image. A map $\omega : x \longrightarrow y$ in $\mathscr{E}$ determines a restriction homomorphism $r : \mathrm{Aut}(y) \longrightarrow \mathrm{Aut}(x)$ characterized by $\nu \circ \omega = \omega \circ r(\nu)$ for $\nu \in \mathrm{Aut}(y)$.*

Proof. The pullback diagram given in (iv) restricts to the pullback diagram

$$\begin{tikzcd} \mathscr{E}(x,y) \times \mathrm{Aut}(x) \arrow[r, "\circ"] \arrow[d, "p"'] & \mathscr{E}(x,y) \arrow[d, "\pi"] \\ \mathscr{E}(x,y) \arrow[r, "\pi"'] & \mathscr{B}(\pi(x), \pi(y)), \end{tikzcd}$$

where p is projection. This implies that $\mathrm{Aut}(x)$ acts freely and transitively on each nonempty fiber of the bundle $\pi\colon \mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x), \pi(y))$. The second statement is immediate from divisibility. $\square$

6. Skeletal, faithful, and discrete bundles of groupoids

The following restricted kinds of bundles of groupoids play a major role in the theory. Recall that a category is *skeletal* if each of its isomorphism classes of objects consists of a single object and is *discrete* if all of its maps are identity maps. Recall that a functor is *faithful* if it maps morphism sets injectively.

Definition 6.1. A groupoid $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ over $\mathscr{B}$ is *skeletal* or *discrete* if each fiber $\mathscr{E}_b$ is skeletal or discrete; it is *faithful* if the functor π is faithful, in which case $\pi\colon \mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y))$ is an inclusion of a union of path components for each pair of objects x and y of $\mathscr{E}$.

Warning 6.2. Observe that a discrete category can admit only the discrete topology on its morphism sets and that the category $\mathscr{E}$ of a discrete groupoid over $\mathscr{B}$ need not be discrete in either the categorical or the topological sense. Henceforward, the word "discrete" will be used only in the sense of categories or of bundles of groupoids; the context will make clear which is intended.

Observe that the morphism space $\mathscr{E}(x,y)$ is a subspace of $\mathscr{B}(\pi(x),\pi(y))$ when $\mathscr{E}$ is faithful. For this reason, we need not pay much attention to the topology when studying faithful groupoids over $\mathscr{B}$. The following basic observations are easily verified; they will be used heavily.

Remarks 6.3. Let $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ be a groupoid over $\mathscr{B}$.

(i) If $\mathscr{E}$ is skeletal, then divisibility implies that the object x asserted to exist in the source lifting property is unique. If $\mathscr{E}$ is both skeletal and faithful, then the morphism asserted to exist in the source lifting property is also unique.

(ii) The fact that $\mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y))$ is a principal $\mathrm{Aut}(x)$-bundle implies that π is faithful if and only if every automorphism of every object in every fiber $\mathscr{E}_b$ is an identity map.

(iii) If $\omega\colon x \longrightarrow y$ is a morphism of $\mathscr{E}$ such that $\pi(\omega)$ is an isomorphism, then ω is an isomorphism, as we see by an application of divisibility to the equality $\pi(\omega)\pi(\omega)^{-1} = \mathrm{id}$. Since every endomorphism of any object of $\mathscr{B}$ is an isomorphism, every endomorphism of any object of $\mathscr{E}$ is also an isomorphism.

(iv) We can construct a faithful groupoid $\mathscr{E}/\pi$ over $\mathscr{B}$ with the same objects as $\mathscr{E}$, but with $(\mathscr{E}/\pi)(x,y) = \mathrm{Im}(\mathscr{E}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y)))$. The quotient functor $\mathscr{E} \longrightarrow \mathscr{E}/\pi$ over $\mathscr{B}$ is the universal functor from $\mathscr{E}$ into a faithful groupoid over $\mathscr{B}$.

(v) We can construct a skeletal subgroupoid $\mathscr{E}' \subset \mathscr{E}$ over $\mathscr{B}$ by choosing a skeleton of each fiber $\mathscr{E}_b$ and taking the full subcategory of $\mathscr{E}$ whose objects are in the chosen skeleta of fibers. The inclusion $\mathscr{E}' \longrightarrow \mathscr{E}$ is an equivalence of groupoids over $\mathscr{B}$ whose left inverse is a retraction over $\mathscr{B}$. We call $\mathscr{E}'$ a *skeleton* of $\mathscr{E}$.

(vi) By (iii), the passage from $\mathscr{E}$ to $\mathscr{E}/\pi$ creates no new isomorphisms, so that we can make the same choices of objects for $\mathscr{E}$ and for $\mathscr{E}/\pi$ when forming skeleta. Then $\mathscr{E}'/\pi = (\mathscr{E}/\pi)'$. This gives a canonical way of passing from any groupoid over $\mathscr{B}$ to an associated discrete groupoid over $\mathscr{B}$.

Discrete groupoids over $\mathscr{B}$ are central to our work. It is clear from the definitions that if $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ is skeletal and faithful, then it is discrete. Remark 6.3(ii) implies the converse.

Lemma 6.4. *$\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ is discrete if and only if it is skeletal and faithful.*

An obvious but useful observation is that the 2-category structure trivializes for maps into discrete groupoids over $\mathscr{B}$.

Lemma 6.5. *Let $\mathscr{F}$ be discrete. If F, $F'\colon \mathscr{E} \longrightarrow \mathscr{F}$ are functors over $\mathscr{B}$ and $\eta\colon F \longrightarrow F'$ is an isomorphism over $\mathscr{B}$, then $F = F'$ and η is the identity.*

In fact, discrete groupoids over $\mathscr{B}$ are actually quite simple and familiar objects.

Lemma 6.6. *The category of discrete groupoids over $\mathscr{B}$ and functors over $\mathscr{B}$ is equivalent to the category of continuous (= locally constant) set-valued contravariant functors on $\mathscr{B}$ and their natural isomorphisms.*

Proof. Given $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$, define a functor $\Gamma\colon \mathscr{B} \longrightarrow \mathrm{Sets}$ by letting $\Gamma(b)$ be the set of objects of the fiber $\mathscr{E}_b$. For a morphism $\alpha\colon a \to b$ of $\mathscr{B}$ and an object y of $\mathscr{E}_b$, let $\Gamma(\alpha)(y)$ be the unique object x of $\mathscr{E}_a$ that is the source of a map $x \longrightarrow y$ covering α. Remark 6.3(i) implies that the inverse image in $\mathscr{B}(a,b)$ of a function $f = \Gamma(\alpha)$ is the union of components $\pi(\mathscr{E}(f(y), y))$ and is thus open and closed. Conversely, given Γ, define $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ as follows. The objects of $\mathscr{E}$ are the pairs (y, b) where b is an object of $\mathscr{B}$ and $y \in \Gamma(b)$. A morphism $(x, a) \longrightarrow (y, b)$ is a morphism $\alpha\colon a \longrightarrow b$ of $\mathscr{B}$ such that $\Gamma(\alpha)(y) = x$. The functor π projects onto the second coordinate and restricts to an injection of $\mathscr{E}((x,a),(y,b))$ onto an open and closed subset of $\mathscr{B}(a,b)$; we give $\mathscr{E}((x,a),(y,b))$ the subspace topology. These constructions specify functors that give the claimed equivalence of categories. $\square$

Although reassuring, this result is not useful to us because our theory focuses on a comparison between general groupoids over $\mathscr{B}$ and discrete ones. The germ of the comparison is the fact that the categories $\mathscr{B}/b$ of objects over b give discrete groupoids over $\mathscr{B}$ whose represented functors in the 2-category of groupoids over $\mathscr{B}$ detect the fiber groupoids of arbitrary groupoids over $\mathscr{B}$. We use the following result to show this.

Lemma 6.7. *Let $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$ be a groupoid over $\mathscr{B}$, let b be an object of $\mathscr{B}$, and let y be an object of $\mathscr{E}$ such that $\pi(y) = b$. Consider the commutative diagram*

$$\begin{tikzcd} \mathscr{E}/y \arrow[r, "\lambda_y"] \arrow[d, "\pi_y"'] & \mathscr{E} \arrow[d, "\pi"] \\ \mathscr{B}/b \arrow[r, "\lambda_b"'] & \mathscr{B}, \end{tikzcd}$$

where λ_b and λ_y are the canonical functors and π_y is induced by π. Then $\mathscr{B}/b$ is a discrete groupoid over $\mathscr{B}$, $\mathscr{E}/y$ is a groupoid over $\mathscr{B}$, and λ_y and π_y are maps over $\mathscr{B}$. The functor π_y has a section σ. If $\mathscr{E}$ is discrete, then σ is unique and π_y is an isomorphism of categories with inverse σ.

Proof. The fiber $(\mathscr{B}/b)_a$ is the discrete category whose objects are the maps $a \longrightarrow b$ in $\mathscr{B}$, so that λ_b is discrete. The rest of the first statement is straightforward. By the source lifting property, for each map $\alpha\colon a \longrightarrow b$ of $\mathscr{B}$, there is an object x_α of $\mathscr{E}$ and a map $\sigma(\alpha)\colon x_\alpha \longrightarrow y$ such that $\pi(\sigma(\alpha)) = \alpha$. We may choose $\sigma(\mathrm{id}_b) = \mathrm{id}_y$. By the divisibility property, for maps $\alpha'\colon a' \longrightarrow b$ and $\lambda\colon a \longrightarrow a'$ in $\mathscr{B}$ such that $\alpha' \circ \lambda = \alpha$, there is a unique map $\sigma(\lambda)\colon x_\alpha \longrightarrow x_{\alpha'}$ in $\mathscr{E}$ such that $\pi(\sigma(\lambda)) = \lambda$ and $\sigma(\alpha') \circ \sigma(\lambda) = \sigma(\alpha)$. The pullback diagram in the divisibility property specializes to show that the resulting function $\sigma\colon \mathscr{B}/b(\alpha,\alpha') \longrightarrow \mathscr{E}/y(\sigma(\alpha),\sigma(\alpha'))$ is continuous, and the uniqueness of divisibility implies that σ is a functor. This gives the section σ, and it is the inverse isomorphism to π_y if π is discrete by Remark 6.3(i). $\square$

Proposition 6.8. *Let $\mathscr{E}$ be a groupoid over $\mathscr{B}$ and let $\varepsilon\colon \mathscr{B}(\mathscr{B}/b,\mathscr{E}) \longrightarrow \mathscr{E}_b$ be the functor that sends functors F and isomorphisms $\eta\colon F \longrightarrow F'$ over $\mathscr{B}$ to their evaluations on the object id_b of $\mathscr{B}/b$. Then ε is an equivalence of groupoids. If $\mathscr{E}$ is discrete, then, for an object y of $\mathscr{E}_b$, there is a unique functor $\tilde{y}\colon \mathscr{B}/b \longrightarrow \mathscr{E}$ over $\mathscr{B}$ such that $\tilde{y}(\mathrm{id}_b) = y$, and therefore ε is an isomorphism of groupoids.*

Proof. Observe that a map $\alpha\colon a \longrightarrow b$ of $\mathscr{B}$ gives both an object α of $\mathscr{B}/b$ and a morphism $\overline{\alpha}\colon \alpha \longrightarrow \mathrm{id}_b$ of $\mathscr{B}/b$. The functor ε is full and faithful by the divisibility property of $\pi\colon \mathscr{E} \longrightarrow \mathscr{B}$. Indeed, for a morphism $\omega\colon F(\mathrm{id}_b) \longrightarrow F'(\mathrm{id}_b)$ of $\mathscr{E}_b$, let $\eta(\alpha)\colon F(\alpha) \longrightarrow F'(\alpha)$ be the unique morphism of $\mathscr{E}_a$ such that $F'(\overline{\alpha}) \circ \eta(\alpha) = \omega \circ F(\overline{\alpha})$. The $\eta(\alpha)$ give the unique morphism $\eta\colon F \longrightarrow F'$ of $\mathscr{B}(\mathscr{B}/b,\mathscr{E})$ such that $\varepsilon(\eta) = \omega$. By [**19**, p. 93], to prove that ε is an equivalence of categories, it suffices to show that for each object y of $\mathscr{E}_b$, there is an object $\tilde{y}\colon \mathscr{B}/b \longrightarrow \mathscr{E}$ of $\mathscr{B}(\mathscr{B}/b,\mathscr{E})$ such that $\tilde{y}(\mathrm{id}_b) = y$, and we can take $\tilde{y} = \lambda_y \circ \sigma$ for a section σ of π_y. If $\mathscr{E}$ is discrete, then $\tilde{y} = \lambda_y \circ \pi_y^{-1}$ is unique. $\square$

7. Representations and orientations of bundles of groupoids

We think of the $\pi\colon \mathscr{V}_G(n) \longrightarrow \mathscr{O}_G$ as target groupoids over $\mathscr{O}_G$ for a kind of representation theory, and we note that we have chosen these groupoids over $\mathscr{O}_G$ to be skeletal. It is convenient to change our point of view on bundles of groupoids over $\mathscr{B}$ by focusing attention on a fixed target $\mathscr{R}$ for maps of groupoids over $\mathscr{B}$. We adopt the following language. Remember that we write π generically for the projections of groupoids over $\mathscr{B}$.

Definition 7.1. Fix a skeletal groupoid $\mathscr{R}$ over $\mathscr{B}$ and consider groupoids $\mathscr{E}$ and $\mathscr{F}$ over $\mathscr{B}$. We define "representations", "maps", and "homotopies" that give the 2-category of representations in $\mathscr{R}$.

(i) A *representation R of $\mathscr{E}$ in $\mathscr{R}$* is a functor $R\colon \mathscr{E} \longrightarrow \mathscr{R}$ over $\mathscr{B}$. We denote a representation as a pair $(\mathscr{E}, R)$ when $\mathscr{R}$ is understood.

(ii) A *map* from a representation $(\mathscr{E}, R)$ to a representation $(\mathscr{F}, S)$ is a pair (F, ϕ), where $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ is a functor over $\mathscr{B}$ and $\phi\colon S \circ F \longrightarrow R$ is an isomorphism over $\mathscr{B}$:

$$\begin{tikzcd}
\mathscr{E} \arrow[rr, "F"] \arrow[dr, "R"] \arrow[ddr, "\pi"'] & & \mathscr{F} \arrow[dl, "S"'] \arrow[ddl, "\pi"] \\
& \mathscr{R} \arrow[d, "\pi"] & \\
& \mathscr{B} &
\end{tikzcd} \qquad \phi \colon S \circ F \longrightarrow R.$$

The composite of (F, ϕ) and $(K, \kappa)\colon (\mathscr{F}, S) \longrightarrow (\mathscr{T}, T)$ is $(K \circ F, \phi \circ (\kappa \circ F))$. We say that (F, ϕ) is a strict map and write $(F, \phi) = F$ if ϕ is given by identity maps, so that $S \circ F = R$.

(iii) A *homotopy* between maps of representations (F, ϕ) and (F', ϕ') from $(\mathscr{E}, R)$ to $(\mathscr{F}, S)$ is an isomorphism $\eta\colon F \longrightarrow F'$ over $\mathscr{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
S \circ F \arrow[rr, "S \circ \eta"] \arrow[dr, "\phi"'] & & S \circ F' \arrow[dl, "\phi'"] \\
& R. &
\end{tikzcd}$$

If F and F' are strict maps, this means that $S \circ \eta = \mathrm{id} : R \longrightarrow R$.

(iv) We say that representations $(\mathscr{E}, R)$ and $(\mathscr{F}, S)$ are *equivalent* if there are maps $(F, \phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ and $(F^{-1}, \psi)\colon (\mathscr{F}, S) \longrightarrow (\mathscr{E}, R)$ whose composites are homotopic to the respective identity maps; (F, ϕ) is then called an *equivalence*. We say that $(\mathscr{E}, R)$ and $(\mathscr{F}, S)$ are *isomorphic* if there are maps (F, ϕ) and (F^{-1}, ϕ^{-1}) whose composites are equal to the respective identity maps.

(v) A representation $(\mathscr{E}, R)$ is *skeletal, faithful, or discrete* if the groupoid $\mathscr{E}$ over $\mathscr{B}$ is skeletal, faithful, or discrete.

(vi) A representation $(\mathscr{E}, R)$ is *orientable* if $R(\omega) = R(\omega')$ for any pair of maps $\omega, \omega'\colon x \longrightarrow y$ in $\mathscr{E}$ such that $\pi(\omega) = \pi(\omega')$; equivalently, by Remark 6.3(iv), R must factor through the faithful quotient $\mathscr{E}/\pi$.

The following observation is easily verified. The analogue for skeletal representations is not valid.

Lemma 7.2. *If one of two equivalent representations is either faithful or discrete, then so is the other.*

Orientations will be maps into certain discrete representations, and we have the following immediate implication of Lemma 6.5.

Lemma 7.3. *Let $\eta\colon (F, \phi) \longrightarrow (F', \phi')$ be a homotopy between maps of representations $(\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$, where $(\mathscr{F}, S)$ is discrete. Then $(F, \phi) = (F', \phi')$ and η is given by identity maps. Thus equivalent discrete representations are isomorphic.*

We shall define an *orientation* of an orientable representation $(\mathscr{E}, R)$ to be a map of representations from it to the "universal orientable representation" $(\mathscr{S}\mathscr{R}, S)$. We shall construct $(\mathscr{S}\mathscr{R}, S)$ in the following two sections. We give some intuition here. Since our interest now is in orientable representations and an orientable representation of $\mathscr{E}$ factors through its faithful quotient $\mathscr{E}/\pi$, we focus on faithful representations. Since any bundle of groupoids over $\mathscr{B}$ is equivalent to a skeletal bundle of groupoids (see Remarks 6.3(v) and (vi)), we may as well focus on discrete (= skeletal and faithful) representations. We seek a discrete representation that has as many morphisms as possible, to increase the chance that other representations will map into it. We shall call such a representation *saturated* and will give a precise definition in the next section. This will allow the following definition and theorem.

Definition 7.4. Let $\mathscr{R}$ be a skeletal groupoid over $\mathscr{B}$. A *universal orientable representation* $(\mathscr{S}\mathscr{R}, S)$ in $\mathscr{R}$ is a saturated representation such that, for every faithful representation $(\mathscr{E}, R)$, there is a map $(F, \phi) : (\mathscr{E}, R) \longrightarrow (\mathscr{S}\mathscr{R}, S)$.

We emphasize that this is not a universal property: (F, ϕ) is not unique.

Theorem 7.5. *Any skeletal groupoid $\mathscr{R}$ over $\mathscr{B}$ has a universal orientable representation $(\mathscr{S}\mathscr{R}, S)$, and $(\mathscr{S}\mathscr{R}, S)$ is unique up to isomorphism of representations.*

We shall prove the theorem in §9, where we give several characterizations of the representation $(\mathscr{S}\mathscr{R}, S)$. Of course, if $\mathscr{R}$ itself is faithful and thus discrete, then $(\mathscr{S}\mathscr{R}, S) = (\mathscr{R}, \mathrm{Id})$ and the theory trivializes. We think of $S\colon \mathscr{S}\mathscr{R} \longrightarrow \mathscr{R}$ as the best possible approximation of $\mathscr{R}$ by a discrete groupoid over $\mathscr{B}$.

Definition 7.6. Let $(\mathscr{E}, R)$ be a representation in $\mathscr{R}$. An *orientation* of $(\mathscr{E}, R)$ is a map of representations $(F, \phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{S}\mathscr{R}, S)$.

Since $\mathscr{S}\mathscr{R}$ is discrete and since any faithful representation maps into it, the following reassuring result is immediate from the definition.

Corollary 7.7. *A representation is orientable if and only if it has an orientation.*

We obtain the definition of an orientation of a G-bundle by specializing to the case $\mathscr{R} = \mathscr{V}_G(n)$, starting with the following reinterpretation of Proposition 5.3.

Proposition 7.8. *For an n-plane G-bundle $p\colon E \longrightarrow B$, $(\Pi_G B, p^*)$ is a representation in $\mathscr{V}_G(n)$. For a G-bundle map $(\tilde{f}, f)\colon p \longrightarrow q$, where $q\colon E' \longrightarrow B'$,*

$$(f_*, \tilde{f}_*)\colon (\Pi_G B, p^*) \longrightarrow (\Pi_G B', q^*)$$

is a map of representations. For a G-bundle homotopy $(\tilde{h}, h)\colon (\tilde{f}, f) \simeq (\tilde{f}', f')$ between G-bundle maps $p \longrightarrow q$,

$$h_*\colon (f_*, \tilde{f}_*) \longrightarrow (f'_*, \tilde{f}'_*)$$

is a homotopy between maps of representations.

This result suggests that there is a substantial analogy between the homotopy theory of representations and topological homotopy theory, and we shall say more about that point of view in §15.

Definition 7.9. Let $p\colon E \longrightarrow B$ be an n-plane G-bundle. An *orientation* of p is an orientation of the representation $(\Pi_G B, p^*)$ of $\Pi_G B$ in $\mathscr{V}_G(n)$. That is, an orientation of p is a map of representations

$$(F,\phi)\colon (\Pi_G B, p^*) \longrightarrow (\mathscr{S}\mathscr{V}_G(n), S).$$

If (F',ϕ') is an orientation of an n-plane bundle $q\colon E' \longrightarrow B'$, then a G-bundle map $(\tilde{f}, f)\colon p \longrightarrow q$ is *orientation preserving* if $(F,\phi) = (F',\phi')\circ(f_*, \tilde{f}_*)$; we then say that (F,ϕ) is the *pullback* of (F',ϕ') along f and denote it by $f^*(F',\phi')$.

In view of Lemma 7.3, our notion of an orientation is homotopy invariant.

Lemma 7.10. *If $(\tilde{h}, h)\colon (\tilde{f}, f) \simeq (\tilde{f}', f')$ is a G-bundle homotopy between maps $p \longrightarrow q$ of G-bundles and (F',ϕ') is an orientation of q, then $f^*(F',\phi') = f'^*(F',\phi')$.*

Proof. A little diagram chase shows that $F'\circ h_*$ is a homotopy from $f^*(F',\phi')$ to $f'^*(F',\phi')$. Since its target is discrete, it must be the identity isomorphism. □

8. Saturated and supersaturated representations

We fix a skeletal groupoid $\mathscr{R}$ over $\mathscr{B}$ throughout this section and the next. Representations will mean representations in $\mathscr{R}$. We give some simple definitions and observations about groupoids over $\mathscr{B}$ before defining saturated representations.

Definitions 8.1. Let $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ be a functor over $\mathscr{B}$.

(i) F is an *injection* or *surjection* if it is injective or surjective on both objects and morphisms. An injection is an *inclusion* if it is surjective on morphisms between pairs of objects, so that its image is a full subcategory of $\mathscr{F}$. A *retraction* of groupoids over $\mathscr{B}$ is a functor over $\mathscr{B}$ left inverse to an inclusion.

(ii) A map $(F,\phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ of representations is an *injection, inclusion, surjection,* or *retraction* if the underlying functor $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ is an injection, inclusion, surjection, or retraction of groupoids over $\mathscr{B}$.

(iii) An injection is *strict* if (F,ϕ) is a strict map, so that $\phi = \mathrm{id}$ and $S\circ F = R$. A strict inclusion $F : (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ specifies a *subrepresentation*; that is, F is the inclusion of a full subcategory $\mathscr{E}$ of $\mathscr{F}$ such that $S|\mathscr{E} = R$.

Lemma 8.2. *Let $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ be a functor over $\mathscr{B}$.*

(i) If $\mathscr{E}$ is faithful, then F is a faithful functor.

(ii) If $\mathscr{E}$ is skeletal and $F\colon \mathscr{E}(x,y) \longrightarrow \mathscr{F}(F(x),F(y))$ is a surjection for all objects x and y of $\mathscr{E}$, then F is injective on objects.

(iii) If $\mathscr{F}$ is faithful, then any morphism $\omega'\colon F(x) \longrightarrow F(y)$ in $\mathscr{F}$ that is not in the image of F has the form $\omega' = F(\omega)\circ\xi$, where $\omega\colon z \longrightarrow y$ is a morphism in $\mathscr{E}$, $F(z) \neq F(x)$, and $\xi\colon F(x) \longrightarrow F(z)$ is an isomorphism in $\mathscr{F}$ such that $\pi(\xi)$ is an identity map in $\mathscr{B}$.

(iv) If $\mathscr{F}$ is discrete, then the image of F is a full subcategory.

Proof. Since $\pi \circ F = \pi$ is faithful if $\mathscr{E}$ is faithful, (i) is immediate. For (ii), if $F(x) = F(y)$, then $\mathscr{E}(x, y)$ contains a map covering the identity map of $\pi(x)$. This implies that x is isomorphic to y and hence, since $\mathscr{E}$ is skeletal, that $x = y$. For (iii), source lifting gives a map $\omega\colon z \longrightarrow y$ in $\mathscr{E}$ such that $\pi(\omega) = \pi(\omega')$. Here $F(z) \neq F(x)$ since $\pi(F(\omega)) = \pi(\omega')$, $F(\omega) \neq \omega'$, and $\mathscr{F}$ is faithful. By divisibility, there is a map $\xi\colon F(x) \longrightarrow F(z)$ such that $F(\omega) \circ \xi = \omega'$ and $\pi(\xi)$ is an identity map, so that ξ must be an isomorphism. Part (iv) follows immediately from (iii). $\square$

Definitions 8.3. We define saturated and supersaturated representations.

(i) Fix a set O. Define a partial ordering on the collection of faithful representations $(\mathscr{E}, R)$ such that $\mathscr{E}$ has object set O by letting $(\mathscr{E}, R) \leqslant (\mathscr{E}', R')$ if there is a strict injection $F\colon (\mathscr{E}, R) \longrightarrow (\mathscr{E}', R')$ such that F is the identity on object sets. A faithful representation with object set O is *supersaturated* if it is maximal with respect to this partial ordering.

(ii) A *saturated* representation is a discrete supersaturated representation.

(iii) A *saturation* of a faithful representation $(\mathscr{E}, R)$ is a surjection (F, ϕ) from $(\mathscr{E}, R)$ to a saturated representation $(\mathscr{S}, S)$.

(iv) Two saturations $(F, \phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{S}, S)$ and $(F', \phi')\colon (\mathscr{E}, R) \longrightarrow (\mathscr{S}', S')$ are *isomorphic* if there is a map $(K, \kappa)\colon (\mathscr{S}, S) \longrightarrow (\mathscr{S}', S')$ of representations such that $(F', \phi') = (K, \kappa) \circ (F, \phi)$. It follows from Proposition 8.14 below that (K, κ) is then an isomorphism of representations.

Supersaturated representations have all the maps they can hold while remaining faithful; saturated representations have no redundant objects. The following analogous definition will not be very useful to us, but it helps to clarify ideas.

Definition 8.4. We define replete and superreplete representations.

(i) A faithful representation $(\mathscr{E}, R)$ is *superreplete* if, for all objects x, y of $\mathscr{E}$, every map $\pi(x) \longrightarrow \pi(y)$ in the image of $\pi : \mathscr{R}(R(x), R(y)) \longrightarrow \mathscr{B}(\pi(x), \pi(y))$ is also in the image of $\pi : \mathscr{E}(x, y) \longrightarrow \mathscr{B}(\pi(x), \pi(y))$. That is, every map in $\mathscr{B}$ that might possibly be the image under π of a map in $\mathscr{E}$ is such an image.

(ii) A *replete* representation is a discrete superreplete representation.

(iii) A *repletion* of a faithful representation $(\mathscr{E}, R)$ is a surjection (F, ϕ) from $(\mathscr{E}, R)$ to a replete representation $(\mathscr{F}, S)$.

The following observations are immediate from the definition.

Proposition 8.5. *The following statements hold.*

(i) Any subrepresentation of a superreplete representation is superreplete.

(ii) A representation is superreplete if and only if it has a replete skeleton.

(iii) If one of two equivalent representations is superreplete, then so is the other.

(iv) Any superreplete representation is supersaturated.

We would prefer to work with replete rather than saturated representations, but we have not been able to prove that enough of them exist for a workable theory.

One might guess that the converse of (iv) holds. We state this guess formally, as a guide to future work, although we do not believe that it is true in general. We do think that it may hold under restrictive hypotheses on $\mathscr{B}$ and $\mathscr{R}$.

Conjecture 8.6. *Any supersaturated representation is superreplete.*

We give some idea of the starting point towards a verification in special cases.

Remark 8.7. Suppose that, for all objects x, y of $\mathscr{R}$, either $\mathscr{R}(x,y)$ is empty or $\pi : \mathscr{R}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y))$ is a surjection. By Lemma 2.3, this holds when $\mathscr{R} = \mathscr{V}_G(n)$ and $\mathscr{B} = \mathscr{O}_G$ for an *Abelian* compact Lie group G. Then a representation $(\mathscr{F}, S)$ is superreplete if and only if, for all objects x, y in $\mathscr{F}$, either $\mathscr{F}(x,y)$ is empty or $\pi : \mathscr{F}(x,y) \longrightarrow \mathscr{B}(\pi(x),\pi(y))$ is a bijection. This condition on $\mathscr{F}$ is independent of $\mathscr{R}$. It implies that if $(F,\phi) : (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ is a repletion, then, up to isomorphism, $\mathscr{F}$ must be the quotient category of $\mathscr{E}$ whose objects and morphisms are the equivalence classes of objects and morphisms of $\mathscr{E}$, where objects x and x' are equivalent if $\pi(x) = \pi(x')$ and $x \cong x'$ and morphisms $\omega : x \longrightarrow y$ and $\omega' : x' \longrightarrow y'$ are equivalent if x is equivalent to x', y is equivalent to y', and $\pi(\omega) = \pi(\omega')$. The functor $F : \mathscr{E} \longrightarrow \mathscr{F}$ must send objects and morphisms to their equivalence classes, and $\pi : \mathscr{F} \longrightarrow \mathscr{B}$ must be given by $\pi(F(x)) = \pi(x)$ and $\pi(F(\omega)) = \pi(\omega)$. These specifications give a well-defined discrete groupoid $\mathscr{F}$ over $\mathscr{B}$ and map $F : \mathscr{E} \longrightarrow \mathscr{F}$ of groupoids over $\mathscr{B}$; $\mathscr{F}$ will satisfy the required bijectivity of π on non-empty morphism sets if, for every pair of morphisms α, $\beta : a \longrightarrow b$ in $\mathscr{B}$, there is a morphism $\gamma : a \longrightarrow a$ such that $\alpha \circ \gamma = \beta$. For example, this holds for $\mathscr{O}_G$ if G is Abelian. However, further conditions are needed to ensure that R factors up to isomorphism through a functor $S : \mathscr{F} \longrightarrow \mathscr{R}$.

Saturated representations give the closest possible approximations to replete representations that can be constructed in general. The idea of their construction is to adjoin isomorphisms over identity maps in $\mathscr{B}$, as suggested by Lemma 8.2(iii), to expand any faithful representation to a supersaturated one. Unless Conjecture 8.6 holds, we cannot expect to expand all the way to a superreplete representation. We then take a skeleton to obtain a saturated representation. This makes sense in view of the following crucial result, which is the analogue for supersaturated representations of Proposition 8.5(i); it will imply the analogues of (ii) and (iii).

Theorem 8.8. *Any subrepresentation $(\mathscr{E}, R)$ of a supersaturated representation $(\mathscr{S}, S)$ is supersaturated.*

Proof. Suppose for a contradiction that $(\mathscr{E}, R) < (\mathscr{E}', R')$. Then, by Lemma 8.2(iii), $\mathscr{E}'$ is obtained from $\mathscr{E}$ by adjoining isomorphisms over identity maps of $\mathscr{B}$. Let $\mathscr{S}'$ be the category obtained by adjoining these isomorphisms to $\mathscr{S}$. Formally, the category $\mathscr{S}'$ is the pushout of the inclusion $\mathscr{E} \longrightarrow \mathscr{S}$ and the injection $\mathscr{E} \longrightarrow \mathscr{E}'$:

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}' \arrow[d] \\ \mathscr{S} \arrow[r] & \mathscr{S}'. \end{tikzcd}$$

The objects of $\mathscr{S}'$ are the objects of $\mathscr{S}$, and the morphism sets are constructed in a manner similar to the construction of amalgamated free products of groups. We do not assume familiarity with pushouts of categories such as this, and we will give an explicit construction of $\mathscr{S}'$ below. By the pushout property, there is a unique functor $\pi\colon \mathscr{S}' \longrightarrow \mathscr{B}$ that restricts to the functors π on $\mathscr{S}$ and $\mathscr{E}'$ and a unique functor $S'\colon \mathscr{S}' \longrightarrow \mathscr{R}$ over $\mathscr{B}$ that restricts to S on $\mathscr{S}$ and to R' on $\mathscr{E}'$. We will show that $\mathscr{S}'$ is a faithful groupoid over $\mathscr{B}$. This will contradict the maximality of $(\mathscr{S}, S)$ and complete the proof.

We claim first that, assuming $\mathscr{S}'$ exists, any morphism $x \longrightarrow y$ in it must admit a factorization of the form $\omega\psi\xi$, where ξ is an isomorphism in $\mathscr{S}$ over an identity map in $\mathscr{B}$, ψ is one of the adjoined isomorphisms of $\mathscr{E}'$ over identity maps of $\mathscr{B}$, and ω is a morphism in $\mathscr{S}$. For any composite $\psi\omega\psi'$ in which ψ and ψ' are in $\mathscr{E}'$ and ω is in $\mathscr{S}$, the source and target of ω are in $\mathscr{E}$ and therefore, since $\mathscr{E}$ is a full subcategory, ω is in $\mathscr{E}$. Thus the composite is in $\mathscr{E}'$. This implies that we can reduce the length of any word in morphisms of $\mathscr{S}$ and $\mathscr{E}'$ to the form $\omega'\psi'\nu'$ with ω' and ν' in $\mathscr{S}$ and ψ' in $\mathscr{E}'$. As in the proof of Lemma 8.2(iii), we can use the source lifting property in $\mathscr{E}$ and the divisibility property in $\mathscr{S}$ to write $\nu' = \mu\xi$, where μ is a map in $\mathscr{E}$ and ξ is an isomorphism in $\mathscr{S}$ over an identity map. By Lemma 8.2(iii), we can write $\psi'\mu = \omega''\psi$, where ω'' is a map in $\mathscr{E}$ and ψ is one of the adjoined isomorphisms. With $\omega = \omega'\omega''$, this gives the claimed factorization.

We construct $\mathscr{S}'$ by letting the morphisms $x \longrightarrow y$ be the equivalence classes of formal composites

$$x \xrightarrow{\xi} s \xrightarrow{\psi} t \xrightarrow{\omega} y$$

as above, where $\omega\psi\xi$ is equivalent to $\omega'\psi'\xi'$ if $\pi(\omega) = \pi(\omega')$. This condition is equivalent to the existence of unique isomorphisms σ and τ in $\mathscr{E}$ that make the triangles commute in $\mathscr{S}$ and the rectangle commute in $\mathscr{E}'$ in the following diagram:

$$\begin{tikzcd}
 & s \arrow[r, "\psi"] \arrow[d, "\sigma"'] & t \arrow[dr, "\omega"] \arrow[d, "\tau"] & \\
x \arrow[ur, "\xi"] \arrow[dr, "\xi'"'] & & & y \\
 & s' \arrow[r, "\psi'"'] & t' \arrow[ur, "\omega'"'] &
\end{tikzcd}$$

Indeed, if σ and τ make the diagram commute, they must be isomorphisms over identity maps of $\mathscr{B}$ since the ξ's and ψ's are isomorphisms over identity maps of $\mathscr{B}$, and then $\pi(\omega)$ must equal $\pi(\omega')$. Conversely, if $\pi(\omega) = \pi(\omega')$, then by divisibility in $\mathscr{S}$ there is a unique map τ making the right triangle commute. Since $\mathscr{E}$ is a full subcategory of $\mathscr{S}$, τ is in $\mathscr{E}$. By divisibility in $\mathscr{E}'$, there is then a unique map σ in $\mathscr{E}'$ making the middle square commute. Then $\pi(\sigma)$ is an identity map by the diagram, and the left triangle commutes in $\mathscr{E}'$ since $\mathscr{E}'$ is faithful. But then σ is in $\mathscr{E}$. Using the proof of the first claim above, it is an exercise to show that two such formal composites $x \longrightarrow y$ and $y \longrightarrow z$ can be spliced together to give a formal composite $x \longrightarrow z$, well-defined up to equivalence. This makes $\mathscr{S}'$ into a category with $\mathscr{E}$ and

$\mathscr{S}$ as subcategories.

The universal property required of a pushout is immediate from the diagrammatic description of equivalence between morphisms. For example, $\pi\colon \mathscr{S}' \longrightarrow \mathscr{B}$ can and must be defined by $\pi(\omega\psi\xi) = \pi(\omega)$, and it is immediate from the specification of equivalence that π is a faithful functor. Similarly, $S'\colon \mathscr{S}' \longrightarrow \mathscr{R}$ can and must be defined by $S'(\omega\psi\xi) = S(\omega)R'(\psi)S(\xi)$. Since the objects of $\mathscr{S}'$ are those of $\mathscr{S}$, source lifting is inherited from $\mathscr{S}$. To show divisibility, suppose that $\mu = \omega\psi\xi\colon x \longrightarrow y$ and $\mu' = \omega'\psi'\xi'\colon x' \longrightarrow y$ are morphisms in $\mathscr{S}'$, decomposed as usual, and suppose that $\pi(\mu) = \pi(\mu')\beta$. By divisibility in $\mathscr{S}$, there is a ν such that $\omega'\nu = \omega$ and $\pi(\nu) = \beta$, and then $\mu' \circ ((\xi')^{-1}(\psi')^{-1}\nu\psi\xi) = \mu$ in $\mathscr{S}'$. This completes the proof that $\mathscr{S}'$ is a faithful groupoid over $\mathscr{B}$. □

Corollary 8.9. *A representation is supersaturated if and only if it has a saturated skeleton.*

Proof. The theorem implies that a skeleton of a supersaturated representation is saturated, and the converse is obvious. □

Corollary 8.10. *If one of two equivalent representations $(\mathscr{E}, R)$ and $(\mathscr{F}, S)$ is supersaturated, then so is the other.*

Proof. If $(\mathscr{E}, R)$ is supersaturated, then $(\mathscr{F}, S)$ is faithful, by Lemma 7.2, and $(\mathscr{E}, R)$ and $(\mathscr{F}, S)$ have equivalent and therefore isomorphic skeleta, by Lemma 7.3. Thus a skeleton of $(\mathscr{F}, S)$ is saturated and $(\mathscr{F}, S)$ is supersaturated. □

Corollary 8.11. *The image of any map from any representation into a saturated representation is saturated. Therefore, any map from a faithful representation $(\mathscr{E}, R)$ into a saturated representation factors through a saturation of $(\mathscr{E}, R)$.*

Proof. This is now immediate in view of Lemma 8.2(iv) and the fact that a subcategory of a skeletal category is skeletal. □

Now we can produce saturations of any faithful representation.

Lemma 8.12. *Any faithful representation $(\mathscr{E}, R)$ has a saturation.*

Proof. An immediate application of Zorn's lemma shows that there exists a supersaturated representation $(\mathscr{E}', R') \geqslant (\mathscr{E}, R)$. A skeleton of $(\mathscr{E}', R')$ is still supersaturated, by Theorem 8.8, and is therefore saturated. The composite of the injection of $(\mathscr{E}, R)$ into $(\mathscr{E}', R')$ with a retraction of $(\mathscr{E}', R')$ onto a skeleton is clearly surjective on objects, hence on morphisms by Lemma 8.2(iv), so is a saturation. □

This is not the only way to produce a saturated representation from a faithful one. Here is another construction.

Lemma 8.13. *Any faithful representation $(\mathscr{E}, R)$ contains a saturated subrepresentation that is maximal with respect to inclusion.*

Proof. This is just a direct application of Zorn's lemma to the set of saturated subrepresentations, partially ordered by inclusion. This set is nonempty because the empty representation is saturated, and it is easy to check that the union of a chain of saturated subrepresentations is again saturated. □

It is interesting that neither of these two ways of producing a saturated representation from a faithful one gives a unique result in general. In any case, we have plenty of saturated representations. They admit the following characterization.

Proposition 8.14. *Let $(\mathscr{E}, R)$ be a faithful representation. If every map from $(\mathscr{E}, R)$ to a faithful representation is an injection, then $(\mathscr{E}, R)$ is saturated. If $(\mathscr{E}, R)$ is saturated, then every map from $(\mathscr{E}, R)$ to a faithful representation is an inclusion.*

Proof. We first show that if $(\mathscr{E}, R)$ is not saturated, then there is a map from it to a faithful representation that is not an injection. If $(\mathscr{E}, R)$ is not skeletal, then a retraction to a skeleton gives a map to a faithful representation that is not an injection. If $(\mathscr{E}, R)$ is skeletal but not supersaturated, then a saturation of $(\mathscr{E}, R)$ cannot be an injection since it would then be an isomorphism.

Conversely, let $(\mathscr{E}, R)$ be saturated, and let $(F, \phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ be a map into a faithful representation. By Lemma 8.2(i), F is faithful. Let $\mathscr{E}'$ be the groupoid over $\mathscr{B}$ that has the same objects as $\mathscr{E}$, but has morphism sets $\mathscr{E}'(x, y) = \mathscr{F}(F(x), F(y))$. Certainly $\mathscr{E}'$ is faithful. Extend $R\colon \mathscr{E} \longrightarrow \mathscr{R}$ to $R'\colon \mathscr{E}' \longrightarrow \mathscr{R}$ by letting $R' = R$ on objects and letting $R'(\omega) = \phi(y)S(\omega)\phi^{-1}(x)$ for a map $\omega\colon F(x) \longrightarrow F(y)$. By the maximality of $(\mathscr{E}, R)$, $\mathscr{E}'$ can have no more maps than $\mathscr{E}$, hence $F\colon \mathscr{E}(x, y) \longrightarrow \mathscr{F}(F(x), F(y))$ must be a surjection for all x and y. By Lemma 8.2(ii), (F, ϕ) is an inclusion. $\square$

9. Universal orientable representations

We are nearly ready to construct the universal orientable representation. It is not described in terms of the usual kind of universal property, but is instead characterized by one of several different equivalent properties.

Definition 9.1. A faithful representation $(\mathscr{E}, R)$ is an *absolute retract* if every map from $(\mathscr{E}, R)$ into a faithful representation has a left inverse.

If $(\mathscr{E}, R)$ is an absolute retract, then every map of it into a faithful representation must be an injection, and hence, by Proposition 8.14, $(\mathscr{E}, R)$ must be saturated.

Definition 9.2. A representation $(\mathscr{E}, R)$ is said to be *injective* if, for any inclusion $(\mathscr{F}, S) \longrightarrow (\mathscr{F}', S')$ of discrete representations, each map $(\mathscr{F}, S) \longrightarrow (\mathscr{E}, R)$ extends to a map $(\mathscr{F}', S') \longrightarrow (\mathscr{E}, R)$.

We shall construct the universal orientable representation by first constructing a universal discrete representation that is characterized by a categorical universal property and is injective, but is not saturated. Its saturation will inherit the injectivity and will be the representation we want.

Construction 9.3. We construct the *universal discrete representation* $(\tilde{\mathscr{R}}, D)$ in $\mathscr{R}$. Recall the freeness property of the groupoids $\mathscr{B}/b$ over $\mathscr{B}$ given in Proposition 6.8. The objects of the category $\tilde{\mathscr{R}}$ are the functors $F\colon \mathscr{B}/b \longrightarrow \mathscr{R}$ over $\mathscr{B}$. Define $\pi\colon \tilde{\mathscr{R}} \longrightarrow \mathscr{B}$ and $D\colon \tilde{\mathscr{R}} \longrightarrow \mathscr{R}$ on objects by $\pi(F) = b$ and $D(F) = F(\mathrm{id}_b)$. The morphisms $F \longrightarrow F'$, $F'\colon \mathscr{B}/b' \longrightarrow \mathscr{R}$, of $\tilde{\mathscr{R}}$ are the strict maps of representations

$\tilde{\alpha}: \mathscr{B}/b \longrightarrow \mathscr{B}/b'$, so that $F' \circ \tilde{\alpha} = F$. Define π on morphisms by $\pi(\tilde{\alpha}) = \alpha : b \longrightarrow b'$, where $\alpha = \tilde{\alpha}(\mathrm{id}_b)$. Regarding α as a map $\overline{\alpha}: \alpha \longrightarrow \mathrm{id}_{b'}$ in $\mathscr{B}/b'$, define D on morphisms by

$$D(\tilde{\alpha}) = F'(\overline{\alpha}) : F(\mathrm{id}_b) = F'(\alpha) \longrightarrow F'(\mathrm{id}_{b'}).$$

Composition in $\tilde{\mathscr{R}}$ is given by composition of functors. It is easy to check that $\tilde{\mathscr{R}}$ is a well-defined category, that π and D are functors, and that $\pi = \pi \circ D$, so that D is a map over $\mathscr{B}$. By Proposition 6.8, the functor $\tilde{\alpha}$ is determined by α, and this implies that $\tilde{\mathscr{R}}$ is a discrete groupoid over $\mathscr{B}$.

The name "universal discrete representation" is justified by the following result.

Proposition 9.4. *Any discrete representation $(\mathscr{E}, R)$ in $\mathscr{R}$ lifts uniquely to a strict map $\tilde{R}: \mathscr{E} \longrightarrow \tilde{\mathscr{R}}$.*

Proof. We are given $R: \mathscr{E} \longrightarrow \mathscr{R}$, and we define $\tilde{R}: \mathscr{E} \longrightarrow \tilde{\mathscr{R}}$ as follows. Since $\mathscr{E}$ is discrete over $\mathscr{B}$, Proposition 6.8 shows that an object y of $\mathscr{E}$ with $\pi(y) = b$ uniquely determines a functor $\tilde{y}: \mathscr{B}/b \longrightarrow \mathscr{E}$ such that $\tilde{y}(\mathrm{id}_b) = y$. We let $\tilde{R}(y) = R \circ \tilde{y}$. For a morphism $\omega: y \longrightarrow y'$ of $\mathscr{E}$ with $\pi(\omega) = \alpha : b \longrightarrow b'$, we let $\tilde{R}(\omega) = \tilde{\alpha}: \mathscr{B}/b \longrightarrow \mathscr{B}/b'$, so that $\tilde{y}'\tilde{\alpha} = \tilde{y}$. Then $\tilde{R}$ is a well-defined map over $\mathscr{B}$ such that $D \circ \tilde{R} = R$, and it is the only such map. $\square$

Lemma 9.5. *The representation $(\tilde{\mathscr{R}}, D)$ is injective.*

Proof. Let $(I, \iota): (\mathscr{S}, S) \longrightarrow (\mathscr{S}', S')$ be an inclusion of discrete representations and let $(F, \phi): (\mathscr{S}, S) \longrightarrow (\tilde{\mathscr{R}}, D)$ be a map. Thus $\iota: S' \circ I \longrightarrow S$ and $\phi: D \circ F \longrightarrow S$ are isomorphisms over $\mathscr{B}$. We claim that there is a representation $S'': \mathscr{S}' \longrightarrow \mathscr{R}$ such that $S'' \circ I = D \circ F$ and an isomorphism $\phi': S'' \longrightarrow S'$ over $\mathscr{B}$ such that $\iota \circ (\phi' \circ I) = \phi$. Taking $F' = \tilde{S}''$, this will give a map $(F', \phi'): (\mathscr{S}', S') \longrightarrow (\tilde{\mathscr{R}}, D)$ such that $(F', \phi') \circ (I, \iota) = (F, \phi)$. To check the claim, note that S'' and ϕ' are already defined on the full subcategory $I(\mathscr{S})$ of $\mathscr{S}'$. Define $S''(y) = S'(y)$ and let $\phi'(y)$ be the identity map for an object y of $\mathscr{S}'$ that is not in $\mathscr{S}$. For a morphism $\omega: x \longrightarrow y$ of $\mathscr{S}'$, we can and must define $S''(\omega) = \phi'(y)^{-1} \circ S'(\omega) \circ \phi'(x)$. $\square$

We can now construct the universal orientable representation in $\mathscr{R}$.

Theorem 9.6. *The following conditions on a faithful representation $(\mathscr{S}, S)$ in $\mathscr{R}$ are equivalent. Moreover, there exists a representation $(\mathscr{S}\mathscr{R}, S)$ satisfying these conditions, it is unique up to isomorphism, and any map from it to itself is an isomorphism. This representation is called the* universal orientable representation.

(i) $(\mathscr{S}, S)$ *is a maximal saturated subrepresentation of* $(\tilde{\mathscr{R}}, D)$.

(ii) $(\mathscr{S}, S)$ *is a saturation of* $(\tilde{\mathscr{R}}, D)$.

(iii) $(\mathscr{S}, S)$ *is a saturated retract of* $(\tilde{\mathscr{R}}, D)$.

(iv) $(\mathscr{S}, S)$ *is an absolute retract.*

(v) $(\mathscr{S}, S)$ *is a saturated injective representation.*

(vi) $(\mathscr{S}, S)$ *is saturated, and any faithful representation maps into it.*

(vii) $(\mathscr{S}, S)$ *is saturated, and any saturated representation maps into it.*

Proof. We understand the first statement to mean that if $(\mathscr{S}, S)$ satisfies one of the conditions and $(\mathscr{S}', S')$ satisfies another, then $(\mathscr{S}, S)$ is isomorphic to $(\mathscr{S}', S')$. The proofs will show how to obtain the required isomorphisms.

(i) $\Longleftrightarrow$ (ii) and (ii) $\Longleftrightarrow$ (iii): By Lemma 8.13, there exists an $(\mathscr{S}, S)$ satisfying (i). By Lemma 8.12, there also exists a saturation $(F, \phi)\colon (\tilde{\mathscr{R}}, D) \longrightarrow (\mathscr{S}', S')$. By Proposition 8.14, the restriction of (F, ϕ) to $(\mathscr{S}, S)$ is an inclusion. By the injectivity of $(\tilde{\mathscr{R}}, D)$ and Proposition 8.14, the inclusion of $(\mathscr{S}, S)$ in $(\tilde{\mathscr{R}}, D)$ extends to an inclusion of $(\mathscr{S}', S')$. By maximality, the inclusion of $(\mathscr{S}, S)$ in $(\mathscr{S}', S')$ is an isomorphism. This already proves that (i) is equivalent to (ii) and that these conditions imply (iii); (iii) implies (i) trivially.

(iii)$\Longrightarrow$(iv): Let $(\mathscr{S}, S) \longrightarrow (\mathscr{E}', R')$ be a map into a faithful representation and let $(\mathscr{E}', R') \longrightarrow (\mathscr{E}'', R'')$ be a retraction onto a skeleton. By Proposition 8.14, the composite of these maps is an inclusion. By the injectivity of $(\tilde{\mathscr{R}}, D)$, this inclusion extends to a map of $(\mathscr{E}'', R'')$ into $(\tilde{\mathscr{R}}, D)$. Composing with a retraction $(\tilde{\mathscr{R}}, D) \longrightarrow (\mathscr{S}, S)$, we see that $(\mathscr{S}, S)$ is a retract of $(\mathscr{E}', R')$.

(iv)$\Longrightarrow$(iii): There is a map $(\mathscr{S}, S) \longrightarrow (\tilde{\mathscr{R}}, D)$ by the universal property of $(\tilde{\mathscr{R}}, D)$. This map admits a retraction since $(\mathscr{S}, S)$ is an absolute retract.

(iii)$\Longrightarrow$(v): A retract of an injective representation is injective.

(v)$\Longrightarrow$(vi) and (vi) $\Longleftrightarrow$ (vii): Any faithful representation retracts to a skeletal and thus discrete representation. For discrete representations, condition (vi) is just the special case of the injectivity condition (v) in which the given domain representation is empty. Obviously (vi)$\Longrightarrow$(vii), and (vii)$\Longrightarrow$(vi) by Corollary 8.11.

(vi)$\Longrightarrow$(i): By (vi) and Proposition 8.14, there is an inclusion of any given maximal saturated subrepresentation $(\mathscr{S}', S')$ of $(\tilde{\mathscr{R}}, D)$ in $(\mathscr{S}, S)$. By the injectivity of $(\tilde{\mathscr{R}}, D)$ and Proposition 8.14, this inclusion extends to an inclusion of $(\mathscr{S}, S)$ in $(\tilde{\mathscr{R}}, D)$. By the maximality of $(\mathscr{S}', S')$, the first inclusion must be an isomorphism.

The argument of the last paragraph also gives the uniqueness, up to isomorphism, of $(\mathscr{S}, S)$. Varying the argument by composing the initial inclusion of $(\mathscr{S}', S')$ with any self-map of $(\mathscr{S}, S)$, and noting that such a self-map is an inclusion, we see that any self-map of $(\mathscr{S}, S)$ is an isomorphism. $\square$

Criterion (vi) is the one on which our definition of an orientation is based. Criterion (ii) is the one most useful for the actual construction of specific examples. One implication of the theorem is that the set of self-maps of $(\mathscr{S}\mathscr{R}, S)$ is a group under composition. This is not true for a general saturated representation, as is shown by the following example.

Example 9.7. Take $\mathscr{B}$ to be the "unit interval" category $\mathscr{I}$. It has two objects, $\mathbf{0}$ and $\mathbf{1}$, and one non-identity morphism $\mathbf{I} : \mathbf{0} \longrightarrow \mathbf{1}$. Clearly functors $\pi\colon \mathscr{E} \longrightarrow \mathscr{I}$ are determined by the images of objects. Let $\mathscr{R}$ be the groupoid over $\mathscr{I}$ with a single object $\tilde{\mathbf{0}}$ over $\mathbf{0}$, a single object $\tilde{\mathbf{1}}$ over $\mathbf{1}$, and morphism sets $\mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{0}}) = \mathbb{Z}$, $\mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{1}}) = \mathbb{Z}$, and $\mathscr{R}(\tilde{\mathbf{1}}, \tilde{\mathbf{1}}) = \{\mathrm{id}\}$. Composition is defined so that $\mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{0}}) = \mathbb{Z}$ as a group and $\mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{0}})$ acts on $\mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{1}})$ by addition.

(i) Let $\mathscr{E}$ be the groupoid over $\mathscr{I}$ with a single object x over $\mathbf{0}$, infinitely many objects y_i, $i \geqslant 1$, over $\mathbf{1}$, and a map $\omega_i \colon x \longrightarrow y_i$ for each i as its only non-identity maps. Define a representation $R \colon \mathscr{E} \longrightarrow \mathscr{R}$ by sending ω_i to $i \in \mathscr{R}(\tilde{\mathbf{0}}, \tilde{\mathbf{1}})$. Clearly $(\mathscr{E}, R)$ is discrete and replete, hence saturated. There is a map $(F, \phi) \colon (\mathscr{E}, R) \longrightarrow (\mathscr{E}, R)$ with $F(x) = x$, $F(y_i) = y_{i+1}$ for each i, $\phi(x) = 1 : \tilde{\mathbf{0}} \longrightarrow \tilde{\mathbf{1}}$, and $\phi(y_i) = \mathrm{id}_{\tilde{\mathbf{1}}}$. The map (F, ϕ) is not an isomorphism.

(ii) Unusually, in this example $(\tilde{\mathscr{R}}, D)$ is its own saturation. Its description is similar to that of $(\mathscr{E}, R)$, except that there is now an object y_i for every integer i. Its group of self-maps is $\mathbb{Z}$.

Of course, there may be many different orientations $(F, \phi) \colon (\mathscr{E}, R) \longrightarrow (\mathscr{S}\mathscr{R}, S)$ of a given orientable representation $(\mathscr{E}, R)$. We have the following observations.

Proposition 9.8. *Let $\Omega = \Omega(\mathscr{S}\mathscr{R}, S)$ be the group of automorphisms of the universal orientable representation $(\mathscr{S}\mathscr{R}, S)$ and let $\Omega(\mathscr{E}, R)$ be the set of orientations of an orientable representation $(\mathscr{E}, R)$. Then Ω acts on $\Omega(\mathscr{E}, R)$ by composition. This action has the following properties.*

(i) If $(\mathscr{E}, R)$ is saturated, then Ω acts transitively on $\Omega(\mathscr{E}, R)$, hence $\Omega(\mathscr{E}, R)$ is isomorphic as an Ω-set to Ω/Λ, where Λ is the isotropy group of any chosen orientation of $(\mathscr{E}, R)$.

(ii) If $(\mathscr{E}, R)$ is faithful, then any orientation $(\mathscr{E}, R) \longrightarrow (\mathscr{S}\mathscr{R}, S)$ factors through a saturation of $(\mathscr{E}, R)$. Therefore,

$$\Omega(\mathscr{E}, R) \cong \coprod_{[\mathscr{E}', R']} \Omega(\mathscr{E}', R')$$

as Ω-sets, where the union runs over one representative from each isomorphism class of saturations $(\mathscr{E}, R) \longrightarrow (\mathscr{E}', R')$ of $(\mathscr{E}, R)$.

Proof. This follows from Corollary 8.11, Proposition 8.14, and the fact that $(\mathscr{S}\mathscr{R}, S)$ is an injective representation whose self-maps are all automorphisms. ☐

Our explicit examples of universal orientable representations will be replete. In such cases, Conjecture 8.6 certainly holds. Indeed, the following result is immediate from Propositions 8.14 and 8.5.

Proposition 9.9. *If the universal orientable representation $(\mathscr{S}\mathscr{R}, S)$ is replete, then every supersaturated representation $(\mathscr{E}, R)$ is superreplete.*

Part III. Examples of universal orientable representations

10. Cyclic groups of prime order

We here determine $(\mathscr{S}\mathscr{V}_G(n), S)$ explicitly for $G = \mathbb{Z}/p$. We first insert the following observation. Recall from Remark 1.3 that $\mathscr{O}_G/(G/K) \cong \Pi_G(G/K)$ when G is finite. Also, recall Construction 9.3.

Proposition 10.1. *If G is finite, then the groupoids $\tilde{\mathscr{V}}_G(n)$ and $\mathscr{S}\mathscr{V}_G(n)$ over $\mathscr{O}_G$ have only finitely many objects and finitely many morphisms.*

Proof. It suffices to consider $\tilde{\mathscr{V}}_G(n)$. Since $\tilde{\mathscr{V}}_G(n)$ is faithful over $\mathscr{O}_G$ and $\mathscr{O}_G$ is finite, it suffices to show that $\tilde{\mathscr{V}}_G(n)$ has finitely many objects. The objects are the representations $F\colon \mathscr{O}_G/(G/K) \longrightarrow \mathscr{V}_G(n)$. Such functors are consistent families of homotopy classes of maps $G\times_H V \longrightarrow G\times_K W = F(\mathrm{id}_{G/K})$ of G-vector bundles, where W is a fixed K-representation and H runs over the subconjugates of K. We are working in the skeletal category $\mathscr{V}_G(n)$, and there are only finitely many choices for W, finitely many H, finitely many choices of V for each H, and finitely many choices of maps for each H and V. $\square$

In the proof just given, W actually determines all of the representations V. However, it does not determine the maps $G\times_H V \longrightarrow G\times_K W$, since we can precompose a given map with any map $G\times_H V \longrightarrow G\times_H V$ over the identity map of G/H. This variability is crucial to understanding $\tilde{\mathscr{V}}_G(n)$. Postcomposition with maps $G\times_K W \longrightarrow G\times_K W$ over the identity map of G/K leads to natural transformations $\beta \longrightarrow \beta'$ between objects of $\tilde{\mathscr{V}}_G(n)$ with the same W. This variability is crucial to understanding the passage from $(\tilde{\mathscr{V}}_G(n), D)$ to its saturated retract, the universal orientable representation $(\mathscr{S}\mathscr{V}_G(n), S)$.

Let $G = \mathbb{Z}/p$, where p is a prime, and let t be a generator of G. The orbit category $\mathscr{O}_G$ has only the two objects $G = G/e$ and $P = G/G$. The maps $G \longrightarrow G$ form a copy of G, there is a unique quotient map $q\colon G \longrightarrow P$, and the only map $P \longrightarrow P$ is the identity. We consider the cases $p = 2$ and $p > 2$ separately. In both cases, it turns out that $(\mathscr{S}\mathscr{V}_G(n), S)$, and therefore every saturated representation, is replete. For this reason, the description of $(\mathscr{S}\mathscr{V}_G(n), S)$ can easily be guessed. However, for illustrative purposes, we follow the outline of the general theory.

Example 10.2. $p = 2$. In this example, we write γ generically for a self-map of a representation that reverses (non-equivariant) orientation and satisfies $\gamma^2 = 1$.

(a) The category $\mathscr{V}_G(n)$.

- (i) $\mathscr{V}_G(n)$ has a single object $U = G\times \mathbb{R}^n$ over G. This object has four self-maps, namely 1 and γ (that is, $1\times\gamma$) over the identity map of G, and t (that is, action by t) and $t\gamma = \gamma t$ over $t\colon G \longrightarrow G$.
- (ii) For $0 \leqslant k \leqslant n$, let $V_k = \mathbb{R}^{n-k}\oplus \mathbb{L}^k$ denote the sum of $n-k$ copies of the trivial one-dimensional representation $\mathbb{R}$ and k copies of the non-trivial one-dimensional representation $\mathbb{L}$. These are the objects of $\mathscr{V}_G(n)$ over P. There are no maps from V_k to $V_{k'}$ unless $k = k'$, in which case there are either the two maps 1 and γ if $k = 0$ or n, or there are the four maps 1, $1\oplus\gamma$, $\gamma\oplus 1$, and $\gamma\oplus\gamma$ if $0 < k < n$; $\gamma\oplus\gamma$ is the composite of $1\oplus\gamma$ and $\gamma\oplus 1$ in either order.
- (iii) There are two maps $U \longrightarrow V_k$, namely the action μ of G on V_k and $\mu\gamma$. We have $(\gamma\oplus 1)\mu = \mu\gamma = (1\oplus\gamma)\mu$. More interestingly, $\mu t = \mu$ if k is even, but $\mu t = \mu\gamma$ if k is odd.

(b) The category $\tilde{\mathscr{V}}_G(n)$.

- (i) The objects over G are the representations $\Pi_G G \longrightarrow \mathscr{V}_G(n)$, and $\Pi_G G$ has two objects over G connected by an isomorphism t over $t\colon G \longrightarrow G$. There are two such representations, U_+ and U_-; the first sends t to t and the second sends t to $t\gamma$. There are no maps between these objects, and each has a self-map $\tilde{t}$ over t.

(ii) The objects over P are the representations $\mathscr{O}_G = \Pi_G P \longrightarrow \mathscr{V}_G(n)$. For each k there are two representations, $(V_k)_+$ and $(V_k)_-$, sending P to V_k. The first sends q to μ and the second sends q to $\mu\gamma$. In both cases, t must be sent to t if k is even and to $t\gamma$ if k is odd. There are no non-identity maps among these objects.

(iii) There is a map $\tilde{q}\colon U_+ \longrightarrow (V_k)_\pm$ over q if and only if k is even; there is a map $\tilde{q}\colon U_- \longrightarrow (V_k)_\pm$ over q if and only if k is odd.

(c) Saturations of $\tilde{\mathscr{V}}_G(n)$.

The only maps that we may add to $\tilde{\mathscr{V}}_G(n)$ are maps between $(V_k)_+$ and $(V_k)_-$. For each k we have a choice of how to send such an added map to $\mathscr{V}_G(n)$: we may send it to $1\oplus\gamma$ or to $\gamma\oplus 1$. There is only one choice, γ, when $k = 0$ or n. Thus there are $2n-2$ possible supersaturations. We obtain a saturation by passing to a skeleton, which means throwing out one of $(V_k)_+$ or $(V_k)_-$ for each k. For definiteness, we discard $(V_k)_-$. Changing notation, we have the following description of the universal orientable representation.

(d) The universal orientable representation $(\mathscr{S}\mathscr{V}_G(n), S)$.

(i) $\mathscr{S}\mathscr{V}_G(n)$ has two objects, u_+ and u_-, over G, and S sends each of them to $U = G \times \mathbb{R}^n$. Each has a self map t over $t\colon G \longrightarrow G$; S sends $t\colon u_+ \longrightarrow u_+$ to t and sends $t\colon u_- \longrightarrow u_-$ to $t\gamma$.

(ii) $\mathscr{S}\mathscr{V}_G(n)$ has $n+1$ objects v_k, $0 \leqslant k \leqslant n$, over P, and $S(v_k) = V_k$.

(iii) There is a map $m\colon u_+ \longrightarrow v_k$ over $q\colon G \longrightarrow P$ if and only if k is even; there is a map $m\colon u_- \longrightarrow v_k$ if and only if k is odd; $S(m) = \mu$ in both cases.

Pictorially, the universal orientable representation looks like this:

$$\begin{tikzcd}
G \arrow[loop left, "t"] \arrow[dd, "q"'] & & u_+ \arrow[loop left, "t"] \arrow[dd, "m"'] & u_- \arrow[loop right, "t"] \arrow[dd, "m"'] & & U \arrow[loop left, "t"] \arrow[r, equal] \arrow[dd, "\mu"] & U \arrow[loop right, "t\gamma"] \arrow[dd, "\mu"] \\
 & {} & {} \arrow[l, "\pi"'] & {} \arrow[r, "S"] & {} & & \\
P & & v_{2k} & v_{2k+1} & & V_{2k} & V_{2k+1}
\end{tikzcd}$$

$\mathscr{O}_G$ $\qquad\qquad$ $\mathscr{S}\mathscr{V}_G(n)$ $\qquad\qquad$ in $\mathscr{V}_G(n)$

(e) The group $\Omega_n = \Omega(\mathscr{S}\mathscr{V}_G(n), S)$.

Ω_n is an elementary abelian 2-group of order 2^{n+1}. Since (d)(iii) rules out the possibilities $A(u_+) = u_-$ or $A(u_-) = u_+$, every automorphism (A, α) has $A = \mathrm{id}\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{S}\mathscr{V}_G(n)$, hence is specified by giving the isomorphisms $\alpha(x)\colon S(x) \longrightarrow S(x)$ for objects x in $\mathscr{S}\mathscr{V}_G(n)$. As generators for Ω_n we can take the collection $\{\alpha_+, \alpha_-, \alpha_1, \ldots, \alpha_{n-1}\}$, where, specifying only the behavior of each of these on objects not assigned identity maps, we have

$$\alpha_+(u_+) = \gamma \text{ and } \alpha_+(v_{2k}) = \gamma \oplus 1 \text{ (or } \gamma \text{ if } 2k = 0 \text{ or } n\text{)},$$

$$\alpha_-(u_-) = \gamma \text{ and } \alpha_-(v_{2k+1}) = \gamma \oplus 1 \text{ (or } \gamma \text{ if } 2k+1 = n\text{), and}$$

$$\alpha_k(v_k) = \gamma \oplus \gamma.$$

(f) An example of non-unique saturation.

We exhibit two saturations of the representation $(\Pi_G G, U_+)$ given in (b)(i) above that are not isomorphic by displaying two maps, (F, ϕ) and (F', ϕ'), from $(\Pi_G G, U_+)$ into $(\mathscr{S}\mathscr{V}_G(n), S)$ whose images do not differ by an automorphism in Ω_n. In fact, let F send both objects of $\Pi_G G$ to u_+, and let ϕ be the identity; let F' send both objects to u_-, and let ϕ' send both to the map $\gamma\colon G \times \mathbb{R}^n \longrightarrow G \times \mathbb{R}^n$.

Observe that the category $\mathscr{S}\mathscr{V}_G(n)$ has two components, although $\mathscr{V}_G(n)$ has the initial object U and is therefore connected. Observe too that the order of Ω_n increases as n increases. The following simple example illustrates the fact that even trivial bundles can have many more equivariant than nonequivariant orientations. As we shall see, none of these phenomena can occur for a finite group of odd order.

Example 10.3. With $G = \mathbb{Z}/2$ still, let S^1 be the circle with G acting by complex conjugation, so that the action of G has two fixed points. Take one of them as a basepoint. For any integer $k \geqslant 2$, let B be the wedge of $k-1$ copies of S^1, so that B is nonequivariantly connected and has k fixed points. Let $E = B \times (\mathbb{R} \oplus \mathbb{L})$ and let $p\colon E \longrightarrow B$ be the projection. Then p has 2^{k+1} distinct orientations. To see this, we can work with a skeleton of $\Pi_G B$ consisting of one object b with $\pi(b) = G$ and k objects x_1 through x_k with $\pi(x_i) = P$. An orientation (F, ϕ) of p must satisfy $F(b) = u_-$ and $F(x_i) = v_1$ for each i. There are two possible choices of $\phi : U = S \circ F(b) \longrightarrow p^*(b) = U$, namely 1 and γ corresponding to the two nonequivariant orientations of p. Having chosen one of these, for each x_i there are then two choices of maps $\phi : V_1 = S \circ F(x_i) \longrightarrow p^*(x_i) = V_1$, differing by composition with $\gamma \oplus \gamma$. Since these choices are independent, we obtain 2^{k+1} different orientations, as claimed.

If, instead of taking a wedge of finitely many copies of S^1, we take the wedge of countably infinitely many copies, we obtain a (trivial) bundle over a connected basespace having uncountably infinitely many different orientations. Note also that, in all of these examples, the group Ω_2 acts neither freely nor transitively on the set of orientations of the bundle.

Example 10.4. p odd. In this example we write γ generically for an orientation reversing map of $\mathbb{R}^k$, $k \geqslant 1$, with $\gamma^2 = 1$. A representation that contains no trivial summands admits no orientation reversing self G-map.

(a) The category $\mathscr{V}_G(n)$.

(i) $\mathscr{V}_G(n)$ has a single object $U = G \times \mathbb{R}^n$ over G. This object has $2p$ self maps, namely 1 and γ (that is, $1 \times \gamma$) over the identity map of G, and t^i (that is, action by t^i) and $t^i\gamma = \gamma t^i$ over $t^i : G \longrightarrow G$.

(ii) G has the trivial irreducible representation $\mathbb{R}$ and $(p-1)/2$ non-trivial irreducible two-dimensional representations. The objects over P are the distinct n-dimensional sums of these. Write V for a typical such object; it has a non-identity map γ if and only if it contains a trivial summand.

(iii) There are two maps μ and $\mu\gamma$ from U to each V. We have $\mu t = \mu$ and, if V contains a trivial summand, $\mu\gamma = \gamma\mu$.

(b) The category $\tilde{\mathscr{V}}_G(n)$.

(i) The objects over G are the representations $\Pi_G G \longrightarrow \mathscr{V}_G(n)$, and $\Pi_G G$ has p objects $\{e_1, \ldots, e_p\}$ over G cyclically permuted by isomorphisms t over $t\colon G \longrightarrow G$. For each sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_p)$, $\epsilon_i = 0$ or 1 and $\sum \epsilon_i$ even, there is a representation $U_\epsilon\colon \Pi_G G \longrightarrow \mathscr{V}_G(n)$ that sends $t\colon e_i \longrightarrow e_{i+1}$ to $t\gamma^{\epsilon_i}$. We write U_0 when all $\epsilon_i = 0$. There is a self-map $\tilde{t}$ over t of U_0. If t acts on the sequences ϵ by cyclic permutation, there are maps $\tilde{t}\colon U_\epsilon \longrightarrow U_{t\epsilon}$ over t. We also have the iterates $\tilde{t}^i$.

(ii) The objects over P are the representations $\mathscr{O}_G = \Pi_G P \longrightarrow \mathscr{V}_G(n)$. For each V, there are two such representations, V_+ and V_-, that send P to V. The first sends q to μ and the second sends q to $\mu\gamma$; both send t to t. There are no non-identity maps among these objects.

(iii) For each V, there are maps $\tilde{q}\colon U_0 \longrightarrow V_\pm$ over q, and there are no other maps over q.

(c) Saturations of $\tilde{\mathscr{V}}_G(n)$.

When V contains a trivial summand, we may add an isomorphism between V_+ and V_- that maps to γ in $\mathscr{V}_G(n)$. We may also add an isomorphism over the identity map of G between U_0 and any other U_ϵ. This isomorphism may be chosen to map to either the identity or γ in $\mathscr{V}_G(n)$. This choice then determines isomorphisms between U_0 and the other $U_{\epsilon'}$ such that ϵ' is a cyclic permutation of ϵ. We pass to a saturation by discarding V_- when V contains a trivial summand, and discarding all U_ϵ with $\epsilon \neq 0$. Changing notation, we have the following description of the universal orientable representation.

(d) The universal orientable representation $(\mathscr{S}\mathscr{V}_G(n), S)$.

(i) $\mathscr{S}\mathscr{V}_G(n)$ has one object u over G, and $S(u) = U = G \times \mathbb{R}^n$. u has self-maps t^i over $t^i\colon G \longrightarrow G$, and $S(t^i) = t^i$.

(ii) For each representation V that contains a trivial summand, $\mathscr{S}\mathscr{V}_G(n)$ has one object v over P with $S(v) = V$. For each representation V that contains no trivial summand, $\mathscr{S}\mathscr{V}_G(n)$ has two objects v_+ and v_- over P, and $S(v_\pm) = V$.

(iii) For each V that contains a trivial summand, there is a map $m\colon u \longrightarrow v$ over q with $S(m) = \mu$. For each V that contains no trivial summand, there are maps $m_+\colon u \longrightarrow v_+$ and $m_-\colon u \longrightarrow v_-$ over q; $S(m_+) = \mu$ and $S(m_-) = \mu\gamma$.

Pictorially, the universal orientable representation looks like this:

$$
\begin{tikzcd}
G \arrow[loop left, "t^i"] \arrow[dd, "q"'] & & & & u \arrow[loop left, "t^i"] \arrow[ddl, "m"'] \arrow[dd, "m_+"] \arrow[ddr, "m_-"] & & & & U \arrow[loop left, "t^i"] \arrow[ddl, "\mu"'] \arrow[dd, "\mu"] \arrow[ddr, "\mu\gamma"] & \\
& & \arrow[l, "\pi"'] & & & & \arrow[r, "S"] & & & \\
P & & & v & v_+ & v_- & & V & V & V
\end{tikzcd}
$$

$(V^G \neq 0)$ $\quad$ $(V^G = 0)$

$\mathscr{O}_G$ $\quad$ $\mathscr{S}\mathscr{V}_G(n)$ $\quad$ in $\mathscr{V}_G(n)$

(e) The group $\Omega_n = \Omega(\mathscr{S}\mathscr{V}_G(n), S)$.

$\Omega_n = \mathbb{Z}/2$. The non-trivial automorphism (A, α) has $A(u) = u$ and $\alpha(u) = \gamma$; $A(v) = v$ and $\alpha(v) = \gamma$ if V contains a trivial summand; and $A(v_+) = v_-$, $A(v_-) = v_+$, and $\alpha(v_+) = \alpha(v_-) = \mathrm{id}_V$ if V does not contain a trivial summand.

11. Orientations of V-dimensional G-bundles

Let $p\colon E \longrightarrow B$ be a G-bundle. Our general theory has been set up to deal with the problem that the fiber representations of p can vary. The fiber F_x over a point $x \in B$ with isotropy group G_x is only a G_x-representation V_x, and the bundle over the orbit $G \cdot F_x \cong G/G_x$ is isomorphic to $G \times_{G_x} V_x$. The universal orientable representation gives a universal model for describing all possible ways that such local data can be oriented consistently. It is sometimes possible to restrict to G-bundles with a single type of fiber representation.

Definition 11.1. Let V be a representation of G. A G-bundle $p\colon E \longrightarrow B$ has *dimension* V if each of its fiber representations V_x is isomorphic to V, regarded as a representation of G_x. This means that $p^*\colon \Pi_G B \longrightarrow \mathscr{V}_G$ factors through $\mathscr{V}_G(V)$, where $\mathscr{V}_G(V)$ is the full subgroupoid over $\mathscr{O}_G$ of $\mathscr{V}_G$ whose objects are the G-bundles isomorphic to $G/H \times V \cong G \times_H V$ for some $H \subset G$. Thus $(\Pi_G B, p^*)$ is a representation in $\mathscr{V}_G(V)$.

For example, p must be V-dimensional for some V if B is G-connected, in the sense that all of its fixed point spaces are non-empty and path connected. One might think that our theory becomes trivial or unnecessary for G-bundles of dimension V but, as we now explain, that is not the case.

Define an orientation of a fiber F_x to be a choice of a homotopy class $\phi(x)$ of G_x-linear isometries $F_x \longrightarrow V$. This directly generalizes the nonequivariant idea that an orientation of a fiber is a choice of one of the two homotopy classes of isomorphisms with a fixed copy of $\mathbb{R}^n$. Define a *naive orientation* of the G-bundle p to be a compatible collection $\{\phi(x)\}$ of orientations of the fibers F_x. Here, by "compatible", we mean that the following two conditions hold. First, noting that $G_{gx} = gG_xg^{-1}$ for $x \in B$ and $g \in G$ and that g^{-1} maps F_{gx} to F_x, the $\phi(x)$ are G-invariant in the sense that $\phi(gx) = g\phi(x)g^{-1}$. Second, if $x \in B^H$, $y \in B^K$ and (ω, α) is a map from x to y in $\Pi_G B$, then $\phi(x) = \phi(y) \circ \tilde{\omega}$, where $\tilde{\omega}\colon : F_x \to F_y$ is the homotopy class of H-linear isometries determined by (ω, α). This definition can be reformulated representation theoretically as follows. To avoid pedantry, we ignore the choice of isomorphism between $G/H \times V$ and the unique isomorphic G-bundle in the skeletal $\mathscr{O}_G$-groupoid $\mathscr{V}_G$ in what follows.

Definition 11.2. Define $I\colon \mathscr{I}\mathscr{V}_G(V) \longrightarrow \mathscr{V}_G(V)$ to be the injection of the subgroupoid over $\mathscr{O}_G$ that contains all of the objects of $\mathscr{V}_G(V)$ but only the maps that are of the form $\alpha \times \mathrm{id}\colon G/H \times V \longrightarrow G/K \times V$ for a map $\alpha\colon G/H \longrightarrow G/K$ in $\mathscr{O}_G$. A *naive orientation* of a V-dimensional G-bundle $p\colon E \longrightarrow B$ is a map of representations

$$(F, \phi)\colon (\Pi_G B, p^*) \longrightarrow (\mathscr{I}\mathscr{V}_G(V), I).$$

As in the nonequivariant case, the functor F here is uniquely determined. It is an instructive exercise to verify that our two definitions of a naive orientation coincide; that is, a naive orientation as defined in the previous definition determines and is determined by a set of compatible orientations of fibers.

However, if we insist that every orientable G-bundle must have an orientation, then the naive definition of an orientation of a V-dimensional bundle is inadequate, at least when $V^G = 0$. Since the representation $(\mathscr{I}\mathscr{V}_G(V), I)$ is clearly discrete, it maps into the universal orientable representation $(\mathscr{S}\mathscr{V}_G(V), S)$ in $\mathscr{V}_G(V)$. We may view a map $(\mathscr{I}\mathscr{V}_G(V), I) \longrightarrow (\mathscr{S}\mathscr{V}_G(V), S)$ as an orientation of $(\mathscr{I}\mathscr{V}_G(V), I)$. However, $\mathscr{I}\mathscr{V}_G(V)$ has too few morphisms for this orientation to be an isomorphism of representations in $\mathscr{V}_G(V)$. Therefore, a naive orientation gives an orientation, but not conversely. When $V^G \neq 0$, a V-dimensional G-bundle does admit a naive orientation, but it may have more orientations than naive orientations.

To display an orientable V-dimensional G-bundle that has no naive orientation, let G be the circle group S^1 and let G act on S^2 by rotation about the axis through the poles. Let V be the tangent representation at the north pole n. We can view S^2 as the one-point compactification of the standard representation of the unit circle of complex numbers on $\mathbb{C}$, taking the origin as the north pole and the point at ∞ as the south pole. Thus V is just a copy of $\mathbb{C}$ with its standard action by S^1. The tangent representation at the south pole s is isomorphic to V, and the tangent bundle τ of S^2 is V-dimensional. Moreover, τ is obviously orientable since there is only one path class connecting any two objects of the equivariant fundamental groupoid $\Pi_G S^2$. However, τ does not have a naive orientation. There is only one homotopy class of G-linear isomorphisms from V to itself, so there is only one choice of equivariant orientation at both the north and the south poles. These orientations are not compatible since any nonequivariant path from the south to the north pole induces a map of tangent planes that reverses orientation.

Now let G be a cyclic group of order p embedded as usual as a subgroup of the circle group. By restriction, we can regard the S^1-bundle just displayed as a G-bundle. It is still V-dimensional and orientable, and it still admits no naive orientation. In the following example, we use the universal orientable G-bundles constructed in the previous section to display orientations of this bundle.

Example 11.3. Let $G = \mathbb{Z}/p$, let $V = \mathbb{L}^2 = V_2$ if $p = 2$, and let V be an irreducible two-dimensional representation if p is odd. Consider the sphere S^V obtained by one-point compactification of V. A skeleton $\Pi'_G(S^V)$ of the fundamental groupoid has one object x over G, and two objects n and s, the north and south poles, over P. The representation $\tau^* \colon \Pi'_G(S^V) \longrightarrow \mathscr{V}_G(2)$ induced by the tangent bundle of S^V sends x to $U = G \times \mathbb{R}^2$ and both n and s to V. There are maps $x \longrightarrow n$ and $x \longrightarrow s$ over q, and they are not sent to the same map in $\mathscr{V}_G(2)$ due to the incompatibility phenomenon that we have observed. Rather, one is sent to μ and the other is sent to $\mu\gamma$. There are two orientations $(F, \phi) \colon (\Pi'_G(S^V), \tau^*) \longrightarrow (\mathscr{S}\mathscr{V}_G(2), S)$. If $p = 2$, they both have $F(x) = u_+$ and $F(n) = v_2 = F(s)$; one has $\phi(x) = 1$ and the other has $\phi(x) = \gamma$. If $p > 2$, they both have $F(x) = u$, one has $F(n) = v_+$ and $F(s) = v_-$, and the other has $F(n) = v_-$ and $F(s) = v_+$; both have $\phi(n) = \mathrm{id} = \phi(s)$, while one has $\phi(x) = 1$ and the other has $\phi(x) = \gamma$.

There is a still more naive notion of an orientation of a G-vector bundle, namely a nonequivariant orientation such that G acts by orientation preserving maps. This notion is widely used in the literature when G is a finite group of odd order. Here the following observation shows that the requirement that the action preserve the orientation is a negligible restriction on nonequivariantly orientable G-bundles.

Lemma 11.4. *Let G be finite of odd order and let $p : E \longrightarrow B$ be a nonequivariantly orientable G-vector bundle.*

(i) If B is path connected, then G acts on p by orientation preserving maps for either of the orientations of p.

(ii) If B/G is path connected, then p admits two orientations such that G acts on p by orientation preserving maps.

(iii) In general, p admits orientations such that G acts by orientation preserving maps.

Proof. For (i), an orientation of p is given by a Thom class in $\tilde{H}^n(T; \mathbb{Z}) \cong \mathbb{Z}$, where T is the Thom space of p, and action by an odd order group element must preserve this class. For (ii), pick a path component B_0 of B with isotropy group G_0 and fix an orientation of the restriction of E over B_0. This is a G_0-bundle, and G_0 acts by orientation preserving maps by (i). By translation by group elements, we obtain orientations on the rest of the path components of B such that the action by G is orientation preserving, and this is the only way that such orientations can arise. Part (iii) follows, since B is the disjoint union of G-spaces with path connected orbit spaces. $\square$

Remark 11.5. In view of (i), the action of $\mathbb{Z}/p \subset S^1$ on S^2 in Example 11.3 shows that a nonequivariantly oriented V-dimensional G-bundle with an orientation preserving action by G need not be naively G-oriented in the sense of Definition 11.2. However, if G is finite of odd order and B is G-connected, then any nonequivariantly oriented V-dimensional G-bundle $p : E \longrightarrow B$ does have a naive G-orientation, as is easily deduced from Lemma 12.1 below. Example 11.3 shows that the G-connectivity of B is essential to the conclusion.

12. Complex Bundles and Odd-Order Groups

There is of course an analog of our theory of orientations in which we restrict attention to complex representations and bundles throughout. However, the resulting theory is quite trivial. Define $\mathscr{U}_G(n)$ in the same way as $\mathscr{V}_G(n)$, using complex n-dimensional bundles over G-orbits. Then $\mathscr{U}_G(n)$ is discrete over $\mathscr{O}_G$ since there is at most one homotopy class of complex G-bundle maps covering a given map in $\mathscr{O}_G$. This implies that the identity functor of $\mathscr{U}_G(n)$ is itself the universal orientable complex representation. In fact, this functor is obviously replete, and thus saturated, and any representation $R: \mathscr{E} \longrightarrow \mathscr{U}_G(n)$ specifies a strict map $(\mathscr{E}, R) \longrightarrow (\mathscr{U}_G(n), \mathrm{Id})$. Moreover, it is also clear that there are no nontrivial automorphisms of $(\mathscr{U}_G(n), \mathrm{Id})$. Therefore, every complex representation has a unique complex orientation. Notice that we need not assume that the underlying groupoid $\mathscr{E}$ is faithful. We conclude that complex G-bundles over arbitrary G-spaces admit unique complex orientations.

We have an evident "realification" functor $r: \overline{\mathscr{U}}_G(n) \longrightarrow \overline{\mathscr{V}}_G(2n)$ between the categories of all complex and all real G-vector bundles over orbits. Using the chosen retraction equivalences from these categories to $\mathscr{U}_G(n)$ and $\mathscr{V}_G(2n)$, we obtain a realification representation $r: \mathscr{U}_G(n) \longrightarrow \mathscr{V}_G(2n)$. Nonequivariantly, this just corresponds to choosing an identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We consider an n-dimensional complex representation $(\mathscr{E}, R)$ as a $2n$-dimensional real representation by composing it with r, and we orient an n-dimensional complex bundle as a $2n$-dimensional real bundle by composing its complex orientation with a fixed choice of a real orientation $(\mathscr{U}_G(n), r) \longrightarrow (\mathscr{S}\mathscr{V}_G(2n), S)$. There exists such an orientation since r is an orientable representation in $\mathscr{V}_G(2n)$.

We turn now to another simple case: orientations of real representations when G is a finite group of odd order. We work with a fixed given odd order finite group G in the rest of the section. We can generalize the description of the universal orientable representation for $\mathbb{Z}/p$, p odd, given in Example 10.4 to obtain a description in the general case. The essential point is that if V is a representation of G with $V^G = 0$, then V admits a complex structure, and so its group of orthogonal G-linear isometries is connected. This observation can be codified as follows.

Lemma 12.1. *Let $f : V \longrightarrow V$ be an orthogonal G-map. Then the following three statements are equivalent.*

(i) f is linearly homotopic to the identity.

(ii) f is G-linearly homotopic to the identity.

(iii) $f^G : V^G \longrightarrow V^G$ is linearly homotopic to the identity.

Proof. Let V_G be the orthogonal complement of V^G. Then $f = f^G \oplus f_G$, and $f_G : V_G \longrightarrow V_G$ is G-linearly homotopic to the identity. Since f^G is linearly homotopic to the identity if and only if it is G-linearly homotopic to the identity, the conclusion follows. □

Therefore, for objects $G \times_H V$ and $G \times_K W$ of $\mathscr{V}_G(n)$, there are at most two morphisms $G \times_H V \longrightarrow G \times_K W$ in $\mathscr{V}_G(n)$ over a given map $G/H \longrightarrow G/K$ in $\mathscr{O}_G$. If there is one such morphism, then whether there are one or two morphisms is entirely determined by whether or not V^H is empty. Again, it turns out that $(\mathscr{S}\mathscr{V}_G(n), S)$, and thus every saturated representation, is replete.

Construction 12.2. Let G be a finite group of odd order. We can describe the universal orientable representation $S\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{V}_G(n)$ as follows. The category $\mathscr{S}\mathscr{V}_G(n)$ has one object $v = v(H)$ for each object $G \times_H V$ in $\mathscr{V}_G(n)$ such that $V^H \neq 0$; it has two objects $v_+ = v_+(H)$ and $v_- = v_-(H)$ for each $G \times_H V$ such that $V^H = 0$. Of course, $\pi\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{O}_G$ and $S\colon \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{V}_G(n)$ send these objects to G/H and $G \times_H V$. For each object $G \times_H V$ of $\mathscr{V}_G(n)$, choose a map $\mu\colon G \times \mathbb{R}^n \longrightarrow G \times_H V$ over the quotient map $G \longrightarrow G/H$. We think of μ as specifying an orientation of V. Suppose that there exists a G-bundle map $G \times_H V \longrightarrow G \times_K W$ over a G-map $\alpha\colon G/H \longrightarrow G/K$, where $\alpha(eH) = gK$. Write $\tilde{\alpha}_+$ for a map $G \times_H V \longrightarrow G \times_K W$ covering α and satisfying $\tilde{\alpha}_+ \circ \mu = \mu \circ g$, if such exists, where $g\colon G \times \mathbb{R}^n \longrightarrow G \times \mathbb{R}^n$ is given by right multiplication by g on the G coordinate. Write $\tilde{\alpha}_-$ for a map covering α and satisfying $\tilde{\alpha}_- \circ \mu = \mu \circ \gamma \circ g$, if such exists, where, as usual, γ is the orientation reversing map on $G \times \mathbb{R}^n$. We think of $\tilde{\alpha}_+$ as preserving orientation, and $\tilde{\alpha}_-$ as reversing it. If $V^H \neq 0$, then both maps occur. If $V^H = 0$, then only one of them occurs. If $W^K \neq 0$ and therefore $V^H \neq 0$, then $\mathscr{S}\mathscr{V}_G(n)$ has a map $m_+\colon v \longrightarrow w$. If $W^K = 0$ and $V^H \neq 0$, then $\mathscr{S}\mathscr{V}_G(n)$ has maps $m_+\colon v \longrightarrow w_+$ and $m_-\colon v \longrightarrow w_-$. Finally, if $V^H = 0$, then if $\tilde{\alpha}_+$ exists, there are maps $m_+\colon v_+ \longrightarrow w_+$ and $m_+\colon v_- \longrightarrow w_-$, while if it is $\tilde{\alpha}_-$ that exists, there are maps $m_-\colon v_+ \longrightarrow w_-$ and $m_-\colon v_- \longrightarrow w_+$. Whenever composites are defined, they satisfy the generic rules

$$m_+ \circ m_+ = m_+,\ m_+ \circ m_- = m_-,\ m_- \circ m_+ = m_-, \text{ and } m_- \circ m_- = m_+.$$

The functor π sends any of these maps to the underlying map α of G-orbits that is used to define it, while the functor S sends m_+ to $\tilde{\alpha}_+$ and m_- to $\tilde{\alpha}_-$. It is easy to check that $\mathscr{S}\mathscr{V}_G(n)$ is a well-defined $\mathscr{O}_G$-groupoid and S is a well-defined functor over $\mathscr{O}_G$. Intuitively, the last is just the observation that the composite of two orientation preserving maps preserves orientation, the composite of two orientation reversing maps preserves orientation, and the composite of an orientation preserving and an orientation reversing map reverses orientation.

The structure of $(\mathscr{S}\mathscr{V}_G(n), S)$ perhaps becomes clearer if we notice that the group of automorphisms of any object maps isomorphically to the group of automorphisms of G/H, and that S maps the automorphism over any $\alpha\colon G/H \longrightarrow G/H$ to $\tilde{\alpha}_+$. One only encounters orientation reversing maps between representations that do not contain trivial summands when a change of subgroups is involved. However, there is no getting around such maps by relabeling, as the following example shows.

Example 12.3. Let $G = H \times K$, where H and K are cyclic of order 3 with generators s and t. Let V be the two-dimensional representation of G on which both s and t act by rotation by $2\pi/3$; let W be the two-dimensional representation of G on which s acts by rotation by $2\pi/3$ and t acts by rotation by $-2\pi/3$. Then V

and W restrict to the same representation Y of H and to the same representation Z of K. The category $\mathscr{S}\mathscr{V}_G(2)$ has corresponding objects $v_\pm$, $w_\pm$, $y_\pm$, and $z_\pm$. We can choose orientations so that there are maps $y_+ \longrightarrow v_+$, $y_+ \longrightarrow w_+$, and $z_+ \longrightarrow v_+$ in $\mathscr{S}\mathscr{V}_G(2)$, but then we are forced to have a map $z_+ \longrightarrow w_-$. Moreover, no relabeling will result in only orientation preserving maps in $\mathscr{S}\mathscr{V}_G(2)$.

The following theorem validates Construction 12.2.

Theorem 12.4. *If G has odd order, then the representation $(\mathscr{S}\mathscr{V}_G(n), S)$ is the universal orientable representation.*

Proof. It is clear from the construction that $\mathscr{S}\mathscr{V}_G(n)$ is replete and thus saturated. By Theorem 9.6, it suffices to show that any faithful representation $(\mathscr{E}, R)$ maps into $(\mathscr{S}\mathscr{V}_G(n), S)$. Since $(\mathscr{E}, R)$ factors through a skeleton, we may assume without loss of generality that $\mathscr{E}$ is skeletal and thus discrete.

We must construct a functor $F: \mathscr{E} \longrightarrow \mathscr{S}\mathscr{V}_G(n)$ over $\mathscr{O}_G$ and an isomorphism $\phi: S \circ F \longrightarrow R$ over $\mathscr{O}_G$. Let $\pi^{-1}(G/e)$ be the full subcategory of $\mathscr{E}$ containing all objects x with $\pi(x) = G/e$. For any object $x \in \pi^{-1}(G/e)$, we must take $F(x) = r$. Choose an object x_0 in each component of $\pi^{-1}(G/e)$, and let $\phi(x_0)$ be the identity map of $G \times \mathbb{R}^n$. This initial choice will determine F and ϕ. By source lifting and discreteness, there is a unique map $\omega_{g,x}: x \longrightarrow x_0$ over $g: G \longrightarrow G$ for each $g \in G$; divisibility implies that every x in the component of x_0 is the domain of such a map $\omega_{g,x}$ and that every map in the component is a composite of such maps and their inverses. We must take $F(\omega_{g,x})$ to be the unique map $r \longrightarrow r$ over g; we let $\phi(x) = \mathrm{id}$ if $R(\omega_{g,x}) = g$ and $\phi(x) = \gamma$ if $R(\omega_{g,x}) = g \circ \gamma$. This is the unique choice such that $\phi(x_0) \circ S(F(\omega_{g,x})) = R(\omega_{g,x}) \circ \phi(x)$. This specifies the restrictions of F and ϕ to $\pi^{-1}(G/e)$.

Now let y be an object of $\mathscr{E}$ such that $R(y) = G \times_H V$ with $H \neq e$. By source lifting and discreteness, there is a unique map $\omega: x \longrightarrow y$ over the quotient map $G \longrightarrow G/H$. In the naturality relation $\phi(y) \circ S(F(\omega)) = R(\omega) \circ \phi(x)$, $\phi(x)$ and $R(\omega)$ are already specified, and $F(\omega)$ will be determined once we specify $F(y)$. If $V^H \neq 0$, then we must let $F(y) = v$. Here, if $\phi(x) = \mathrm{id}$ and $R(\omega) = \mu$, or if $\phi(x) = \gamma$ and $R(\omega) = \mu \circ \gamma$, we must take $\phi(y) = \mathrm{id}$; in the remaining two cases we must take $\phi(y) = \gamma$, the unique non-identity map of $G \times_H V$. On the other hand, if $V^H = 0$, then $\phi(y)$ must be the identity map of $G \times_H V$. Here, if $\phi(x) = \mathrm{id}$ and $R(\omega) = \mu$, or if $\phi(x) = \gamma$ and $R(\omega) = \mu \circ \gamma$, we must take $F(y) = v_+$; in the remaining two cases we must take $F(y) = v_-$.

Finally, suppose given a map $\xi: y \longrightarrow z$ in $\mathscr{E}$ over $\alpha: G/H \longrightarrow G/K$, where $\alpha(eH) = gK$. Let $\omega: x \longrightarrow y$ and $\omega': x' \longrightarrow z$ be the unique maps over the quotient maps $G \longrightarrow G/H$ and $G \longrightarrow G/K$. By divisibility and discreteness, there is a unique map $\zeta: x \longrightarrow x'$ over $g: G \longrightarrow G$ such that $\omega'\zeta = \xi\omega$. A diagram chase (comparing preservation and reversal of orientations) shows that in all cases there is a unique map $F(y) \longrightarrow F(z)$ over α in $\mathscr{S}\mathscr{V}_G(n)$; we can and must take $F(\xi)$ to be this map. It is then clear that F is a well-defined functor. The naturality relations for ζ, ω, and ω' imply that the naturality relation $\phi(z) \circ S(F(\xi)) = R(\xi) \circ \phi(y)$ holds when precomposed with $S(F(\omega))$ and therefore holds as written. $\square$

In the proof just given, we could instead have chosen $\phi(x_0) = \gamma$. This would force changes from $\phi(y) = \mathrm{id}$ to $\phi(y) = \gamma$ and vice-versa when $R(y) = G \times_H V$ with $V^H \neq 0$, and from $F(y) = v_+$ to $F(y) = v_-$ and vice-versa when $V^H = 0$. Moreover, we have an independent such choice for each component of $\pi^{-1}(G/e)$. Said another way, making a choice for each component fixes a saturation of $(\mathscr{E}, R)$, and making the opposite choice in each component gives an isomorphic saturation. Proposition 9.8 gives the following interpretation.

Corollary 12.5. *Let G have odd order and let $(\mathscr{E}, R)$ be a faithful representation in $\mathscr{V}_G(n)$ such that the subcategory $\pi^{-1}(G/e)$ of $\mathscr{E}$ is connected.*

(i) The group $\Omega(\mathscr{S}\mathscr{V}_G(n), S)$ is cyclic of order two.

(ii) $(\mathscr{E}, R)$ has a unique isomorphism class of saturations.

(iii) $(\mathscr{E}, R)$ has exactly two orientations, and an orientation (F, ϕ) is determined by $\phi(x_0)$ for any chosen object x_0 over G/e.

This has the following implication for orientations of G-vector bundles, which should be compared with Example 10.3.

Corollary 12.6. *If G has odd order and B/G is path connected, then an orientable G-vector bundle p over B admits exactly two orientations, and an orientation of p is completely determined by a choice of nonequivariant orientation of the restriction of p to any path component of B.*

Thus our equivariant orientations encode information implicit in nonequivariant orientations. The following complementary observation generalizes and makes precise the folklore result (see for example [**11**, p. 16]) that, for actions of odd order groups, nonequivariant orientability implies equivariant orientability.

Theorem 12.7. *Let G be finite of odd order. A G-vector bundle $p : E \longrightarrow B$ is equivariantly orientable if and only if it is nonequivariantly orientable. The equivariant orientations of p are in bijective correspondence with the nonequivariant orientations on which G acts by orientation preserving maps.*

Proof. Clearly equivariant orientability implies nonequivariant orientability. Suppose that p is nonequivariantly orientable. Let $(\xi, \mathrm{id}) : x \longrightarrow x$ be a map in $\Pi_G B$ and thus in $\Pi(B^H)$. Regarding ξ as a path in B and working nonequivariantly, we have $p^*(\xi) = \mathrm{id}$ by the nonequivariant orientability of p. By Lemmas 2.3 and 12.1, we conclude that $p^*(\xi, \mathrm{id}) = \mathrm{id}$ equivariantly. By Remark 2.9, this implies that p is equivariantly orientable. The second statement follows from the first together with Lemma 11.4, which shows that p always admits nonequivariant orientations on which G acts by orientation preserving maps, and Corollary 12.6. $\square$

In Definition 11.2 we defined naive orientations of V-dimensional G-bundles for any G. Using our explicit description of $\mathscr{S}\mathscr{V}_G$, we can define naive orientations of arbitrary G-bundles when $|G|$ is odd. The notions coincide when both are defined.

Definition 12.8. Let $\mathscr{I}\mathscr{V}_G$ be the subcategory of $\mathscr{S}\mathscr{V}_G$ obtained by omitting the objects v_- and the maps to and from them. Deleting subscripts $+$ from the notation, the category $\mathscr{I}\mathscr{V}_G$ has an object $v = v(H)$ for each $G \times_H V$ and a map $m\colon v \longrightarrow w$ over $\alpha\colon G/H \longrightarrow G/K$ if there is an orientation preserving G-bundle map $G \times_H V \longrightarrow G \times_K W$ over α. Let $I\colon \mathscr{I}\mathscr{V}_G \longrightarrow \mathscr{V}_G$ be the restriction of S. A *naive orientation* (F, ϕ) of a G-bundle $p\colon E \longrightarrow B$ is a functor $F\colon \Pi_G B \longrightarrow \mathscr{I}\mathscr{V}_G$ over $\mathscr{O}_G$ and an isomorphism $\phi\colon I \circ F \longrightarrow p^*$ over $\mathscr{O}_G$.

Remarks 12.9. We compare naive orientations and orientations for $|G|$ odd.

(i) Clearly, we may view a naive orientation (F, ϕ) as a restricted kind of orientation. Therefore, when p admits a naive orientation, its orientations and naive orientations determine one another.

(i) We can interpret Example 12.3 as showing that $\mathscr{I}\mathscr{V}_G$ need not be a groupoid over $\mathscr{O}_G$ because it need not satisfy the source lifting property. Thus it is only the genuine orientations that fit naturally into our definitional framework.

(ii) The full subcategories of $\mathscr{I}\mathscr{V}_G$ and $\mathscr{S}\mathscr{V}_G$ with one object $v = v(H)$ for each $G \times_H V$ such that $V^H \neq 0$ coincide, hence naive orientations and orientations coincide for G-bundles all of whose fiber representations contain trivial summands. In particular, this holds stably.

Remark 12.10. For a general compact Lie group G, there is an essentially similar generalization of Construction 12.2 that applies when $\mathscr{V}_G$ is replaced by its full subcategory whose objects are those $G \times_H V$ such that the group of H-linear isometries of V_H is connected, where V_H is the orthogonal complement of V^H. That is, we only take the representations $\mathbb{R}^q \oplus W$ such that W contains no irreducible summands of real type, only summands of complex and quaternionic type.

13. Abelian compact Lie groups

We explore our theory for Abelian compact Lie groups G and give a conjectural description of universal orientable representations when G is an elementary Abelian 2-group $(\mathbb{Z}/2)^k$. More precisely, we first construct a replete, hence saturated, representation $(\mathscr{S}(n), S)$ for any Abelian compact Lie group G and any $n \geqslant 1$. For $G = (\mathbb{Z}/2)^k$, we then show that $(\mathscr{S}(n), S)$ is universal for replete representations. Thus it is universal if Conjecture 8.6 is true for $\mathscr{R} = \mathscr{V}_G(n)$. However, Remark 8.7 is relevant, and we doubt that the conjecture is true even in this simple case.

Since G is Abelian, The orbit category $\mathscr{O}_G$ has a very simple form. The function that sends gH to the G-map $g : G/H \longrightarrow G/H$ is an isomorphism of Lie groups

$$G/H \longrightarrow \mathscr{O}_G(G/H, G/H).$$

We regard it as an identification. We have a subcategory $\mathscr{Q}_G$ of $\mathscr{O}_G$ whose objects are the orbits G/H and whose morphisms are the quotient maps $q : G/H \longrightarrow G/K$ associated to inclusions $H \subset K$. There is a map $\alpha : G/H \longrightarrow G/K$ in $\mathscr{O}_G$ if and only if $H \subset K$, and then α is the composite of q and a map $g : G/H \longrightarrow G/H$. The relations in the category are generated by the obvious commutative diagrams

$$\begin{tikzcd} G/H \arrow[r, "g"] \arrow[d, "q"'] & G/H \arrow[d, "q"] \\ G/K \arrow[r, "g"'] & G/K. \end{tikzcd}$$

The skeletal category $\mathscr{V}_G(n)$, $n \geqslant 1$, also admits a simple description since G is Abelian, as we see by use of Lemmas 2.3 and 2.6. Consider the set of maps

$$\omega : G \times_H V \longrightarrow G \times_K W.$$

This set is non-empty when $H \subset K$ and there is an H-linear isometry $\iota : V \longrightarrow W$. Any other H-linear isometry $V \longrightarrow W$ is a composite of ι and an H-linear isometry $V \longrightarrow V$. We deduce that any map ω is the composite of a self-map of $G \times_H V$ and the map $\tilde{q} : G \times_H V \longrightarrow G \times_K W$ over $q : G/H \longrightarrow G/K$ that is obtained from ι by passage to orbits. Note that $\pi_0(O_H(V)) = \mathrm{Aut}(G \times_H V)$ and, by Proposition 5.4, a map $\omega : G \times_H V \longrightarrow G \times_K W$ induces a restriction homomorphism

$$\pi_0(O_K(W)) \longrightarrow \pi_0(O_H(V)).$$

Taking $\omega = \tilde{q}$, this map is the evident composite

$$r : \pi_0(O_K(W)) \longrightarrow \pi_0(O_H(W)) \cong \pi_0(O_H(V)).$$

It is independent of the choice of ι and thus of the choice of $\tilde{q}$ over q since inner H-linear automorphisms of V are homotopic to the identity.

Any self-map of $G \times_H V$ is a composite of a map over the identity map of G/H induced by an element of $\pi_0(O_H(V))$ and the map given by multiplication by g, which covers $g : G/H \longrightarrow G/H$. These two types of maps commute with one another. An element $h \in H$ gives an H-linear isometry $h : V \longrightarrow V$. By passage to homotopy, this gives a homomorphism $\lambda_V : H \longrightarrow \pi_0(O_H(V))$. Via Lemma 2.3, we deduce an isomorphism of groups

$$\mathscr{V}_G(G \times_H V, G \times_H V) \cong G \times_H \pi_0(O_H(V)),$$

where H acts on $\pi_0(O_H(V))$ via λ_V. Moreover, viewing G as $G \times_H H$, we see that λ_V extends to a homomorphism $\tilde{\lambda}_V : G \longrightarrow \mathscr{V}_G(G \times_H V, G \times_H V)$ that induces a map of extensions

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r] \arrow[d, "\lambda_V"'] & G \arrow[r, "p"] \arrow[d, "\tilde{\lambda}_V"] & G/H \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & \pi_0(O_H(V)) \arrow[r] & \mathscr{V}_G(G \times_H V, G \times_H V) \arrow[r, "\pi"'] & \mathscr{O}_G(G/H, G/H) \arrow[r] & 1, \end{tikzcd}$$

where $p : G \longrightarrow G/H$ is the quotient homomorphism.

Now let $\mathscr{E}$ be a discrete groupoid over $\mathscr{O}_G$ and consider a representation R of $\mathscr{E}$ in $\mathscr{V}_G(n)$. We consider general features of the functor $R : \mathscr{E} \longrightarrow \mathscr{V}_G(n)$. Let x be an object of $\mathscr{E}$ with $\pi(x) = G/H$ and $R(x) = G \times_H V$. Since $\mathscr{E}$ is faithful,

$$\pi : \mathscr{E}(x, x) \longrightarrow \mathscr{O}_G(G/H, G/H) \cong G/H$$

is an inclusion. We are especially interested in the case when π is an isomorphism, so that $\mathscr{E}(x,x) \cong G/H$. Let

$$G_x = p^{-1}\mathscr{E}(x,x) \subset G \quad \text{and} \quad V_x = \pi^{-1}\mathscr{E}(x,x) \subset \mathscr{V}_G(R(x), R(x)).$$

The map of extensions displayed above restricts to a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, "\lambda_V"'] & G_x \arrow[r, "p"] \arrow[d, "\tilde{\lambda}_x"] & \mathscr{E}(x,x) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_0(O_H(V)) \arrow[r] & V_x \arrow[r, "\pi"'] & \mathscr{E}(x,x) \arrow[r] & 1.
\end{tikzcd}$$

Since $\pi \circ R = \pi$, the bottom extension is split by $R : \mathscr{E}(x,x) \longrightarrow V_x$. Define

$$\rho_x : V_x \longrightarrow \pi_0(O_H(V))$$

by $\rho_x(\omega) = \omega \cdot R(\pi(\omega))^{-1}$. Then the homomorphism

$$\tau_x = \rho_x \circ \tilde{\lambda}_x : G_x \longrightarrow \pi_0(O_H(V))$$

extends λ_V. We can turn this observation around. Clearly

$$V_x \cong G_x \times_H \pi_0(O_H(V)),$$

and it follows that an extension $\tau_x : G_x \longrightarrow \pi_0(O_H(V))$ of λ_V determines a splitting $R : \mathscr{E}(x,x) \longrightarrow V_x$. Explicitly, R is given by $R(q(g)) = (g, \tau_x(g)^{-1})$ for $g \in G_x$. Looked at another way, R gives the inclusion of the first coordinate of a splitting

$$V_x \cong \mathscr{E}(x,x) \times \pi_0(O_H(V))$$

under which $\tilde{\lambda}_x$ has coordinates p and τ_x.

We now consider maps between distinct objects of $\mathscr{E}$. Taking x as above, let $\omega : x \longrightarrow y$ be a map such that $\pi(\omega) = \alpha : G/H \longrightarrow G/K$ and $R(y) = G \times_K W$. Then $H \subset K$ and there is an H-linear isomorphism $\iota : V \longrightarrow W$. By divisibility, for any map $\nu : y \longrightarrow y$, there is a map $\mu : x \longrightarrow x$ such that $\nu \circ \omega = \omega \circ \mu$ covering each map $\xi : G/H \longrightarrow G/H$ such that $\alpha \circ \xi = \pi(\nu) \circ \alpha$. In particular, when $H = K$ and thus $V = W$, we see that the existence of a map $x \longrightarrow y$ implies that $G_x = G_y$ and $\tau_x = \tau_y$. When $H \neq K$, unique source lifting (see Remark 6.3(i)) and divisibility imply that each factorization $\alpha = q \circ \xi$, $\xi : G/H \longrightarrow G/H$, is uniquely covered by a factorization $\omega = \tilde{q} \circ \tilde{\xi}$. That is, ω is a composite of a map $\tilde{\xi} : x \longrightarrow x'$ over ξ and the unique map $\tilde{q}_y : x' \longrightarrow y$ with target y over $q : G/H \longrightarrow G/K$. When $\mathscr{E}(x,x) \cong G/H$, Lemma 8.2(ii) implies that $x' = x$, so that every morphism $x \longrightarrow y$ is a composite of a self-map of x and a canonical map $\tilde{q}_y : x \longrightarrow y$. In particular, $(\mathscr{E}, R)$ is replete in the sense of Definition 8.4 when $\mathscr{E}(x,x) \cong G/H$ for all H and all x over G/H. In general, since for any map $\nu : y \longrightarrow y$ there is a map $\mu : x \longrightarrow x$ such that $\nu \circ \tilde{q} = \tilde{q} \circ \mu$, we see that $G_y \subset G_x$. By a diagram chase from the functoriality of R, we find that the following diagram commutes:

$$\begin{tikzcd}
G_y \arrow[r] \arrow[d, "\tau_y"'] & G_x \arrow[d, "\tau_x"] \\
\pi_0(O_K(W)) \arrow[r, "r"'] & \pi_0(O_H(V)).
\end{tikzcd}$$

Thus we can think of τ_x as an extension of $r \circ \tau_y$. When $(\mathscr{E}, R)$ is replete, we conclude that the functor R is determined by maps $R(\tilde{q})$ that give a functor over $\mathscr{Q}_G \subset \mathscr{O}_G$ together with homomorphisms τ_x that extend the λ_V and make these diagrams commute. The point is that the commutativity of these diagrams ensures that relations $\nu \circ \tilde{q} = \tilde{q} \circ \mu$ as above are carried to relations $R(\nu) \circ R(\tilde{q}) = R(\tilde{q}) \circ R(\mu)$. This gives an inductive description of replete representations.

We can describe maps $(F, \phi) : (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ between replete representations in terms of these data. Since $\mathscr{V}_G(n)$ is skeletal, for there to be an isomorphism $\phi : S \circ F \longrightarrow R$ over $\mathscr{O}_G$, we must specify F on objects so that $S(F(x)) = R(x)$ for all $x \in \mathscr{E}$. Since $(\mathscr{F}, S)$ is replete, we must then specify F on non-empty morphism sets to be the composite

$$\mathscr{E}(x, y) \xrightarrow{\pi} \mathscr{O}_G(\pi(x), \pi(y)) \xrightarrow{\pi^{-1}} \mathscr{F}(F(x), F(y)).$$

To have $S \circ F = R$ on sets of endomorphisms, we must have $\tau_x = \tau_{F(x)}$ for all objects x of $\mathscr{E}$. Finally, we must choose isomorphisms $\phi : S(F(x)) = R(x) \longrightarrow R(x)$ that make the evident naturality diagrams over $\mathscr{Q}_G$ commute:

$$\begin{tikzcd} S(F(x)) \arrow[r, "\phi"] \arrow[d, "S(F(\tilde{q}))"'] & R(x) \arrow[d, "R(\tilde{q})"] \\ S(F(y)) \arrow[r, "\phi"'] & R(y). \end{tikzcd} \tag{13.1}$$

Here $R(\tilde{q}) = S(F(\tilde{q})) \circ \psi_q$ for some $\psi_q : R(x) \longrightarrow R(x)$, and the diagram can be rewritten by replacing the top arrow with $\psi_q \circ \phi$ and the right arrow with $S(F(\tilde{q}))$. This relates the construction of the maps ϕ to the behavior of restriction maps $r : \pi_0(O_K(W)) \longrightarrow \pi_0(O_H(W))$. The following observation is relevant.

Proposition 13.2. *Let K be a subgroup of G that is not topologically cyclic and let W be a K-space.*

(i) Restriction maps induce an isomorphism $\pi_0(O_K(W)) \longrightarrow \lim_H \pi_0(O_H(W))$, where the limit is taken over the proper subgroups $H \subset K$.

(ii) Assume given a homomorphism $\tau_H : G \longrightarrow \pi_0(O_H(W))$ for each proper subgroup H of K such that τ_H extends $\lambda_V : H \longrightarrow \pi_0(O_H(V))$ and τ_H restricts to τ_J, $\tau_J = r^H_J \circ \tau_H$, when $J \subset H$. Then there is a unique homomorphism $\tau_K : G \longrightarrow \pi_0(O_K(W))$ that extends λ_K and restricts to τ_H for each $H \subset K$.

Proof. Let $SO_H(W)$ be the identity component of $O_H(W)$. Since the closure of the subgroup generated by any element of K is a proper subgroup, we have

$$O_K(W) = \cap_H O_H(W) = \lim_H O_H(W) \text{ and } SO_K(W) = \cap_H SO_H(W) = \lim_H SO_H(W).$$

Here $\pi_0(O_K(W)) \cong \lim O_H(W)/SO_K(W)$ since $\pi_0(O_K(W)) = O_K(W)/SO_K(W)$. Since $\lim^1$ vanishes on countable systems of compact groups, the countably many short exact sequences

$$1 \longrightarrow SO_H(W)/SO_K(W) \longrightarrow O_H(W)/SO_K(W) \longrightarrow \pi_0(O_H(W)) \longrightarrow 1$$

give an exact sequence on passage to limits. This gives (i), and (ii) follows by letting τ_K be the homomorphism obtained from the τ_H by passage to limits. □

Of course, we can replace the $O_H(W)$ in this result by the spaces of H-linear isometries $V_H \longrightarrow W$ for any choices of V_H that are H-isomorphic to W.

We now construct the promised replete representation $(\mathscr{S}(n), S)$.

Construction 13.3. Let $\mathscr{S}(n)$ have objects all pairs $x = (G \times_H V, \tau_x)$, where H is a (closed) subgroup of G, V is an H-space such that $G \times_H V$ is an object of $\mathscr{V}_G(n)$, and $\tau_x : G \longrightarrow \pi_0(O_H(V))$ is a homomorphism that extends $\lambda_V : H \longrightarrow \pi_0(O_H(V))$. The functors $\pi : \mathscr{S}(n) \longrightarrow \mathscr{O}_G$ and $S : \mathscr{S}(n) \longrightarrow \mathscr{V}_G(n)$ send x to G/H and to $G \times_H V$. Let $y = (G \times_K W, \tau_y)$. There are no morphisms $x \longrightarrow y$ unless $H \subset K$, V is H-isomorphic to W (without a specified choice of isomorphism), and τ_y restricts to τ_x. When these conditions hold, we require

$$\pi : \mathscr{S}(n)(x, y) \longrightarrow \mathscr{O}_G(G/H, G/K)$$

to be a bijection. Composition is dictated by the functoriality of π. In particular,

$$\pi : \mathscr{S}(n)(x, x) \longrightarrow \mathscr{O}_G(G/H, G/H) \cong G/H$$

is an isomorphism, $G_x = G$, and the discussion above shows that the homomorphism τ_x determines a homomorphism

$$S : \mathscr{S}(x, x) \longrightarrow \mathscr{V}_G(n)(S(x), S(x))$$

such that $\pi \circ S = \pi$. These homomorphisms specify S on self-maps in $\mathscr{S}(n)$. For each object $y = (G \times_K W, \tau_y)$ and each proper subgroup H of K we have a unique H-space V_H such that $G \times_H V_H$ is in $\mathscr{V}_G(n)$ and V_H is H-isomorphic to W. The retraction equivalence from $\overline{\mathscr{V}}_G(n)$ to $\mathscr{V}_G(n)$ chosen in Definition 2.2 fixes a choice of isomorphism $\iota_H : V_H \longrightarrow W$. Let $y|_H$ denote the object $(G \times_H V_H, r^K_H \circ \tau_y)$. We define S on the unique morphism $\tilde{q} : y|_H \longrightarrow y$ that covers the quotient map $q : G/H \longrightarrow G/K$ to be the map $S(y|_H) \longrightarrow S(y)$ determined by ι_H and passage to orbits. Clearly this specification of maps is functorial over $\mathscr{Q}_G$. By the discussion above, this completes the specification of the functor S. It is immediate that $(\mathscr{S}(n), S)$ is replete, hence saturated.

Theorem 13.4. *Let $G = (\mathbb{Z}/2)^k$. Then every replete representation maps into $(\mathscr{S}(n), S)$. Therefore $(\mathscr{S}(n), S)$ is a universal orientable representation in $\mathscr{V}_G(n)$ if and only if Conjecture 8.6 holds for $\mathscr{R} = \mathscr{V}_G(n)$.*

Proof. The second statement follows from the first by Theorem 9.6(vi) and Proposition 9.9. To prove the first, let $(\mathscr{E}, R)$ be replete. We must construct a map $(F, \phi) : (\mathscr{E}, R) \longrightarrow (\mathscr{S}(n), S)$. By the discussion above, the construction is largely tautological. For an object x of $\mathscr{E}$ with $\pi(x) = G/H$ and $R(x) = G \times_H V$, we must define $F(x) = (R(x), \tau_x)$ on objects x, and it remains only to specify isomorphisms $\phi : S(F(x)) = R(x) \longrightarrow R(x)$ that make the diagrams (13.1) commute. We proceed by induction on the rank of H. If $H = \{e\}$, then $R(x) = G \times \mathbb{R}^n$ and we choose ϕ arbitrarily. Suppose next that H is of rank one. There is a unique map $\tilde{q} : x' \longrightarrow x$ over $q : G \longrightarrow G/H$. Since $r : \pi_0(O_H(V)) \longrightarrow \pi_0(O(n))$ is an epimorphism (see

Example 10.2(ii)), we can choose $\phi : G \times_H V \longrightarrow G \times_H V$ making the relevant diagram (13.1) commute. Now assume that H is of rank s and ϕ has been specified for all objects over all orbits G/J with rank $J < s$. We deduce from Proposition 13.2(i) that there is a unique way to specify $\phi : S(F(x)) \longrightarrow R(x)$ such that all of the relevant diagrams (13.1) commute. □

14. The universal orientable representation $\mathscr{S}\mathscr{V}_{D_6}(2)$

We give one non-Abelian even order example to illustrate ideas. Let $G = D_6$ be the dihedral group of order 6. It contains a normal subgroup K of order 3 and three conjugate subgroups H_1, H_2, H_3 of order 2. Let $J = G/K$. Schematically, with doubleheaded arrows indicating conjugations, the orbit category $\mathscr{O}_G$ looks like

$$
\begin{tikzcd}
 & G/e \arrow[loop above] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
G/K \arrow[loop above] \arrow[dr] & & G/H_1 \arrow[dl] \arrow[r, leftrightarrow] & G/H_2 \arrow[dll] \arrow[r, leftrightarrow] & G/H_3. \arrow[dlll] \\
 & G/G & & &
\end{tikzcd}
$$

The group G has three irreducible real representations, namely the trivial representation $\mathbb{R}$, the representation $\mathbb{L}$ obtained by pullback of the non-trivial irreducible representation of J, and a representation $\mathbb{V}$ whose restriction to K is isomorphic to the standard representation of K on $\mathbb{C}$. The restriction of $\mathbb{V}$ to each H_i is the sum of a trivial and a non-trivial representation.

We describe $\mathscr{S}\mathscr{V}_G(2)$, which has features of both cases $G = \mathbb{Z}/2$ and $G = \mathbb{Z}/p$ from §10. Recall from Definition 11.1 that an n-dimensional representation V of G determines the full subgroupoid $\mathscr{V}_G(V)$ of $\mathscr{V}_G(n)$ with objects isomorphic to the $G \times_H V$. Here $G \times_H V$ is isomorphic to $G/H \times V$ since G acts on V. The reader will appreciate that examples where subgroups have representations that are not restrictions of representations of G will be substantially more complicated.

We have the four two dimensional representations $\mathbb{R}^2$, $\mathbb{R} \oplus \mathbb{L}$, $\mathbb{L}^2$, and $\mathbb{V}$, the first three of which are trivial as representations of K. The elements of order 2 act in an orientation preserving way on $\mathbb{R}^2$ and $\mathbb{L}^2$ and in an orientation reversing way on $\mathbb{R} \oplus \mathbb{L}$ and $\mathbb{V}$. The category $\mathscr{V}_G(2)$ has two components. For the first three representations $\mathbb{U}$, there is a copy of the orbit category in $\mathscr{S}\mathscr{V}_G(2)$ that maps to the subcategory of $\mathscr{V}_G(2)$ that is depicted schematically as follows, where downward pointing arrows restrict to the identity map on representations.

$$\begin{tikzcd}
& G\times\mathbb{R}^2 \arrow[loop left] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
G/K\times\mathbb{R}^2 \arrow[loop left] \arrow[dr] & & G/H_1\times\mathbb{U} \arrow[r, leftrightarrow] \arrow[dl] & G/H_2\times\mathbb{U} \arrow[r, leftrightarrow] \arrow[dll] & G/H_3\times\mathbb{U}. \arrow[dlll] \\
& G/G\times\mathbb{U} & & &
\end{tikzcd}$$

One component is generated by the two subcategories with $\mathbb{U}=\mathbb{R}^2$ and $\mathbb{U}=\mathbb{L}^2$, with the parts of the two categories with objects labelled by $\mathbb{R}^2$ rather than by $\mathbb{U}$ identified; in this component, all maps by elements of order 2 are orientation preserving; that is there is no twisting by maps γ of degree -1 on representations. The other component is generated by the subcategory with $\mathbb{U}=\mathbb{R}\oplus\mathbb{L}$ and another subcategory that maps to the subcategory of $\mathscr{V}_G(2)$ generated by $\mathbb{V}$ in a manner that we depict schematically as follows:

$$\begin{tikzcd}
& G\times\mathbb{R}^2 \arrow[loop left] \arrow[dl, "\mu"'] \arrow[d, "\mu\gamma"] \arrow[dr, "\mu"] \arrow[drr, "\mu"] \arrow[drrr, "\mu"] & & & \\
G/K\times\mathbb{V} \arrow[loop left] \arrow[dr, "\mu"'] & G/K\times\mathbb{V} \arrow[loop left] \arrow[dr, "\mu"'] & G/H_1\times\mathbb{V} \arrow[r, leftrightarrow] \arrow[dl, "\mu"'] \arrow[d, "\mu\gamma"] & G/H_2\times\mathbb{V} \arrow[r, leftrightarrow] \arrow[dll, "\mu"'] \arrow[dl, "\mu\gamma"] & G/H_3\times\mathbb{V}. \arrow[dlll, "\mu"'] \arrow[dll, "\mu\gamma"] \\
& G/G\times\mathbb{V} & G/G\times\mathbb{V} & &
\end{tikzcd}$$

The copies of $G\times\mathbb{R}^2$ in these two subcategories are identified, and all elements of order 2 act in an orientation reversing way on the resulting object and also on $G/K\times\mathbb{R}^2$ in the subcategory for $\mathbb{U}=\mathbb{R}\oplus\mathbb{L}$ and on the two copies of $G/K\times\mathbb{V}$ in the last subcategory (compare Example 10.2(d)). Vertical arrows involve twistings γ as indicated by the labels of arrows. It is clear that this does specify a well-defined groupoid $\mathscr{S}\mathscr{V}_G(2)$ over $\mathscr{O}_G$ together with a representation $S:\mathscr{S}\mathscr{V}_G(2)\longrightarrow\mathscr{V}_G(2)$. This representation is replete, hence saturated, and it is not hard to check as in §10 that it is universal.

Part IV. Refinements and variants of the theory

15. Fibrations over $\mathscr{B}$ and fibrant representations

The analogy between topological and categorical homotopy theory, in particular the categorical notion of a fibration, is illuminating to the categorical representation theory of Part II. Although we shall not make essential use of this material, we here show how to use fibrations to express orientations in terms of strict maps of representations. To begin with, we return to the framework of §5.

With $\mathscr{I}$ as in Example 9.7, an isomorphism over $\mathscr{B}$ between functors $\mathscr{E} \longrightarrow \mathscr{F}$ over $\mathscr{B}$ can be regarded as a map $\eta\colon \mathscr{E} \times \mathscr{I} \longrightarrow \mathscr{F}$. In fact, $\mathscr{B} \times \mathscr{I}$ is a base category, $\mathscr{E} \times \mathscr{I}$ is a groupoid over $\mathscr{B} \times \mathscr{I}$, and η is a functor over the projection $\mathscr{B} \times \mathscr{I} \longrightarrow \mathscr{B}$. Here we must take $\mathscr{B} \times \mathscr{I}$ and not $\mathscr{B}$ as the base of $\mathscr{E} \times \mathscr{I}$ since otherwise the source lifting and divisibility properties would fail. Using this, we can mimic topological definitions and constructions.

Definition 15.1. Let $\mathscr{F}$ and $\mathscr{R}$ be groupoids over $\mathscr{B}$. A functor $S\colon \mathscr{F} \longrightarrow \mathscr{R}$ over $\mathscr{B}$ is a *fibration over* $\mathscr{B}$ if it satisfies the *categorical CHP*: for maps $R\colon \mathscr{E} \longrightarrow \mathscr{R}$ and $F\colon \mathscr{E} \longrightarrow \mathscr{F}$ of groupoids over $\mathscr{B}$ and an isomorphism $\phi\colon S \circ F \longrightarrow R$ over $\mathscr{B}$, there is a functor $F'\colon \mathscr{E} \longrightarrow \mathscr{F}$ over $\mathscr{B}$ such that $R = S \circ F'$ and an isomorphism $\tilde{\phi}\colon F \longrightarrow F'$ over $\mathscr{B}$ such that $\phi = S \circ \tilde{\phi}$. In terms of diagrams of functors, this takes the following familiar form, in which $R = \phi \circ i_1$ and $F' = \tilde{\phi} \circ i_1$:

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "i_0"'] & \mathscr{F} \arrow[d, "S"] \\
\mathscr{E} \times \mathscr{I} \arrow[r, "\phi"'] \arrow[ur, dashed, "\tilde{\phi}"] & \mathscr{R}.
\end{tikzcd}$$

Since *objects* in the category of representations in $\mathscr{R}$ are *maps* into $\mathscr{R}$ in the category of groupoids over $\mathscr{B}$, we think of fibrations as fibrant representations.

Definition 15.2. A representation $(\mathscr{F}, S)$ in $\mathscr{R}$ is *fibrant* if $S\colon \mathscr{F} \longrightarrow \mathscr{R}$ is a fibration over $\mathscr{B}$. This means that, for any map $(F, \phi)\colon (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ of representations, there is a strict map $F'\colon (\mathscr{E}, R) \longrightarrow (\mathscr{F}, S)$ and a homotopy $\tilde{\phi}\colon F \longrightarrow F'$.

As in topology, a functor over $\mathscr{B}$ is a fibration if and only if it satisfies the path lifting property (PLP). Let $\mathscr{R}^{\mathscr{I}}$ be the functor category of maps $\mathscr{I} \longrightarrow \mathscr{R}$ over identity maps of $\mathscr{B}$. Its objects are maps $\xi = \xi(\mathbf{I})\colon \xi(\mathbf{0}) \longrightarrow \xi(\mathbf{1})$ in $\mathscr{R}$, and ξ must be an isomorphism since $\pi(\xi) = \mathrm{id}$. A map $(\rho_0, \rho_1)\colon \xi \longrightarrow \zeta$ is a pair of maps $\rho_i\colon \xi(\mathbf{i}) \longrightarrow \zeta(\mathbf{i})$ such that the evident diagram commutes. We have the pullback $\mathscr{F} \times_{\mathscr{R}} \mathscr{R}^{\mathscr{I}}$ of $S\colon \mathscr{F} \longrightarrow \mathscr{R}$ and $p_0\colon \mathscr{R}^{\mathscr{I}} \longrightarrow \mathscr{R}$. The universal property gives a functor

$$\Psi\colon \mathscr{F}^{\mathscr{I}} \longrightarrow \mathscr{F} \times_{\mathscr{R}} \mathscr{R}^{\mathscr{I}}$$

whose projections are $p_0\colon \mathscr{F}^{\mathscr{I}} \longrightarrow \mathscr{F}$ and $S^{\mathscr{I}}\colon \mathscr{F}^{\mathscr{I}} \longrightarrow \mathscr{R}^{\mathscr{I}}$.

Lemma 15.3. *A functor $S\colon \mathscr{F} \longrightarrow \mathscr{R}$ is a fibration if and only if it satisfies the* categorical PLP*: there is a functor*

$$\Lambda\colon \mathscr{F} \times_{\mathscr{R}} \mathscr{R}^{\mathscr{I}} \longrightarrow \mathscr{F}^{\mathscr{I}}$$

such that $\Psi \circ \Lambda = \mathrm{Id}$.

This criterion and the path lifting property for (Hurewicz) fibrations of G-spaces imply the following relationship between topological and categorical fibrations.

Proposition 15.4. *If* $p\colon E \longrightarrow B$ *is a fibration of* G*-spaces, then* $\Pi_G p\colon \Pi_G E \longrightarrow \Pi_G B$ *is a fibration of groupoids over* $\mathscr{B}$.

We can replace general functors over $\mathscr{B}$ with equivalent fibrations over $\mathscr{B}$.

Construction 15.5. Let $R\colon \mathscr{E} \longrightarrow \mathscr{R}$ be a functor over $\mathscr{B}$. We construct the associated fibration $R'\colon \mathscr{E}' \longrightarrow \mathscr{R}$ via the following commutative diagram, in which the right square is a pullback of categories and the lower triangle specifies R':

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "J"] \arrow[d, "R"'] & \mathscr{E}' \arrow[r, "P"] \arrow[d] \arrow[dl, "R'"'] & \mathscr{E} \arrow[d, "R"] \\
\mathscr{R} & \mathscr{R}^{\mathscr{I}} \arrow[l, "p_1"] \arrow[r, "p_0"'] & \mathscr{R}.
\end{tikzcd}$$

The objects of $\mathscr{E}'$ are the pairs (x, ξ), where x is an object of $\mathscr{E}$ and ξ is an isomorphism with $\xi(\mathbf{0}) = R(x)$ and $\pi(\xi) = \mathrm{id}_{\pi(x)}$. A map $(\omega, \rho_1)\colon (x, \xi) \longrightarrow (y, \zeta)$ consists of maps $\omega\colon x \longrightarrow y$ in $\mathscr{E}$ and ρ in $\mathscr{R}$ such that $(R(\omega), \rho_1)$ is a map $\xi \longrightarrow \zeta$; that is, $R(\omega) = \zeta^{-1}\rho_1\xi$. The functor J sends x to (x, id) and ω to (ω, id). It is straightforward to verify the following claims.

(i) $\mathscr{E}'$ is a groupoid over $\mathscr{B}$, so that $(\mathscr{E}', R')$ is a representation in $\mathscr{R}$.

(ii) The map $R'\colon \mathscr{E}' \longrightarrow \mathscr{R}$ is a fibration over $\mathscr{B}$, so that $(\mathscr{E}', R')$ is fibrant.

(iii) $R = R' \circ J$, so that $J\colon (\mathscr{E}, R) \longrightarrow (\mathscr{E}', R')$ is a strict map of representations, and $\psi\colon R \circ P \longrightarrow R'$ is a map over $\mathscr{B}$, so that $(P, \psi)\colon (\mathscr{E}', R') \longrightarrow (\mathscr{E}, R)$ is a map of representations, where

$$\psi(x, \xi) = \xi\colon (R \circ P)(x, \xi) = R(x) = \xi(\mathbf{0}) \longrightarrow \xi(\mathbf{1}) = R'(x, \xi).$$

(iii) $P \circ J = \mathrm{Id}\colon \mathscr{E} \longrightarrow \mathscr{E}$ and the maps $(\mathrm{id}, \xi)\colon (x, \mathrm{id}) \longrightarrow (x, \xi)$ specify a homotopy $J \circ P \longrightarrow \mathrm{Id}$, so that the representation $(\mathscr{E}, R)$ is a deformation retract of the fibrant representation $(\mathscr{E}', R')$.

In view of Corollary 8.9 and the categorical CHP, we have the following alternative description of orientations. Recall Definition 7.1(iii).

Corollary 15.6. *Let* $(\mathscr{S}\mathscr{R}, S)$ *be the universal orientable representation and let* $(\mathscr{S}'\mathscr{R}, S')$ *be an equivalent fibrant representation. Then* $(\mathscr{S}'\mathscr{R}, S')$ *is supersaturated (but not saturated), and orientations of a representation* $(\mathscr{E}, R)$ *are in bijective correspondence with homotopy classes of strict maps* $F : (\mathscr{E}, R) \longrightarrow (\mathscr{S}'\mathscr{R}, S')$.

16. Functoriality of universal orientable representations

We have started with a fixed skeletal groupoid $\mathscr{R}$ over $\mathscr{B}$. There are variants of our theory that deal with other types of G-bundles and with G-fibrations, and still other variants that deal with special types of G-bundles, for example V-dimensional ones for a fixed representation V of G. To make comparisons, we must understand the functoriality of our constructions with respect to changes of $\mathscr{R}$.

Proposition 16.1. *Let $\rho\colon \mathscr{R} \longrightarrow \mathscr{R}'$ be a functor over $\mathscr{B}$, where $\mathscr{R}$ and $\mathscr{R}'$ are skeletal groupoids over $\mathscr{B}$.*

(i) There is a unique map $\tilde{\rho}\colon \tilde{\mathscr{R}} \longrightarrow \tilde{\mathscr{R}}'$ of universal discrete representations covering the map ρ, $D' \circ \tilde{\rho} = \rho \circ D$.

(ii) There is a map $\sigma\colon \mathscr{S}\mathscr{R} \longrightarrow \mathscr{S}\mathscr{R}'$ of universal orientable representations covering ρ; it is unique up to isomorphism and can be chosen to cover $\tilde{\rho}$.

(iii) If $\rho'\colon \mathscr{R}' \longrightarrow \mathscr{R}''$ is another functor over $\mathscr{B}$, then we can choose a map $\sigma'\colon \mathscr{S}\mathscr{R}' \longrightarrow \mathscr{S}\mathscr{R}''$ covering ρ' so that $\sigma' \circ \sigma$ covers $\rho' \circ \rho$.

(iv) By composition with σ, an orientation (F, ϕ) of a representation $(\mathscr{E}, R)$ in $\mathscr{R}$ induces an orientation of $(\mathscr{E}, R)$ in $\mathscr{R}'$:

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[dr, "R"'] & \mathscr{S}\mathscr{R} \arrow[rr, "\sigma"] \arrow[d, "S"] & & \mathscr{S}'\mathscr{R}' \arrow[dl, "S'"] \\
& \mathscr{R} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \mathscr{R}' \arrow[d, "\pi"] & \\
& & \mathscr{B}. &
\end{tikzcd}$$

Proof. The essential point is that the notions of skeletal, discrete, and faithful representations are specified entirely in terms of underlying functors over $\mathscr{B}$. Thus (i) is immediate from the universal property of $D'\colon \tilde{\mathscr{R}}' \longrightarrow \mathscr{R}'$. For (ii), it is convenient to apply the characterization of Theorem 9.6(i). We take $S = D \circ I$, where $I\colon \mathscr{S}\mathscr{R} \longrightarrow \tilde{\mathscr{R}}$ is the inclusion of a maximal saturated subrepresentation. The image of $\tilde{\rho} \circ I$ is a saturated subrepresentation of $\tilde{\mathscr{R}}'$. We expand the image to obtain an inclusion $I'\colon \mathscr{S}\mathscr{R}' \longrightarrow \tilde{\mathscr{R}}'$ of a maximal saturated subrepresentation, and we take $S' = I' \circ D'$. With this construction, σ is given by $\tilde{\rho} \circ I$ and we have $S' \circ \sigma = \rho \circ S$. This can be interpreted as giving a strict map of representations in $\mathscr{R}'$. Taking this construction as the starting point for the analogous construction of σ', part (iii) is clear. Part (iv) is also clear. □

To study change of groups or to restrict to the study of G-spaces of restricted isotropy types, we must also change the base category $\mathscr{B}$.

Proposition 16.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be base categories, let $\iota\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a continuous functor, and let $\pi\colon \mathscr{R} \longrightarrow \mathscr{B}$ be a groupoid over $\mathscr{B}$. Define $\iota^*\mathscr{R}$ to be the pullback displayed in the diagram*

$$\begin{tikzcd}
\iota^*\mathscr{R} \arrow[r] \arrow[d, "\lambda"'] & \mathscr{R} \arrow[d, "\pi"] \\
\mathscr{A} \arrow[r, "\iota"'] & \mathscr{B}.
\end{tikzcd}$$

Then $\iota^*\mathscr{R}$ is a groupoid over $\mathscr{A}$ that is skeletal or discrete if $\mathscr{R}$ is skeletal or discrete. If ι is a faithful functor and $\mathscr{R}$ is faithful over $\mathscr{B}$, then $\iota^*\mathscr{R}$ is faithful over $\mathscr{A}$.

Proof. It is elementary to check that $\lambda\colon \iota^*\mathscr{R} \longrightarrow \mathscr{A}$ satisfies conditions (i)–(iv) of Definition 5.1. Note for this that the fibers of λ are copies of fibers of π, a fact that also explains why $\iota^*\mathscr{A}$ is skeletal or discrete if π is so. The last statement is a standard property of pullbacks of categories. □

Remark 16.3. Assume the hypotheses of the proposition.

(i) Lemma 6.4 implies that if $\mathscr{R}$ is discrete, then $\iota^*\mathscr{R}$ is discrete and therefore faithful. This holds whether or not ι is faithful.

(ii) If ι is an injection, then $\iota^*\mathscr{R}$ just restricts $\mathscr{R}$ to the subcategory given by the objects and maps that π sends to objects and maps in the image of $\mathscr{A}$.

(iii) Suppose given a groupoid $\mathscr{Q}$ over $\mathscr{A}$ and a commutative diagram

$$\begin{tikzcd} \mathscr{Q} \arrow[r, "\rho"] \arrow[d, "\lambda"'] & \mathscr{R} \arrow[d, "\pi"] \\ \mathscr{A} \arrow[r, "\iota"'] & \mathscr{B}. \end{tikzcd}$$

Then ρ factors uniquely through a functor $\overline{\rho}\colon \mathscr{Q} \longrightarrow \iota^*\mathscr{R}$ over $\mathscr{A}$. If $\mathscr{R}$ and $\mathscr{Q}$ are skeletal over $\mathscr{B}$, then $\iota^*\mathscr{R}$ is skeletal over $\mathscr{A}$ and Proposition 16.1 applies to compare the universal orientable representations of $\mathscr{Q}$ and $\iota^*\mathscr{R}$.

17. Orientations and change of groups

So far, we have restricted attention to a fixed ambient group G. Various natural constructions on equivariant bundles lead one to consider what happens on passage to subquotient groups. For example, let $H \subset G$, let NH be the normalizer of H in G, and let $WH = NH/H$. For a G-bundle $p\colon E \longrightarrow B$, the restriction $p|_{B^H}$ is an NH-bundle that has a subbundle $p^H\colon E^H \longrightarrow B^H$, which is a WH-bundle, together with a complementary NH-subbundle p_H, so that $p^H \oplus p_H \cong p|B^H$. We here describe how orientations of p give rise to orientations of these related bundles. In particular, if a smooth G-manifold is oriented, then its orientation, which is an orientation of its tangent G-bundle, determines an orientation of the WH-manifold M^H and an orientation of the normal NH-bundle of the inclusion of M^H in M. Note that the dimensions of fixed point bundles can vary over components and recall that we write $\mathscr{V}_G$ for $\coprod \mathscr{V}_G(n)$, and similarly for $\mathscr{S}\mathscr{V}_G$.

We first consider subgroups and then consider quotient groups, making heavy use of the observations of the previous section.

Let $i\colon H \longrightarrow G$ be an inclusion. We have the functor

$$i_*\colon \mathscr{O}_H \longrightarrow \mathscr{O}_G$$

given by $i_*(H/K) = G \times_H (H/K) \cong G/K$ on objects and $i_*(\alpha) = G \times_H \alpha$ on morphisms. For a groupoid $\pi\colon \mathscr{E} \longrightarrow \mathscr{O}_G$ over $\mathscr{O}_G$, let $i^*\mathscr{E}$ denote the pullback of $\mathscr{E}$ along i_*. By Proposition 16.2, $i^*\mathscr{E}$ is a groupoid over $\mathscr{O}_H$ that is skeletal, faithful, or discrete over $\mathscr{O}_H$ if $\mathscr{E}$ is skeletal, faithful, or discrete over $\mathscr{O}_G$. The functor i_* is an injection, and i^* just restricts $\mathscr{E}$ to those orbits G/K such that K is a subgroup of H. We have the following observations.

Proposition 17.1. *For a G-space X, $i^*\Pi_G X$ is isomorphic over $\mathscr{O}_H$ to $\Pi_H X$. For any n, $i^*\mathscr{V}_G(n)$ is isomorphic over $\mathscr{O}_H$ to $\mathscr{V}_H(n)$.*

Proof. We do mean isomorphism and not just equivalence of categories over $\mathscr{O}_H$. The result for fundamental groupoids is clear since, for $K \subset H$, a K-fixed point of X can be viewed as either an H-map $H/K \longrightarrow X$ or a G-map $G/K \longrightarrow X$, and similarly for paths. We obtain inverse isomorphisms between our categories of G-bundles by extending H-bundles $H \times_K V$ to G-bundles $G \times_K V$ and restricting G-bundles $G \times_K V$ to their H-subbundles $H \times_K V$. We are using the skeletal categories specified in Definition 2.2, and the respective composites are easily verified to be identity functors. □

Proposition 17.2. *Let $p\colon E \longrightarrow B$ be a G-bundle and let $p|H$ denote p regarded as an H-bundle. The representation $(p|H)^*\colon \Pi_H B \longrightarrow \mathscr{V}_H$ is isomorphic to*

$$i^*p^*\colon \Pi_H B \cong i^*\Pi_G B \longrightarrow i^*\mathscr{V}_G \cong \mathscr{V}_H.$$

Proof. Again, we mean isomorphism and not just equivalence of representations. With the notations of Definition 7.1(iv), we set $F = F' = \mathrm{Id}\colon \Pi_H B \longrightarrow \Pi_H B$ and seek an isomorphism $\phi\colon (p|H)^* \longrightarrow i^*p^*$ of functors $\Pi_H B \longrightarrow \mathscr{V}_H$ over $\mathscr{O}_H$. Recall the proof of Proposition 2.7, remembering the use of the equivalence $\overline{\mathscr{V}}_G \longrightarrow \mathscr{V}_G$ described in Definition 2.2. For a morphism $(\omega, \alpha)\colon x \longrightarrow y$ of $\Pi_H B$, $\alpha\colon H/K \longrightarrow H/L$, $(p|H)^*(\omega, \alpha)$ is the composite

$$H \times_K V \xrightarrow{\xi_H(x)^{-1}} (p|H)^*(x) \xrightarrow{\tilde{\omega}_1} (p|H)^*(y) \xrightarrow{\xi_H(y)} H \times_L W,$$

where $\xi_H(x)$ and $\xi_H(y)$ are two of the chosen isomorphisms used in defining the equivalence $\overline{\mathscr{V}}_H \longrightarrow \mathscr{V}_H$ and $\tilde{\omega}_1$ is obtained from the H-bundle CHP. We may extend $x\colon H/K \longrightarrow B$ to a G-map $G/K \longrightarrow B$, and similarly for y, and we may view (ω, α) as a morphism in $\Pi_G B$. Then $\tilde{\omega}_1$ in our description of $(p|H)^*(\omega, \alpha)$ can be taken to be the restriction of a map $\tilde{\omega}_1$ obtained by use of the G-bundle CHP. The map $i^*p^*(\omega, \alpha)$ is the restriction to H-subbundles of the composite

$$G \times_K V \xrightarrow{\xi_G(x)^{-1}} p^*(x) \xrightarrow{\tilde{\omega}_1} p^*(y) \xrightarrow{\xi_G(y)} G \times_L W$$

Here $\xi_G(x)$ and $\xi_G(y)$ are two of the chosen isomorphisms used in defining the equivalence $\overline{\mathscr{V}}_G \longrightarrow \mathscr{V}_G$. The required isomorphism ϕ is specified by $\phi(x) = \xi_G(x)\xi_H(x)^{-1}$, where $\xi_G(x)$ is restricted to the canonical H-subbundles. The point is that we obtain an isomorphism of representations even though we must make independent choices of the $\xi_G(x)$ and $\xi_H(x)$. □

Let $(\mathscr{S}\mathscr{V}_G, S_G)$ denote the universal orientable representation in $\mathscr{V}_G$. The representation $(i^*\mathscr{S}\mathscr{V}_G, i^*S_G)$ in $\mathscr{V}_H$ is not universal, but it is orientable. Therefore it admits an orientation

$$(F_H^G, \phi_H^G)\colon (i^*\mathscr{S}\mathscr{V}_G, i^*S_G) \longrightarrow (\mathscr{S}\mathscr{V}_H, S_H). \tag{17.3}$$

Although there seems to be no canonical choice of (F^G_H, ϕ^G_H), we may think of it as specifying a universal way to orient the underlying H-bundle of an oriented G-bundle. That is, given an orientation

$$(F, \phi) \colon (\Pi_G B, p^*) \longrightarrow (\mathscr{S}\mathscr{V}_G, S_G)$$

of a G-bundle $p\colon E \longrightarrow B$, we define the *induced orientation* of $p|H$ to be the composite

$$(F^G_H, \phi^G_H) \circ (i^* F, i^*\phi). \tag{17.4}$$

Here we use Propositions 17.1 and 17.2 to interpret the domain of $(i^*F, i^*\phi)$ as $(\Pi_H B, (p|H)^*)$. The dimensions of bundles are unchanged under these constructions, so that we can work equally well with $\mathscr{V}_H(n)$ and $\mathscr{V}_G(n)$.

To study bundles associated to a subquotient group $WH = NH/H$, we can use the observations above to first restrict down from G to NH. Thus, thinking of the normal subgroup H of NH with quotient group WH, we change notations and consider a normal subgroup N of a group G with quotient group J. Let $q\colon G \longrightarrow J$ be the quotient homomorphism.

We have the functor

$$q^* \colon \mathscr{O}_J \longrightarrow \mathscr{O}_G$$

given by $q^* J/K = G/H$ on objects, where $H = q^{-1}K$. Since q induces an isomorphism of G-spaces from G/H to J/K regarded as a G-space via q, we can define q^* on maps by regarding a J-map of orbits as a G-map of orbits. For a groupoid $\pi\colon \mathscr{E} \longrightarrow \mathscr{O}_G$ over $\mathscr{O}_G$, let $q_*\mathscr{E}$ denote the pullback of $\mathscr{E}$ along q^*. By Proposition 16.2, $q_*\mathscr{E}$ is a groupoid over $\mathscr{O}_J$ that is skeletal, faithful, or discrete over $\mathscr{O}_J$ if $\mathscr{E}$ is skeletal, faithful, or discrete over $\mathscr{O}_G$. The functor q^* is an injection, and q_* just restricts $\mathscr{E}$ to those orbits G/H such that H contains N.

We also have the functor

$$q_* \colon \mathscr{O}_G \longrightarrow \mathscr{O}_J$$

that sends a G-orbit G/H to the J-orbit $J \times_G G/H \cong G/HN \cong J/K$, where $K = HN/N$. In contrast to our previous functors between orbit categories, q_* is not an injection. Observe that the composite $q_*q^* \colon \mathscr{O}_G \longrightarrow \mathscr{O}_G$ sends G/H to G/HN. The quotient G-maps $\gamma\colon G/H \longrightarrow G/HN$ specify a natural transformation $\gamma\colon \mathrm{Id} \longrightarrow q_*q^*$ of functors $\mathscr{O}_G \longrightarrow \mathscr{O}_G$. For a groupoid $\mathscr{F}$ over $\mathscr{O}_J$, let $q^*\mathscr{F}$ denote the pullback of $\mathscr{F}$ along q_*. By Proposition 16.2, $q^*\mathscr{F}$ is a groupoid over $\mathscr{O}_G$ that is skeletal or discrete over $\mathscr{O}_G$ if $\mathscr{F}$ is skeletal or discrete over $\mathscr{O}_J$. We shall be interested mainly in the case $\mathscr{F} = q_*\mathscr{E}$, where $\mathscr{E}$ is a groupoid over $\mathscr{O}_G$. An object of $q^*q_*\mathscr{E}$ is a pair $(G/H, x)$, where x is an object of $\mathscr{E}$ such that $\pi(x) = G/HN$; a morphism $(G/H, x) \longrightarrow (G/L, y)$ is a pair (α, ω), where $\alpha\colon G/H \longrightarrow G/L$, $\omega\colon x \longrightarrow y$, and $\pi(\omega)$ is the map $G/HN \longrightarrow G/LN$ induced by α.

For a G-space X, X^N is a G-space with action factoring through J.

Proposition 17.5. *For a G-space X, $q_*\Pi_G X$ is isomorphic over $\mathscr{O}_J$ to $\Pi_J X^N$ and $q^*q_*\Pi_G X$ is isomorphic over $\mathscr{O}_G$ to $\Pi_G X^N$.*

Proof. If $H \supset N$ and $K = q(H)$, then $X^H = (X^N)^K$. An H-fixed point of X can be viewed as either a G-map $G/H \longrightarrow X$ or a J-map $J/K \longrightarrow X^N$, and similarly for paths. This gives the first isomorphism. For a general subgroup H of G, $(X^N)^H = X^{HN}$. An H-fixed point of X^N can be viewed either as G/H together with a G-map $G/HN \longrightarrow X$ or as a G-map $G/H \longrightarrow X^N$, and similarly for paths. This gives the second isomorphism. $\square$

However, $q_*\mathscr{V}_G$ is not isomorphic to $\mathscr{V}_J$ since $q_*\mathscr{V}_G$ is the groupoid over $\mathscr{O}_J$ of G-bundles over N-fixed orbit spaces and the total space $G\times_H V$ of such a G-bundle need not be N-fixed. Letting $K = q(H)$ and taking N-fixed points, we obtain the J-bundle $(G\times_H V)^N \cong J\times_K V^N$ over J/K. Since this gives fixed point bundles of varying dimensions, we must work with $\mathscr{V}_G$ and $\mathscr{V}_J$ rather than working one dimension at a time. Let

$$\Phi^N : q_*\mathscr{V}_G \longrightarrow \mathscr{V}_J$$

denote the functor obtained by passage to N-fixed bundles. The orthogonal complement V_N of V^N is an H-subrepresentation of V, and we also have the functor

$$\Phi_N : q_*\mathscr{V}_G \longrightarrow q_*\mathscr{V}_G$$

that sends $G\times_H V$ to $G\times_H V_N$. Finally, we shall need the functor

$$\Gamma\colon q^*q_*\mathscr{V}_G \longrightarrow \mathscr{V}_G$$

that sends an object $(G/H, G\times_{HN} V \longrightarrow G/HN)$ to the pullback $G\times_H V \longrightarrow G/H$ along the quotient G-map $\gamma\colon G/H \longrightarrow G/HN$.

Proposition 17.6. *Let $p\colon E \longrightarrow B$ be a G-bundle and let p_N be the complementary G-bundle to the N-fixed point J-bundle p^N over B^N, so that $p^N \oplus p_N \cong p|_{B^N}$ as G-bundles. The representation $(p^N)^*\colon \Pi_J B^N \longrightarrow \mathscr{V}_J$ is isomorphic to the composite*

$$\Pi_J B^N \cong q_*\Pi_G X \xrightarrow{q_*p^*} q_*\mathscr{V}_G \xrightarrow{\Phi^N} \mathscr{V}_J.$$

The representation $(p_N)^*\colon \Pi_G B^N \longrightarrow \mathscr{V}_G$ is isomorphic to the composite

$$\Pi_G B^N \cong q^*q_*\Pi_G X \xrightarrow{q^*q_*p^*} q^*q_*\mathscr{V}_G \xrightarrow{q^*\Phi_N} q^*q_*\mathscr{V}_G \xrightarrow{\Gamma} \mathscr{V}_G.$$

Proof. The proof is precisely analogous to that of Proposition 17.2, but with passage to N-fixed point bundles and to complementary G-bundles replacing restriction to H-subbundles. With an evident notation, the isomorphism ϕ from $(p^N)^*$ to the displayed composite is given on objects $x\colon J/K \longrightarrow B^N$ by $\phi(x) = \xi_G(x)^N\xi_J(x)$, and similarly for $(p_N)^*$. $\square$

The representations $(q_*\mathscr{S}\mathscr{V}_G, \Phi^N\circ q_*S_G)$ in $\mathscr{V}_J$ and $(q^*q_*\mathscr{S}\mathscr{V}_G, \Gamma\circ q^*\Phi_N\circ q^*q_*S_G)$ in $\mathscr{V}_G$ are not universal, but they are orientable. Therefore they admit orientations

$$(F^G_J, \phi^G_J)\colon (q_*\mathscr{S}\mathscr{V}_G, \Phi^N\circ q_*S_G) \longrightarrow (\mathscr{S}\mathscr{V}_J, S_J). \tag{17.7}$$

and

$$(\bar{F}^N_G, \bar{\phi}^N_G)\colon (q^*q_*\mathscr{S}\mathscr{V}_G, \Gamma\circ q^*\Phi_N\circ q^*q_*S_G) \longrightarrow (\mathscr{S}\mathscr{V}_G, S_G). \tag{17.8}$$

We think of (F^G_J, ϕ^G_J) and $(\bar{F}^N_G, \bar{\phi}^N_G)$ as specifying universal ways to orient the N-fixed J-bundle p^N and the complementary G-bundle p_N of an oriented G-bundle p.

That is, given an orientation

$$(F,\phi)\colon (\Pi_G B, p^*) \longrightarrow (\mathscr{S}\mathscr{V}_G, S_G)$$

of a G-bundle $p\colon E \longrightarrow B$, we define the *induced orientation* of p^N to be the composite

$$(F_J^G, \phi_J^G) \circ (q_* F, \Phi^N \circ q_*\phi), \tag{17.9}$$

and we define the *induced orientation* of p_N to be the composite

$$(\bar{F}_G^N, \bar{\phi}_G^N) \circ (q^* q_* F, \Gamma \circ q^*\Phi_N \circ q^* q_*\phi). \tag{17.10}$$

Here we use Propositions 17.5 and 17.6 to interpret the domains of $(q_*F, \Phi^N \circ q_*\phi)$ and $(q^*q_*F, \Gamma \circ q^*\Phi^N \circ q^*q_*\phi)$ as $(\Pi_J B, (p^N)^*)$ and $(\Pi_G B, (p_N)^*)$.

18. Variant kinds of orientations

Nonequivariantly, there is only one sensible definition of an orientation of a vector bundle, but this is a calculational fact that does not extend to the equivariant setting. The point is that

$$\mathbb{Z}_2 \cong \pi_0(O(n)) \cong \pi_0(PL(n)) \cong \pi_0(Top(n)) \cong \pi_0(F(n))$$

for all $n \geqslant 1$, including $n = \infty$. Nothing like this holds equivariantly. We should think of these groups as the groups of homotopy classes of linear, PL, topological, or homotopical equivalences of the n-sphere. If we consider instead a G-representation V and its associated G-sphere, we get different answers as the category varies. By forgetting or stabilizing structure, Y we obtain (at least) eight different reasonable orientation theories on G-vector bundles corresponding to the linear, piecewise linear, topological, and homotopical categories and their stable variants. Similarly, there are six orientation theories for PL G-bundles, four for topological G-bundles, and two for spherical G-fibrations. We proceed to make this precise, the crucial point being that our categorical framework is sufficiently general to set up and compare all of these variants with little additional work. The following diagram displays the skeletal groupoids over $\mathscr{O}_G$ that serve as the target categories for the relevant representations.

$$\begin{tikzcd}
\mathscr{V}_G(n) \arrow[r] \arrow[d] & \mathscr{P}\mathscr{L}_G(n) \arrow[r] \arrow[d] & \mathscr{T}op_G(n) \arrow[r] \arrow[d] & \mathscr{F}_G(n) \arrow[d] \\
s\mathscr{V}_G(n) \arrow[r] & s\mathscr{P}\mathscr{L}_G(n) \arrow[r] & s\mathscr{T}op_G(n) \arrow[r] & s\mathscr{F}_G(n)
\end{tikzcd} \tag{18.1}$$

We shall write $\mathscr{C}at$ as shorthand for any one of $\mathscr{V}$, $\mathscr{P}\mathscr{L}$, $\mathscr{T}op$, or $\mathscr{F}$. To form the top row of (18.1), we take skeleta of the categories $\overline{\mathscr{C}at}_G(n)$ of G-vector bundles over orbits, with morphisms being, respectively, the homotopy classes of maps of G-vector bundles, PL G-bundles, topological G-bundles, and spherical G-fibrations. In the last case, we identify objects with their fiberwise one-point compactifications and understand maps to mean maps that preserve the resulting section. We are requiring our fibers to be equivalent, in the appropriate sense, to linear representations since we are interested in locally linear bundle and fibration theories. There are further variants in which more general fibers are allowed, for example based spaces homotopy equivalent to spheres in the fibration case. In all four cases, there is an analogue of Lemma 2.3 that gives a complete description of maps in $\mathscr{C}at_G(n)$ in terms of the homotopy theory of $\mathscr{C}at_G(n)$-maps between representations. The source lifting and divisibility properties required of a groupoid over $\mathscr{O}_G$ are easily checked for these categories. The arrows of the top row of (18.1) are obtained by neglect of structure, together with choices of isomorphisms of representative objects as we pass from more rigid to less rigid types of bundles.

The bottom row of (18.1) is the stabilization of the top row. To control the relevant colimits, we let U be the direct sum of countably many copies of each irreducible representation of G; that is, U is a "complete G-universe". By a subspace of U, we understand a finite dimensional sub inner product space. Observe that U is also a complete H-universe for any $H \subset G$ since any representation of H extends to a representation of G on a possibly larger vector space. We stabilize over U as follows. Consider objects $G \times_H V$ and $G \times_K W$ in $\mathscr{C}at_G(n)$. A stable map between these objects over a given base map $\alpha\colon G/H \longrightarrow G/K$ is the image in the colimit of a homotopy class of $\mathscr{C}at_G(n)$-bundle maps

$$\tilde{\alpha}\colon G \times_H (V \oplus Z) \longrightarrow G \times_K (W \oplus Z)$$

over α, where Z is a K-subspace of U. The maps of the colimit system are given by "suspension": for $Z \subset Z'$, we suspend $\tilde{\alpha}$ to a map

$$G \times_H (V \oplus Z') \longrightarrow G \times_K (W \oplus Z')$$

by taking its product with the identity map of the trivial bundle $Z' - Z \longrightarrow *$, where $Z' - Z$ is the orthogonal complement of Z in Z'. We emphasize that $\oplus$ denotes external direct sum in the previous two displays. Using the objects of $\overline{\mathscr{C}at}_G(n)$ but replacing maps by stable maps, we obtain a groupoid $s\overline{\mathscr{C}at}_G(n)$ over $\mathscr{O}_G$. Taking a skeleton, we obtain the category $s\mathscr{C}at_G(n)$. Choosing isomorphisms between objects of $\mathscr{C}at_G(n)$ and the chosen representatives of their stable equivalence classes and sending maps to their stable equivalence classes, we obtain a functor $\mathscr{C}at_G(n) \longrightarrow s\mathscr{C}at_G(n)$. We obtain the arrows in the bottom row of (18.1) similarly. The four squares then commute up to natural isomorphisms, which are again determined by the chosen isomorphisms between representative objects.

The $\mathscr{C}at_G(n)$ analogs of Proposition 2.7 are valid. A $\mathscr{C}at_G(n)$ G-bundle or fibration $p\colon E \longrightarrow B$ determines a functor $p^*\colon \Pi_G(B) \longrightarrow \mathscr{C}at_G(n)$ over $\mathscr{O}_G$. The proof is based on the appropriate bundle covering homotopy property. Using the functor $\mathscr{C}at_G(n) \longrightarrow s\mathscr{C}at_G(n)$, we obtain the stable analogue. Definition 2.8 then gives us eight notions of orientability corresponding to these eight choices for the target category. Similarly, Theorem 7.5 gives us eight universal orientable representations. If $\mathscr{R}$ denotes any of the eight target categories, we have a corresponding universal orientable representation $(\mathscr{S}\mathscr{R}, S)$. By varying the target groupoid, and so the corresponding universal representation, Definition 7.9 gives us the eight different notions of an orientation of a G-vector bundle, and similarly for PL G-bundles and so on. By Proposition 16.1, if $\rho\colon \mathscr{R} \longrightarrow \mathscr{R}'$ is any one of the functors over $\mathscr{O}_G$

displayed in (18.1), then there is a map $\sigma\colon \mathscr{S}\mathscr{R} \longrightarrow \mathscr{S}\mathscr{R}'$ of groupoids over $\mathscr{O}_G$ that covers ρ. This allows us to compare our various kinds of orientations.

We give a few examples. Observe first that the functor $\mathscr{V}_G(n) \longrightarrow s\mathscr{V}_G(n)$ can be taken to be the identity on objects, since if two representations are stably equivalent, then they have the same characters and are therefore equivalent. This is implied by the fact that $RO(H)$ is the free group on the irreducible representations, and it does not carry over to our other categories. Moreover, we see from the proof of Lemma 2.6 that the functor $\mathscr{V}_G(n) \longrightarrow s\mathscr{V}_G(n)$ is faithful.

Example 18.2. Let $G = \mathbb{Z}/2$. There are two stable self-maps of V_0 and V_n in $s\mathscr{V}_G(n)$ that are not in $\mathscr{V}_G(n)$; that is, the exceptional cases in parts (a)(ii) and (c) of Example 10.2 are not exceptional stably. We see that $\mathscr{S}s\mathscr{V}_G(n) = \mathscr{S}\mathscr{V}_G(n)$ and that $S : \mathscr{S}s\mathscr{V}_G(n) \longrightarrow s\mathscr{V}_G(n)$ is the composite of $S : \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{V}_G(n)$ and the stabilization functor $\mathscr{V}_G(n) \longrightarrow s\mathscr{V}_G(n)$. The main change is that the group of automorphisms $\Omega(\mathscr{S}s\mathscr{V}_G(n), S)$ is an elementary abelian two group of order 2^{n+3}.

Example 18.3. Let G be a finite group of odd order and again consider $s\mathscr{V}_G(n)$. The stable analogue of Construction 12.2 is similar but simpler since, stably, one need not consider the case $V^H = 0$ separately. The category $\mathscr{S}s\mathscr{V}_G(n)$ has one object v over G/H corresponding to each bundle $G \times_H V$ in $\mathscr{V}_G(n)$. There is a map $m\colon v \longrightarrow w$ over $\alpha\colon G/H \longrightarrow G/K$ whenever there is a stable G-bundle map $G \times_H V \longrightarrow G \times_K W$ covering α; the functor S carries m to the unique stable bundle map preserving chosen orientations. While, as in Proposition 16.1, we can construct a functor $\sigma : \mathscr{S}\mathscr{V}_G(n) \longrightarrow \mathscr{S}s\mathscr{V}_G(n)$ that covers the functor $\rho : \mathscr{V}_G(n) \longrightarrow s\mathscr{V}_G(n)$, it is more natural to let σ send both v_+ and v_-, when present, to v, and send maps labeled m, m_+, or m_- to m. We then have a natural isomorphism $\phi\colon \rho \circ S_{\mathscr{V}_G(n)} \longrightarrow S_{s\mathscr{V}_G(n)} \circ \sigma$ that is the identity on objects v or v_+, and is the orientation reversing stable self map of $G \times_H V$ on objects v_-.

Probably the most interesting variant of orientation theory is the one concerning stable spherical G-fibrations, which fits naturally into equivariant stable homotopy theory. Here the Burnside rings $A(H)$ of the subgroups of G come into play. In fact, it is immediate from the definitions and the standard isomorphism between $A(H)$ and the zeroth stable H-homotopy group of S^0 that the self maps in $s\mathscr{F}_G(n)$ of any object $G \times_H S^V$ form a copy of the group of units in $A(H)$. (A more general result may be found in [**9**, 10.2.2].) There are fewer objects in $s\mathscr{F}_G(n)$ than in $\mathscr{V}_G(n)$ since inequivalent representations can have stably homotopy equivalent spheres. For finite groups, an analysis of when this happens may be found in [**9**, §9.1]; by a result of Traczyk, the case of general compact Lie groups reduces to the case of finite groups [**27**]. Since, as in the nonequivariant world, equivariant cohomology theories detect only stable homotopy type, stable spherical orientations are the appropriate ones to relate to ideas of cohomological orientation.

19. Categories of virtual G-bundles

As said in the introduction, we defer comparison between geometric and cohomological oriention theory to later work. However, as is made clear in [**4, 5**], for such work it is useful to have four more variants of the basic theory, in which stable bundles are replaced by virtual bundles. In fact, the diagram (18.1) can be extended by adding another row of skeletal groupoids over $\mathscr{O}_G$.

$$\begin{array}{ccccccc} s\mathscr{V}_G(n) & \longrightarrow & s\mathscr{P}\mathscr{L}_G(n) & \longrightarrow & s\mathscr{T}op_G(n) & \longrightarrow & s\mathscr{F}_G(n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ v\mathscr{V}_G(n) & \longrightarrow & v\mathscr{P}\mathscr{L}_G(n) & \longrightarrow & v\mathscr{T}op_G(n) & \longrightarrow & v\mathscr{F}_G(n) \end{array} \tag{19.1}$$

The bottom row, which is also defined for negative values of n, is obtained from the top row by passage to virtual bundles. Thus $v\mathscr{C}at_G(n)$ is a skeleton of the category $v\overline{\mathscr{C}at}_G(n)$ of virtual G-vector bundles of virtual dimension n and virtual $\mathscr{C}at_G$-maps. Here a *virtual G-vector bundle* of virtual dimension n is a pair of bundles $(E, F) = (G \times_H V_1, G \times_H V_2)$ over the same G-orbit, with $|V_1| - |V_2| = n$. We think of (E, F) as a formal difference $E - F$. Intuitively, a virtual map is a stable pair of maps, but we must be careful with the definition. Define a *virtual map* from $(G \times_H V_1, G \times_H V_2)$ to $(G \times_K W_1, G \times_K W_2)$ over a map of orbits $\alpha\colon G/H \longrightarrow G/K$ to be the equivalence class of a pair of $\mathscr{C}at_G$-maps

$$f_i\colon G \times_H (V_i \oplus Z) \longrightarrow G \times_K (W_i \oplus Z')$$

over α, where Z is an H-subspace of U and Z' is a K-subspace of U such that $|V_1| + |Z| = |W_1| + |Z'|$ and thus also $|V_2| + |Z| = |W_2| + |Z'|$. The equivalence relation is generated by two basic relations, the first being G-bundle homotopy. The second relation is as follows. Let $k\colon G \times_H T \longrightarrow G \times_K T'$ be a $\mathscr{C}at_G$-map over α, where T is an H-subspace of U orthogonal to Z and T' is a K-subspace of U orthogonal to Z'. Then the pair (f_1, f_2) is equivalent to the "suspension" $(f_1 \oplus k, f_2 \oplus k)$, where

$$f_i \oplus k\colon G \times_H (V_i \oplus (Z \oplus T)) \longrightarrow G \times_K (W_i \oplus (Z' \oplus T'))$$

is the obvious fiberwise direct sum of maps. Note that, in $Z \oplus T$, the sum is internal in U. Composition is defined by suspending until the morphisms can be composed as pairs of bundle maps. The following easily verified observation, applied to U regarded as an H-universe, implies that this gives a well-defined category.

Lemma 19.2. *Let U be a complete G-universe, let V, V', W, and W' be G-subspaces of U, and suppose given $\mathscr{C}at_G$-maps $h\colon V \longrightarrow V'$ and $k\colon W \longrightarrow W'$. Then there exist G-subspaces Z and Z' of $\mathscr{V}$ such that $V \subset Z$, $W \subset Z$, $V' \subset Z'$, and $W' \subset Z'$ together with $\mathscr{C}at_G$-maps*

$$j\colon (Z - V) \longrightarrow (Z' - V') \quad \text{and} \quad \ell\colon (Z - W) \longrightarrow (Z' - W')$$

such that $h \oplus j \simeq k \oplus \ell\colon Z \longrightarrow Z'$ as $\mathscr{C}at_G$-maps. If $V = V'$, we can take $j \simeq \mathrm{id}$.

It is not hard to check that $v\mathscr{C}at_G(n)$ is a groupoid over $\mathscr{O}_G$. It is also not hard to check that the set of isomorphism classes of objects over G/H in $v\mathscr{V}_G(n)$ is in bijective correspondence with the set of n-dimensional elements of $RO(H)$; objects of the other categories admit similar descriptions. For $n \geqslant 0$, we define a map

$s\mathscr{C}at_G(n) \longrightarrow v\mathscr{C}at_G(n)$ by passing to skeleta from the evident functor that sends the G-bundle $G \times_H V$ to the pair $(G \times_H V, G \times_H 0)$. This functor is an inclusion of groupoids over $\mathscr{O}_G$. Using this functor, we obtain a theory of virtual orientations of virtual G-vector bundles, by taking pairs of orientations in the evident fashion.

To relate equivariant orientation theory to equivariant classifying spaces, it is convenient to have small categories that, although not skeletal, are defined in terms of the complete universe U and are equivalent to the skeletal categories $\mathscr{C}at_G(n)$, $s\mathscr{C}at_G(n)$, and $v\mathscr{C}at_G(n)$.

Definition 19.3. Fix a complete G-universe U.

(i) Define $\mathscr{C}at_G(n,U)$ to be the full category of $\overline{\mathscr{C}at}_G(n)$ whose objects are the G-vector bundles of the form $G \times_H V$ for any $H \subset G$ and any n-dimensional H-subspace V of U.

(ii) Define $v\mathscr{C}at_G(n,U)$ to be the colimit over the G-subspaces V of U of the categories $\mathscr{C}at_G(n+|V|, V \oplus U)$, where the colimit runs over the functors

$$\mathscr{C}at_G(n+|V|, V \oplus U) \longrightarrow \mathscr{C}at_G(n+|W|, W \oplus U)$$

that are obtained by adding $W - V \subset W$ to objects and maps, where $V \subset W$.

(iii) Define $s\mathscr{C}at_G(n,U)$ to be the image of $\mathscr{C}at_G(n,U)$ in $v\mathscr{C}at_G(n,U)$ obtained by setting $V = \{0\}$ in the colimit system.

It is clear that the composite of the inclusion $\mathscr{C}at_G(n,U) \subset \overline{\mathscr{C}at}_G(n)$ and the retraction $\overline{\mathscr{C}at}_G(n) \longrightarrow \mathscr{C}at_G(n)$ is an equivalence of categories. The analogue for virtual bundles is less obvious.

Proposition 19.4. *The categories $v\mathscr{C}at_G(n)$ and $v\mathscr{C}at_G(n,U)$ are equivalent, and the equivalence restricts to an equivalence between $s\mathscr{C}at_G(n)$ and $s\mathscr{C}at_G(n,U)$.*

Proof. For each H-representation V such that $G \times_H V$ is an object of $\mathscr{C}at_G(n)$, choose an H-space $\bar{V} \subset U$ and an H-linear isomorphism $i_V : V \longrightarrow \bar{V}$. These choices determine a functor $\mathscr{C}at_G(n) \longrightarrow \mathscr{C}at_G(n,U)$ that is an equivalence of categories. Also, choose an H-space $\bar{V}^\perp \subset U$ such that $\bar{V}^\perp$ is orthogonal to $\bar{V}$ and $\bar{V} \oplus \bar{V}^\perp$ is a G-space, and choose a G-isomorphism $j_V : U \longrightarrow (U - (\bar{V} \oplus \bar{V}^\perp))$.

We first define a functor $J : v\mathscr{C}at_G(n,U) \longrightarrow v\mathscr{C}at_G(n)$. An object X of $v\mathscr{C}at_G(n,U)$ is represented by an object $G \times_H V_1$ of some $\mathscr{C}at_G(n+|V_2|, V_2 \oplus U)$, and we let $J(X)$ be the object of $v\mathscr{C}at_G(n)$ isomorphic to the object $(G \times_H V_1, G \times_H V_2)$ of $v\overline{\mathscr{C}at}_G(n)$. That is, we think of X as the virtual G-bundle $G \times_H V_1 - G \times_H V_2$. If $G \times_K W_1$ in $\mathscr{C}at_G(n+|W_2|, W_2 \oplus U)$ represents a second object Y and $\phi : X \longrightarrow Y$ is a map in $v\mathscr{C}at_G(n,U)$, then ϕ is represented by a $\mathscr{C}at_G$-map

$$f_1 : G \times_H (V_1 \oplus Z) \longrightarrow G \times_K (W_1 \oplus Z')$$

in $\mathscr{C}at_G(n+|T|, T \oplus U)$, where Z and Z' are G-subspaces of U such that Z is orthogonal to V_2, Z' is orthogonal to W_2, and $V_2 \oplus Z = W_2 \oplus Z' = T$, say. We let $J(\phi)\colon J(X) \longrightarrow J(Y)$ be the map represented by the pair (f_1, f_2), where

$$f_2 : G \times_H (V_2 \oplus Z) \longrightarrow G \times_K (W_2 \oplus Z')$$

is induced by the base map $G/H \longrightarrow G/K$ of f_1 and the identity map on T.

We next define a functor $R : v\mathscr{C}at_G(n) \longrightarrow v\mathscr{C}at_G(n, U)$. Consider an object $X = (G \times_H V_1, G \times_H V_2)$ of $v\mathscr{C}at_G(n)$. Embed the external direct sum $\bar{V}_2^{\perp} \oplus \bar{V}_1$ in the universe $\bar{V}_2 \oplus \bar{V}_2^{\perp} \oplus U$ by including $\bar{V}_2^{\perp}$ in $\bar{V}_2 \oplus \bar{V}_2^{\perp}$ and including $\bar{V}_1$ in U. This allows us to view $G \times_H (\bar{V}_2^{\perp} \oplus \bar{V}_1)$ as an object of $\mathscr{C}at_G(n + |V_2 \oplus \bar{V}_2^{\perp}|, \bar{V}_2 \oplus \bar{V}_2^{\perp} \oplus U)$, and we let $R(X)$ be the image of this object in the colimit $v\mathscr{C}at_G(n, U)$.

For a virtual map $\phi : (G \times_H V_1, G \times_H V_2) \longrightarrow (G \times_K W_1, G \times_K W_2)$ represented by $\mathscr{C}at_G$-maps

$$f_i : G \times_H (V_i \oplus Z) \longrightarrow G \times_K (W_i \oplus Z'),$$

define $R(\phi)$ as follows. Using Lemma 19.2, we can find an H-space T in U that is orthogonal to both $\bar{V}_2 \oplus \bar{V}_2^{\perp}$ and Z and a K-space T' in U that is orthogonal to both $\bar{W}_2 \oplus \bar{W}_2^{\perp}$ and Z' together with a G-map

$$k : G \times_H (\bar{V}_2^{\perp} \oplus T) \longrightarrow G \times_K (\bar{W}_2^{\perp} \oplus T')$$

(where the sums are external) such that

(i) $Z \oplus T$ and $Z' \oplus T'$ are G-subspaces of U.

(ii) $\bar{V}_2 \oplus \bar{V}_2^{\perp} \oplus j_{V_2}(Z \oplus T) = \bar{W}_2 \oplus \bar{W}_2^{\perp} \oplus j_{W_2}(Z' \oplus T') = S$, say, and

(iii) The $\mathscr{C}at_G$-map $G \times_H S \longrightarrow G \times_K S$ induced by $f_2 \oplus k$ and our chosen isomorphisms is the map induced by the base map $G/H \longrightarrow G/K$ of f_1 and the identity map on S.

We let $R(\phi)$ be the map in $v\mathscr{C}at_G(n, U)$ that is represented in $\mathscr{C}at_G(n + |S|, S \oplus U)$ by the map $G \times_H S \longrightarrow G \times_K S$ induced by $f_1 \oplus k$ and our chosen isomorphisms. Here, in the domain and target respectively, we are thinking of S as

$$\bar{V}_2 \oplus \bar{V}_2^{\perp} \oplus j_{V_2}(Z \oplus T) \quad \text{and} \quad \bar{W}_2 \oplus \bar{W}_2^{\perp} \oplus j_{W_2}(Z' \oplus T').$$

It is laborious, but straightforward, to check that J and R are inverse equivalences of categories. □

The categories just defined are closely related to standard models for equivariant classifying spaces. We let $\mathscr{C} = \coprod_n \mathscr{C}(n)$ for any of the families of groupoids $\mathscr{C}(n)$ over $\mathscr{O}_G$ that we have introduced.

Definition 19.5. Let U be a complete G-universe. Define $BO_G(n, U)$ to be the Grassmann G-space of n-dimensional subspaces of U; G acts via restriction of its action on U. Of course, $BO_G(n, U)$ is the colimit of the Grassmann G-manifolds of n-planes in G-spaces $V \subset U$. Similarly, let

$$BO_G(U) = \operatorname{colim}_{V \subset U}(\amalg_n BO_G(n, V \oplus U)),$$

where the colimit runs over the G-spaces $V \subset U$ and the system of maps

$$BO_G(n, V \oplus U) \longrightarrow BO_G(n + |W - V|, W \oplus U)$$

given by addition of the plane $W - V$ for $V \subset W$.

It is standard [**14, 18**] that $BO_G(n, U)$ classifies n-dimensional G-vector bundles and that $BO_G(U)$ classifies virtual G-vector bundles over finite G-CW complexes X. For the latter, this means that $KO_G(X) = [X, BO_G(U)]_G$. A direct comparison of definitions gives the following relationship between categories of bundles and fundamental groupoids.

Proposition 19.6. *As groupoids over $\mathscr{O}_G$,*

$$\mathscr{V}_G(n,U) \cong \Pi_G(BO_G(n,U)) \quad \textit{and} \quad v\mathscr{V}_G(U) \cong \Pi_G(BO_G(U)).$$

Part V. The classification of oriented G-bundles

20. Introduction: classifying G-spaces

The main purpose of this part is to display classifying G-spaces for oriented G-bundles and G-fibrations. We take the opportunity to explain several related classifying spaces and classification theorems. We shall focus on G-vector bundles, but the arguments apply equally well to topological G-bundles and to spherical G-fibrations. The case of PL G-bundles requires the use of analogous simplicial techniques and will be left to the interested reader.

Our construction of classifying G-spaces relies on the two-sided categorical bar construction of [**21**, §12]. Modulo a slight change of language, B assigns a G-space $B(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ to each triple consisting of a small topological category $\mathscr{D}$, a continuous covariant functor $\mathscr{X} : \mathscr{D} \to G\mathscr{U}$, and a continuous contravariant functor $\mathscr{Y} : \mathscr{D} \to \mathscr{U}$. The G-space $B(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ is the geometric realization of the simplicial G-space whose space $B_n(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ of n-simplices consists of tuples $y[f_n, \ldots, f_1]x$, where $f_i \colon d_{i-1} \to d_i$ is a map in $\mathscr{D}$, $x \in \mathscr{X}(d_0)$, and $y \in \mathscr{Y}(d_n)$; $B_0(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ is the disjoint union over the objects of $\mathscr{D}$ of the spaces $\mathscr{Y}(d) \times \mathscr{X}(d)$. The action of G on $B_n(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ is induced by the action of G on the $\mathscr{X}$-coordinate. The construction is functorial in all three variables.

The G-space $B(\mathscr{Y}, \mathscr{D}, \mathscr{X})$ is a fattened up homotopical version of the coend

$$\mathscr{Y} \otimes_{\mathscr{D}} \mathscr{X} = \int^{d \in \mathscr{D}} \mathscr{Y}(d) \times \mathscr{X}(d),$$

and there is a canonical map

$$\varepsilon : B(\mathscr{Y}, \mathscr{D}, \mathscr{X}) \longrightarrow \mathscr{Y} \otimes_{\mathscr{D}} \mathscr{X}.$$

We sometimes also write ε for its composite with a given G-map

$$\mathscr{Y} \otimes_{\mathscr{D}} \mathscr{X} \longrightarrow Z.$$

We think of $\mathscr{Y}$ as a $\mathscr{D}$-shaped diagram and call it a $\mathscr{D}$-space. We think of $\mathscr{X}$ as a fixed functor used to induce a G-action on a coalescence of $\mathscr{Y}$ to a single space.

For example, Elmendorf [**12**] applied this construction to construct a G-space ΨT associated to an $\mathscr{O}_G$-space T. Here $\Psi T = B(T, \mathscr{O}_G, \iota)$, where $\iota : \mathscr{O}_G \longrightarrow G\mathscr{U}$ is the evident functor that sends an orbit G/H, regarded as an object of the category $\mathscr{O}_G$, to the orbit G-space G/H. Passage to fixed point spaces X^H associates an $\mathscr{O}_G$-space ΦX to a G-space X, and there is a natural spacewise weak equivalence $\Phi\Psi T \longrightarrow T$. It provides the counit of an adjunction

$$[X, \Psi T]_G \cong [\Phi X, T]_{\mathscr{O}_G} \tag{20.1}$$

for G-CW complexes X [**12**], [**13**, V.3.2]. The unit $X \longrightarrow \Psi\Phi X$ of the adjunction is a G-homotopy equivalence. Formally, this adjunction arises from a Quillen equivalence of model categories [**26**], [**20**, III§1].

Passage to components gives a natural discretization map δ from an $\mathscr{O}_G$-space T to a discrete (or set-valued) $\mathscr{O}_G$-space $\pi_0(T)$. In particular, we have $\delta\colon \Phi X \to \pi_0(\Phi X)$. We say that X is *homotopically discrete* if δ is a spacewise weak equivalence, so that each component of each fixed point space X^H is weakly contractible.

Examples of homotopically discrete G-spaces ΨT that are constructed from discrete $\mathscr{O}_G$-spaces T are central to the classification of G-bundles. In particular, recall that a family $\mathscr{F}$ of (closed) subgroups of a topological group G is a set of subgroups closed under passage to conjugates and subgroups and observe that $\mathscr{F}$ determines a discrete $\mathscr{O}_G$-space $T_{\mathscr{F}}$ that takes G/H to a point if $H \in \mathscr{F}$ and to the empty space if $H \notin \mathscr{F}$. The G-space $\Psi T_{\mathscr{F}}$ is denoted $E\mathscr{F}$ and called the *universal $\mathscr{F}$-space.*

We are concerned with categories $\mathscr{D}$ over $\mathscr{O}_G$; that is, we are given a fixed continuous functor $\pi : \mathscr{D} \longrightarrow \mathscr{O}_G$. We compose this functor with $\iota : \mathscr{O}_G \longrightarrow G\mathscr{U}$ to obtain a functor $\iota \circ \pi\colon \mathscr{D} \longrightarrow G\mathscr{U}$. The following definition fixes notations for the G-spaces of greatest interest to us.

Definition 20.2. Define the classifying G-space of a category $\mathscr{D}$ over $\mathscr{O}_G$ to be

$$B\mathscr{D} = B(*, \mathscr{D}, \iota \circ \pi),$$

where $*$ is the constant functor that takes each object of $\mathscr{D}$ to a point. More generally, for a contravariant functor $\mathscr{Y} : \mathscr{D} \longrightarrow \mathscr{U}$, define

$$B(\mathscr{Y}, \mathscr{D}) = B(\mathscr{Y}, \mathscr{D}, \iota \circ \pi).$$

Since $(G/K)^H = \mathscr{O}_G(G/H, G/K)$,

$$B(\mathscr{Y}, \mathscr{D})^H = B(\mathscr{Y}, \mathscr{D}, \mathscr{O}_G(G/H, \iota\pi(-))).$$

In the following four sections, we explain how to use the bar construction to classify G-bundles and G-fibrations, oriented G-bundles, oriented G-fibrations, and representations of fundamental groupoids in a given groupoid $\mathscr{R}$ over $\mathscr{O}_G$. The methods and some of the results are due to Waner [**29, 30**], who gave the equivariant generalization of the nonequivariant classification theory developed by May [**21, 22**]. Since our methods are quite similar to those in the cited references, we will not give complete details of all proofs.

21. The classification of G-bundles and spherical G-fibrations

Our theory of orientations is based on the bundles of groupoids over $\mathscr{O}_G$ that are described in §§1–4. Here the term "bundle" refers to bundles with discrete fibers, in accordance with the definitions of §5. These groupoids over $\mathscr{O}_G$ are obtained by passage to equivalence classes of maps from much richer categories over $\mathscr{O}_G$. For example, when $G = e$, $\mathscr{V}(n)$ is just $\pi_0(O(n))$, regarded as a category with a single object $\mathbb{R}^n$. To classify bundles, we need to use $O(n)$ itself. The equivariant situation is similar. We will use Roman letters for categories that correspond in this way to some of the categories specified by script letters in §18. We continue to write π for the functors from these larger categories to $\mathscr{O}_G$.

Definition 21.1. Define $V_G(n)$ to be the category over $\mathscr{O}_G$ whose objects are the objects of $\mathscr{V}_G(n)$ but whose morphisms $G \times_H V \longrightarrow G \times_K W$ are the maps of G-vector bundles, topologized with the function space topology. Passage to homotopy classes of maps gives a functor $\omega : V_G(n) \longrightarrow \mathscr{V}_G(n)$ over $\mathscr{O}_G$. Define $Top_G(n)$ and $F_G(n)$ similarly, using maps of topological G-bundles and of spherical G-fibrations.

The functor $\pi : V_G(n) \longrightarrow \mathscr{O}_G$ is given by passage to base spaces, and it is continuous. Since $\omega \circ \pi = \pi$, it is clear that $\omega : V_G(n) \longrightarrow \mathscr{V}_G(n)$ is also continuous, where $\mathscr{V}_G(n)$ is topologized as in Proposition 4.1. In fact, by compactness, it follows that the topologies on morphism spaces specified there must coincide with the quotient topologies induced from the topologies on the morphism spaces of $V_G(n)$. Similarly, the functors

$$\omega : Top_G(n) \longrightarrow \mathscr{T}op_G(n) \quad \text{and} \quad \omega : F_G(n) \longrightarrow \mathscr{F}_G(n)$$

are continuous, although in these cases we do not know whether or not the quotient topology coincides with the topologies defined as in Proposition 4.1.

The following theorem gives the relevant special cases of Waner's general classification theorems [**29**] for equivariant bundles and fibrations. In particular, it implies that $BV_G(n)$ is equivalent to the Grassmann classifying G-space $BO_G(n, U)$ specified in Definition 19.5.

Theorem 21.2. *The G-space $BV_G(n)$ classifies G-vector bundles of dimension n. That is, for G-CW complexes X, $[X, BV_G(n)]_G$ is in natural bijective correspondence with the set of equivalence classes of G-n-plane bundles over X. Similarly $BTop_G(n)$ classifies locally linear topological G-bundles of dimension n and $BF_G(n)$ classifies locally linear sectioned spherical G-fibrations of dimension n.*

Here the term "locally linear" refers to our restriction to fibres homeomorphic to $G \times_H V$ or fiber homotopy equivalent to $G \times_H S^V$ for some subgroup H of G and H-representation V. The term "sectioned" refers to our use of fibrations with based fibers whose basepoints give a canonical section.

There is an earlier and perhaps more conceptual proof of the classification theorem that applies to G-vector bundles and topological G-bundles but not to spherical G-fibrations. We describe it in the case of G-vector bundles. For a topological structure group Π, such as $\Pi = O(n)$ or $\Pi = Top(n)$, a principal (G, Π)-bundle is a Π-free $(G \times \Pi)$-space. We think of G as acting from the left and Π as acting from the right, the two actions commuting. Just as in nonequivariant bundle theory, every n-dimensional G-vector bundle $p : E \longrightarrow X$ has an associated principal $(G, O(n))$-bundle $\pi : P \longrightarrow X$, and equivalence classes of G-n-plane bundles over X are in bijective correspondence with equivalence classes of principal $(G, O(n))$-bundles. The analogue for topological G-bundles also holds.

There is a standard construction of a universal principal (G, Π)-bundle $\pi : E_G(\Pi) \longrightarrow B_G(\Pi)$ in terms of the universal $\mathscr{F}$-space for a well chosen family $\mathscr{F}$ of subgroups of $G \times \Pi$.

Definition 21.3. Let $\mathscr{F}_G(\Pi)$ be the family of subgroups Λ of $G \times \Pi$ such that $\Lambda \cap \Pi = \{e\}$. Such a subgroup Λ has the form $\{(h, \lambda(h)) \mid h \in H\}$ for some subgroup $H \subset G$ and homomorphism $\lambda\colon H \to \Pi$. Define

$$E_G(\Pi) = E\mathscr{F}_G(\Pi) \quad \text{and} \quad B_G(\Pi) = E\mathscr{F}_G(\Pi)/\Pi$$

and let $\pi : E_G(\Pi) \longrightarrow B_G(\Pi)$ be obtained by passage to orbits.

The G-map π is the universal principal (G,Π)-bundle; see [**13**, VII§2], for example.

Theorem 21.4. *For G-CW complexes X, $[X, B_G(\Pi)]_G$ is in natural bijective correspondence with the set of equivalence classes of principal (G,Π)-bundles over X.*

Remarks 21.5. (i) When $\Pi = O(n)$, we view a homomorphism $\lambda : H \longrightarrow O(n)$ as specifying an action of H on $\mathbb{R}^n$, and we denote this representation by $V(\lambda)$. Thus groups Λ in $\mathscr{F}_G(O(n))$ correspond to objects $p(\lambda) : G \times_H V(\lambda) \longrightarrow G/H$ of the category $V_G(n)$. This is the beginning of a direct comparison between our two ways of classifying G-n-plane bundles.
(ii) When $\Pi = Top(n)$, we must modify the definition of $\mathscr{F}_G(\Pi)$ to account for our restriction to linear fibers. Viewing subgroups of $G \times O(n)$ as subgroups of $G \times Top(n)$, we here take $\mathscr{F}_G(Top(n))$ to be the family of subgroups of $G \times Top(n)$ that is generated under conjugation by the subgroups in the family $\mathscr{F}_G(O(n))$. See [**17**, §1] for background on the relevant G-bundle theory.

22. The classification of oriented G-bundles

In this section, we give a classification theorem for oriented G-bundles, starting from Theorem 21.4. We treat spherical G-fibrations in the next section. We assume throughout that our base G-spaces X have the G-homotopy types of G-CW complexes. For definiteness, fix a representation $S\colon \mathscr{S} \longrightarrow \mathscr{V}_G(n)$. We could use other of the target categories displayed in (18.1) or (19.1). We have in mind the universal orientable representation $(\mathscr{S}\mathscr{V}_G(n), S)$. However, $(\mathscr{S}, S)$ need not be universal for the theory of this section, and the generality is likely to have applications in the study of restricted types of G-bundles.

We have the notion of an orientation $(F,\phi) : (\Pi_G B, p^*) \longrightarrow (\mathscr{S}, S)$ of a G-n-plane bundle $p : E \longrightarrow B$. We know what it means for a map of G-bundles to be orientation preserving, and we can pull back orientations along G-bundle maps; see Definition 7.9 and Lemma 7.10. We shall classify equivalence classes of oriented G-bundles under the relation given by orientation preserving equivalence of bundles. That is, we shall construct a G-vector bundle $p\colon E_G(O(n), S) \longrightarrow B_G(O(n), S)$ together with an orientation

$$\mu = (F,\phi)\colon (\Pi_G B_G(O(n), S), p^*) \longrightarrow (\mathscr{S}, S)$$

such that (p,μ) is universal in the sense that pullback of bundles and orientations along G-maps specifies a bijection from the set $[X, B_G(O(n), S)]_G$ of G-homotopy classes of G-maps $X \longrightarrow B_G(O(n), S)$ to the set of equivalence classes of oriented G-n-plane bundles over X. We begin by giving a diagrammatic description of orientations. Recall Definition 21.3 and Remark 21.5(i).

Definition 22.1. We define a discrete $\mathscr{O}_{G\times O(n)}$-space T_S. Let $T_S(G\times O(n)/\Lambda)$ be empty if $\Lambda \notin \mathscr{F}_G(O(n))$ and be the set of orientations of $G\times_H V(\lambda) \longrightarrow G/H$ in $(\mathscr{S}, S)$ if $\Lambda \in \mathscr{F}_G(O(n))$. As is easily checked from Lemma 2.3, a $(G\times O(n))$-map from $G\times O(n)/\Lambda$ to $G\times O(n)/\Lambda'$ determines and is determined by a map of G-vector bundles from $G\times_H V(\lambda) \longrightarrow G/H$ to $G\times_{H'} V(\lambda') \longrightarrow G/H'$. Thus, by pullback of orientations, T_S is a well-defined contravariant functor on $\mathscr{O}_{G\times O(n)}$.

Remark 22.2. Taking $\mathscr{S} = \mathscr{S}\mathscr{V}_G(n)$, Proposition 2.9 and Example 2.10 show that $T_S(G\times O(n)/\Lambda)$ is non-empty if G is finite and $\Lambda \in \mathscr{F}_G(O(n))$, but that this fails for general compact Lie groups G.

We have the following basic observation. See for example [**16**, Thm. 12] for the analysis of the fixed point spaces of principal $(G, O(n))$-bundles and their base spaces which is used in its proof.

Lemma 22.3. *Let $p\colon E \longrightarrow B$ be a G-n-plane bundle and let $\pi\colon P \longrightarrow B$ be its associated principal $(G, O(n))$-bundle. An orientation (F,ϕ) of p in $(\mathscr{S}, S)$ determines and is determined by a map of $\mathscr{O}_G$-spaces $\theta\colon \Phi P \longrightarrow T_S$.*

Proof. We have $E = P\times_{O(n)} \mathbb{R}^n$ and $B = P/O(n)$. Let $x\in B^H$. If $\pi(y) = x$, then $y \in P^\Lambda$ for some $\Lambda = \{(h,\lambda(h)) \mid h\in H\}$; Λ and thus λ are determined by x only up to $O(n)$-conjugacy. There is a unique choice such that the corresponding bundle $p(\lambda) : G\times_H V(\lambda) \longrightarrow G/H$ is in the skeletal category $V_G(n)$, and $p^*(x) = p(\lambda)$. Regarding x as a G-map $G/H \longrightarrow B$, the specification of p^* on morphisms depends further on a fixed choice of isomorphism between the pullback of p along x and the bundle $p(\lambda)$. Such an isomorphism is given by a bundle map $\tilde{x}\colon G\times_H V(\lambda) \longrightarrow E$ over x, and the choice of such a map is equivalent to the choice of $y\in P^\Lambda$. Two choices of y differ by right action by an element τ in the centralizer $O(n)^\lambda$ of λ, and the corresponding maps $\tilde{x}$ differ by precomposition with the bundle automorphism of $G\times_H V(\lambda)$ determined by τ. The choice of retraction $\bar{\mathscr{V}}_G(n) \longrightarrow \mathscr{V}_G(n)$ in Definition 2.2 fixes $\tilde{x}$ and thus y.

A map $\theta\colon \Phi P \longrightarrow T_S$ of $\mathscr{O}_G$-spaces is given by a natural choice of maps constant on components from the spaces P^Λ to the sets of orientations of the bundles $p(\lambda)$. Write (F_y, ϕ_y) for the orientation assigned to the component of y. Thus F_y is a functor $\Pi_G(G/H) \longrightarrow \mathscr{S}$ over $\mathscr{O}_G$ and ϕ_y is an isomorphism $S\circ F_y \longrightarrow p(\lambda)^*$ over $\mathscr{O}_G$. We specify the corresponding orientation (F,ϕ) of p by letting the functor $F\colon \Pi_G B \longrightarrow \mathscr{S}$ be given on objects $x = \pi(y) \in B^H$ by $F(x) = F_y(\mathrm{id}_{G/H})$. The specification of F on morphisms and the specification of the isomorphism $\phi\colon S\circ F \longrightarrow p^*$ over $\mathscr{O}_G$ are similarly determined by the choices above together with the functors F_y and isomorphisms ϕ_y specified by θ. The details of the verification that this gives a bijective correspondence are tedious but straightforward. □

Theorem 22.4. *There exists a universal oriented bundle (p,μ). The G-n-plane bundle $p : E_G(O(n), S) \longrightarrow B_G(O(n), S)$ is characterized as follows. If $p\colon E \longrightarrow B$ is a G-n-plane bundle with associated principal $(G, O(n))$-bundle $\pi\colon P \longrightarrow B$, then p admits an orientation μ in $(\mathscr{S}, S)$ such that (p,μ) is universal if and only if P is homotopically discrete and the diagram $\pi_0(\Phi P)$ is isomorphic to T_S.*

Proof. Define $P_G(O(n), S) = \Psi T_S$ and $B_G(O(n), S) = P_G(O(n), S)/O(n)$, and let

$$E_G(O(n), S) = P_G(O(n), S) \times_{O(n)} \mathbb{R}^n.$$

With the evident projections p and π, this gives a G-n-plane bundle whose associated principal $(G, O(n))$-bundle has the prescribed fixed point structure. By the lemma, the discretization map $\delta : \Phi P_G(O(n), S) \longrightarrow T_S$ gives p an orientation μ. Moreover, $P_G(O(n), S)$ is a $(G \times O(n))$-CW complex. The adjunction (20.1) and use of $(G \times O(n))$-CW approximation imply that any other principal $(G, O(n))$-bundle P with the stated fixed point structure is weakly G-equivalent to $P_G(O(n), S)$, and the lemma implies that its associated G-n-plane bundle has a compatible orientation. To check universality, suppose given an oriented G-n-plane bundle over a G-CW complex X and let Q be its associated principal $(G, O(n))$-bundle. By the lemma, the orientation is given by a map $\theta : \Phi Q \longrightarrow T_S$ of $\mathscr{O}_G$-spaces. By the cited adjunction, there results a $(G \times O(n))$-map $Q \longrightarrow P_G(O(n), S)$. Passage to base G-spaces from this map of principal $(G, O(n))$-bundles gives the required classifying G-map $X \longrightarrow B_G(O(n), S)$, and the rest is routine. □

Corollary 22.5. *There is a G-map $f : B_G(O(n), S) \longrightarrow B_G(O(n))$ that represents the forgetful functor from oriented G-n-plane bundles to G-n-plane bundles.*

Proof. Since there is a unique function from any set to a point, there is an evident natural map of discrete $\mathscr{O}_G$-spaces $T_S \longrightarrow T_{\mathscr{F}_G(O(n))}$. We obtain f by first applying Ψ and then passing to orbits over $O(n)$ □

Remark 22.6. We can obtain the analogous results for locally linear topological G-bundles in exactly the same fashion, replacing $O(n)$ by $Top(n)$ and interpreting $\mathscr{F}_G(Top(n))$ as in Remark 21.5(i).

23. The classification of oriented spherical G-fibrations

We now turn to the case of spherical G-fibrations, where we follow the methods of [**21, 22, 29, 30**]. As in the cited references, slight modifications give alternative treatments of the cases of G-vector bundles and topological G-bundles.

We fix n and a representation $S\colon \mathscr{S} \longrightarrow \mathscr{F}_G(n)$; we could instead use the corresponding category of stable or virtual fibrations over orbits. Define a *sphere space* to be a G-map $p\colon E \longrightarrow B$ such that, for each $b \in B^H$, $p\colon Gp^{-1}(b) \longrightarrow Gb$ is a sectioned G-fibration that is fiber G-homotopy equivalent to $G \times_H S^V$ for some subgroup H of G and representation V of H. A map $(\tilde{f}, f)\colon p \longrightarrow p'$ of sphere spaces is a pair of maps $f\colon B \longrightarrow B'$ and $\tilde{f}\colon E \longrightarrow E'$ such that $p' \circ \tilde{f} = f \circ p$ and each $\tilde{f}\colon Gp^{-1}(b) \longrightarrow G(p')^{-1}(fb)$ is a section-preserving fiber G-homotopy equivalence. Define an *S-sphere space* to be a sphere space such that each G-fibration $p\colon Gp^{-1}(b) \longrightarrow Gb$ has a given orientation μ_b in $(\mathscr{S}, S)$; we do not assume any compatibility among these orientations at this point. A map of S-sphere spaces is a map of sphere spaces such that each $\tilde{f}\colon Gp^{-1}(b) \longrightarrow G(p')^{-1}(fb)$ preserves orientation. Thus, given $(\tilde{f}, f)$ and the orientations of the fibers of p', there are unique orientations of the fibers of p such that $(\tilde{f}, f)$ is a map of S-sphere spaces.

An S-sphere space $p\colon E \longrightarrow B$ is said to be an *S-fibration* if it satisfies the S-covering homotopy property (S-CHP): Given an S-sphere space $q\colon D \longrightarrow A$, a G-homotopy $h\colon A\times I \longrightarrow B$, and a G-map $\tilde{h}_0\colon D \longrightarrow E$ such that $(\tilde{h}_0, h_0)$ is a map of S-sphere spaces, there is a G-homotopy $\tilde{h}$ that starts at $\tilde{h}_0$ and covers h and is such that $(\tilde{h}, h)$ is a map of S-sphere spaces. Here $q\times I\colon D\times I \longrightarrow A\times I$ has the evident structure of S-sphere space determined by that of q and thus, via $(\tilde{h}_0, h_0)$, by that of p. We have the following consistency observation.

Lemma 23.1. *An S-sphere space $p\colon E \longrightarrow B$ is an S-fibration if and only if p is a G-fibration with an orientation in $(\mathscr{S}, S)$ that induces the given orientations of the restrictions $p\colon Gp^{-1}(b) \longrightarrow Gb$.*

Proof. If p is a G-fibration with an orientation in $(\mathscr{S}, S)$, then the ordinary G-CHP immediately implies the S-CHP; any covering homotopy is automatically a map of S-spaces because the orientations of orbits of points connected by paths in the fixed point spaces of B are compatible. Conversely, the S-CHP obviously implies the G-CHP, and it also implies that the orientations of orbits have the compatibility on paths required to specify an orientation of p. $\square$

Definitions 23.2. Define a topological category $\mathscr{F}_G(n, S)$ over $\mathscr{O}_G$ as follows. Its objects are pairs (x, μ), where x is an object of $\mathscr{F}_G(n)$ and μ is an orientation of x in $(\mathscr{S}, S)$. Its morphisms are the maps of S-fibrations, with the function space topology. Neglect of orientation gives a functor $f : \mathscr{F}_G(n, S) \longrightarrow \mathscr{F}_G(n)$ over $\mathscr{O}_G$. We define several functors on $\mathscr{F}_G(n, S)$, and we abbreviate notation by writing $\mathscr{F} = \mathscr{F}_G(n, S)$.

(i) Define a covariant functor $\mathscr{E} = \mathscr{E}_G(n, S)$ from $\mathscr{F}$ to G-spaces by sending an object (x, μ) to the total space of x and sending a map in $\mathscr{F}$ to the underlying map of total spaces.

(ii) Define a covariant functor $\mathscr{B} = \mathscr{B}_G(n, S)$ from $\mathscr{F}$ to G-spaces by sending an object (x, μ) to the base space of x and sending a map in $\mathscr{F}$ to the underlying map of base spaces. Thus, with the notation of §20, $\mathscr{B} = \iota\circ\pi$.We have an evident natural transformation $p\colon \mathscr{E} \longrightarrow \mathscr{B}$.

(iii) Given an S-sphere space $p\colon E \longrightarrow B$, define a contravariant functor $\mathscr{P}E$ from $\mathscr{F}$ to spaces by sending $\alpha = (x, \mu)$ to the space of maps of S-fibrations $x \longrightarrow p$; $\mathscr{P}E$ is specified on morphisms by precomposition. This is to be viewed as a principalization construction. We define $\mathscr{P}B$ by sending α to the space of maps on base spaces of maps of S-fibrations $x \longrightarrow p$, and we have the evident natural map $\mathscr{P}E \longrightarrow \mathscr{P}B$. For each α, $\mathscr{P}E(\alpha) \longrightarrow \mathscr{P}B(\alpha)$ is a nonequivariant fibration (as in [**29**, I.3.3]).

(iv) For each object $\alpha = (x, \mu)$ of $\mathscr{F}$, define a covariant functor $\mathscr{E}[\alpha]$ from $\mathscr{F}$ to spaces by sending an object β to the space $\mathscr{F}(\alpha, \beta)$ of maps $\alpha \longrightarrow \beta$ in $\mathscr{F}$; $\mathscr{E}[\alpha]$ is specified on morphisms by postcomposition. There is an analogous functor $\mathscr{B}[\alpha]$ that sends β to the space of maps on base spaces of morphisms in $\mathscr{F}(\alpha, \beta)$, and there is an evident natural map $p[\alpha]\colon \mathscr{E}[\alpha] \longrightarrow \mathscr{B}[\alpha]$.

Theorem 23.3. *The G-space $B\mathscr{F}_G(n, S) = B(*, \mathscr{F}_G(n, S), \mathscr{B}_G(n, S))$ classifies S-fibrations. That is, the set of equivalence classes of S-fibrations over a G-CW* *complex X is in natural bijective correspondence with the set $[X, B\mathscr{F}_G(n,S)]_G$. The G-map $Bf : B\mathscr{F}_G(n,S) \longrightarrow B\mathscr{F}_G(n)$ represents the forgetful functor from equivalence classes of S-fibrations to equivalence classes of spherical G-fibrations.*

Proof. We sketch the argument, referring the reader to [**21**, §9] and [**29**, §2] for more detailed accounts of analogous proofs. The latter source gives the proof of Theorem 21.2, and we are simply elaborating the argument to take account of the orientations. We abbreviate $\mathscr{F} = \mathscr{F}_G(n,S)$, etc. For a sphere space $p\colon E \longrightarrow X$, there is a natural way to construct a spherical G-fibration $\Gamma p\colon \Gamma E \longrightarrow X$ and a natural map $\eta\colon p \longrightarrow \Gamma p$ of sphere spaces; η is a fiber G-homotopy equivalence if p is a G-fibration, in which case a fiber homotopy inverse $\xi\colon \Gamma p \longrightarrow p$ is essentially a path lifting function. See [**21**, §3] and [**29**, 1.2]. If p is an S-sphere space, then Γp is an S-sphere space and η is a map of S-sphere spaces. The map p is said to be a *G-quasifibration* if each p^H is a quasifibration, and the map $\eta\colon E \longrightarrow \Gamma E$ is then a weak G-equivalence (see [**23**] for a modernized treatment of quasifibrations). We say that p is an S-quasifibration if it is a G-quasifibration with an orientation in $(\mathscr{S}, S)$. (While the pullback of a quasifibration need not be a quasifibration, the restriction of a sphere space over an orbit is a G-fibration, and it therefore makes sense to talk about orientations of sphere spaces.)

For any contravariant functor $\mathscr{Y} : \mathscr{F} \longrightarrow \mathscr{U}$, the canonical map

$$p = B(\mathrm{id}, \mathrm{id}, p)\colon B(\mathscr{Y}, \mathscr{F}, \mathscr{E}) \longrightarrow B(\mathscr{Y}, \mathscr{F}, \mathscr{B})$$

is a G-quasifibration. Moreover, it has a canonical orientation in $(\mathscr{S}, S)$. In fact, since G acts only on the $\mathscr{E}$ and $\mathscr{B}$ coordinates, each orbit in the base space is a copy of the base space of a particular object (x, μ) of $\mathscr{F}$, and the inverse image of that orbit is a copy of the total space of x. Remembering μ, we see that we have canonical orientations of the restrictions of p to orbits. Because the morphisms of $\mathscr{F}$ are orientation preserving, these orientations of orbits satisfy the requisite compatibility to specify an orientation of p in $(\mathscr{S}, S)$.

We claim that the S-fibration

$$\Gamma p\colon \Gamma B(\mathscr{S}, \mathscr{F}, \mathscr{E}) \longrightarrow B(\mathscr{S}, \mathscr{F}, \mathscr{B})$$

is universal. Pulling p back along G-maps, we obtain a natural transformation Ψ from the functor $[X, B(\mathscr{S}, \mathscr{F}, \mathscr{B})]_G$ to the functor that assigns to X the set of equivalence classes of S-fibrations over X, and our claim is that Ψ is a natural bijection. We construct an inverse natural transformation Φ. Let $p\colon E \longrightarrow X$ be an S-fibration. We have a pair of maps

$$X \xleftarrow{\varepsilon} B(\mathscr{P}E, \mathscr{F}, \mathscr{B}) \xrightarrow{q} B(*, \mathscr{F}, \mathscr{B}) = B\mathscr{F}.$$

Here q is induced by the unique natural map $\mathscr{P}E \longrightarrow *$ and ϵ is induced by the composite $\mathscr{P}E \otimes_{\mathscr{F}} \mathscr{B} \longrightarrow \mathscr{P}B \otimes_{\mathscr{F}} \mathscr{B} \longrightarrow X$, where the second arrow is given by evaluation of maps of base spaces. We claim that ϵ is a weak G-equivalence. Granting this, our assumption that X is a G-CW complex ensures that there is a map $g\colon X \longrightarrow B(\mathscr{P}E, \mathscr{F}, \mathscr{B})$, unique up to G-homotopy, such that $\epsilon \circ g \simeq \mathrm{id}$. We define $\Phi(p)$ to be the homotopy class of $f = q \circ g$.

We first verify our claim about ϵ, then verify that $\Psi\Phi$ is the identity transformation, and finally verify that $\Phi\Psi$ is a natural automorphism; it follows formally that $\Phi\Psi$ must be the identity.

For each object α of $\mathscr{F}$ and any $\mathscr{Y}$, the canonical map

$$\epsilon\colon B(\mathscr{Y},\mathscr{F},\mathscr{E}[\alpha]) \longrightarrow \mathscr{Y}(\alpha)$$

induced by the evident evaluation map $\mathscr{Y}\otimes_{\mathscr{F}}\mathscr{E}[\alpha] \longrightarrow \mathscr{Y}(\alpha)$ is a homotopy equivalence by the argument of [**22**, Prop. 9.9]. We apply this with $\mathscr{Y} = \mathscr{P}E$ for an S-fibration $p\colon E \longrightarrow X$ to obtain a map of fibrations

$$\begin{tikzcd}
B(\mathscr{P}E,\mathscr{F},\mathscr{E}[\alpha]) \arrow[r, "\varepsilon"] \arrow[d] & \mathscr{P}E(\alpha) \arrow[d] \\
B(\mathscr{P}E,\mathscr{F},\mathscr{B}[\alpha]) \arrow[r, "\varepsilon"] & \mathscr{P}B(\alpha).
\end{tikzcd}$$

The map ε of total spaces is an equivalence, fibers map by equivalences, and thus the map ε of base spaces is a weak equivalence. If $\alpha = (x,\mu)$ and x has base orbit G/H, then $\mathscr{P}B(\alpha)$ is a subspace of B^H; similarly, $B(\mathscr{P}E,\mathscr{F},\mathscr{B}[\alpha])$ is a subspace of $B(\mathscr{P}E,\mathscr{F},\mathscr{B})^H$. Since the maps ϵ are weak equivalences for all α, a little analysis of fixed point spaces shows that $\epsilon^H : B(\mathscr{P}E,\mathscr{F},\mathscr{B})^H \longrightarrow X^H$ is a weak equivalence for all H; see [**29**, 2.3.2]. This verifies our claim that ϵ is a weak G-equivalence.

Now consider $\Psi\Phi(p)$, $p\colon E \longrightarrow X$. We must check that if $f = q\circ g$ is the classifying G-map that we have constructed, then the pullback of the universal S-fibration along f is equivalent to p. The following schematic diagram gives the idea.

$$\begin{tikzcd}
E \arrow[r] \arrow[dd] & \Gamma E \arrow[dd] & & \Gamma B(\mathscr{P}E,\mathscr{F},\mathscr{E}) \arrow[ll, "\tilde\varepsilon"'] \arrow[rr, "\tilde q"] \arrow[dd, "\Gamma p"] & & \Gamma B(*,\mathscr{F},\mathscr{E}) \arrow[dd, "\Gamma p"] \\
 & & g^{-1}(\Gamma p) \arrow[ul, "H_1"'] \arrow[ur, "\tilde g"] \arrow[rr, "K"] \arrow[dd] & & f^{-1}(\Gamma p) \arrow[ur, "\tilde f"'] \arrow[dd] & \\
X \arrow[r, equal] & X \arrow[dr, equal] & & B(\mathscr{P}E,\mathscr{F},\mathscr{B}) \arrow[ll, "\varepsilon"'] \arrow[rr, "q"] & & B(*,\mathscr{F},\mathscr{B}) \\
 & & X \arrow[rr, equal] & & X \arrow[ur, "f=q\circ g"'] &
\end{tikzcd}$$

Here K is given by the universal property of the pullback $f^{-1}(\Gamma p)$ and H_1 is obtained at the end of a homotopy which starts at $\tilde\epsilon\circ\tilde g$ and covers any homotopy from $\epsilon\circ g$ to the identity; H exists since $\Gamma E \longrightarrow X$ satisfies the S-CHP.

Finally, to consider $\Phi\Psi(f)$, $f\colon X \longrightarrow B(\mathscr{S},\mathscr{F},\mathscr{B})$, we consider the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[rr, bend left, "\Phi\Psi(f)"] & B(\mathscr{P}f^{-1}\Gamma B(*,\mathscr{F},\mathscr{E}),\mathscr{F},\mathscr{B}) \arrow[l, "\varepsilon"] \arrow[r, "q"] \arrow[d, "B(\mathscr{P}\tilde{f},\mathrm{id},\mathrm{id})"] & B\mathscr{F} \arrow[d, equal] \\
B\mathscr{F} & B(\mathscr{P}\Gamma B(*,\mathscr{F},\mathscr{E}),\mathscr{F},\mathscr{B}) \arrow[l, "\varepsilon"] \arrow[r, "q"] & B\mathscr{F}.
\end{tikzcd}$$

We have already noted that the arrows ϵ are weak G-equivalences. The maps q are G-quasifibrations, and we claim that the fibers of the bottom map q are (nonequivariantly) weakly contractible. It follows that q is also a weak G-equivalence; there is no problem of equivariance since G acts only on $\mathscr{B}$. By the G-Whitehead theorem, the two weak G-equivalences of the bottom row induce automorphisms of represented functors on G-CW complexes. We conclude from the diagram that the composite $\Phi\Psi$ is an automorphism of the functor $[X, B\mathscr{F}]_G$, as required.

To see that the fibers of the bottom map q are weakly contractible, observe first that $\mathscr{P}\Gamma E(\alpha)$ is weakly equivalent to $\mathscr{P}E(\alpha)$ for any S-quasifibration $p\colon E \longrightarrow X$. Observe next that there is an identification

$$(\mathscr{P}B(\mathscr{Y},\mathscr{F},\mathscr{E}))(\alpha) \cong B(\mathscr{Y},\mathscr{F},\mathscr{E}[\alpha])$$

for any object α of $\mathscr{F}$ and any $\mathscr{Y}$; we have observed that the right side is equivalent to $\mathscr{Y}(\alpha)$. Taking $\mathscr{Y}$ to be the trivial functor $*$, we see that $(\mathscr{P}B(*,\mathscr{F},\mathscr{E}))(\alpha)$ is contractible. This implies the conclusion. $\square$

24. Moore loops and the classification of representations

In this slightly digressive final section, we prove an analogue of Theorem 21.2 for representations of fundamental groupoids. Contrary to our previous conventions, unless otherwise specified we let G be any topological group, not necessarily compact Lie, here. Fix a groupoid $\mathscr{R}$ over $\mathscr{O}_G$. We may as well assume that $\mathscr{R}$ is skeletal. Recall the definition of a representation from Definition 7.1 and restrict attention to representations $R : \Pi_G X \longrightarrow \mathscr{R}$. We say that two such representations R and R' are isomorphic if there is an isomorphism $\phi : R \longrightarrow R'$ over $\mathscr{O}_G$. In terms of the definition of a map of representations in Definition 7.1, we are requiring the functor on the domain category $\Pi_G X$ of R and R' to be the identity. The following result generalizes [**2**, 3.8]. Rigorously, we should only claim it as a conjecture since, except in the discrete case, we have not filled in the details of one step of the proof (see Lemma 24.8 below).

Theorem 24.1. *The G-space $B\mathscr{R}$ classifies representations of $\Pi_G X$ in $\mathscr{R}$. That is, for G-CW complexes X, $[X, B\mathscr{R}]_G$ is in natural bijective correspondence with the set of isomorphism classes of representations $R : \Pi_G X \longrightarrow \mathscr{R}$.*

When G is compact Lie, we have a G-map $B\omega : BV_G(n) \longrightarrow B\mathscr{V}_G(n)$, and similarly for topological G-bundles and spherical G-fibrations. The following expected comparison is checked by comparing the proofs of Theorems 21.2 and 24.1.

Corollary 24.2. *The G-map $B\omega : BV_G(n) \longrightarrow B\mathscr{V}_G(n)$ represents the natural transformation that sends a G-n-plane bundle $p : E \longrightarrow B$ to the representation $p^* : \Pi_G B \longrightarrow \mathscr{V}_G(n)$. The G-maps*

$$B\omega : BTop_G(n) \longrightarrow B\mathscr{T}op_G(n) \quad \text{and} \quad B\omega : BF_G(n) \longrightarrow B\mathscr{F}_G(n)$$

represent the analogous natural transformations on topological G-bundles and spherical G-fibrations.

We shall use the Moore loop category of X to prove Theorem 24.1. This category is related to $\Pi_G X$ as $V_G(n)$ is related to $\mathscr{V}_G(n)$ and is of independent interest.

Definition 24.3. Let X be a G-space. The *Moore loop category* $\Lambda_G X$ is the category whose objects are the G-maps $x\colon G/H \longrightarrow X$ and whose morphisms $x \longrightarrow y$, $y : G/K \longrightarrow X$, are the triples (λ, r, α), where $\alpha\colon G/H \longrightarrow G/K$ is a G-map, $r \geqslant 0$ is a real number, and $\lambda : G/H \times [0,r] \longrightarrow X$ is a path of length r from x to $y \circ \alpha$ in X^H. Composition is induced by concatenation of paths and addition of real numbers; paths of length zero give identity morphisms. Regard paths as defined on $[0,\infty]$ by letting them be constant on $[r,\infty]$ and topologize the set of maps $x \longrightarrow y$ as a subspace of the space of maps $[0,\infty] \longrightarrow X^H$. Thus the space of self-maps of an object x over the identity map of G/H is the Moore loop space $\Lambda(X^H, x)$. Let $\pi\colon \Lambda_G X \longrightarrow \mathscr{O}_G$ be the functor given by $\pi(x) = G/H$ and $\pi(\omega, \alpha) = \alpha$.

Again, the functor π is continuous. We have used paths of varying length to obtain a category, but this makes the construction of a functor $\omega : \Lambda_G X \longrightarrow \Pi_G X$ awkward, especially in view of the use of paths of length zero. One way to proceed is to first extend paths of length r to paths on $[-1,r]$ that are constant on $[-1,0]$, next to use the evident linear isomorphisms $[-1,r] \longrightarrow [0,1]$ to rescale paths to paths defined on $[0,1]$, and finally to pass to equivalence classes of paths. This gives a continuous functor ω, and it induces a G-map $B\omega : B\Lambda_G X \longrightarrow B\Pi_G X$.

Remark 24.4. A quite different topologization of $\Lambda_G X$ is studied in [**24**]. It is used to construct a G-map $f : X \longrightarrow Y$, where Y is a kind of "$K(\Pi_G X, 1)$", namely a G-space Y such that $f_* : \Pi_G X \longrightarrow \Pi_G Y$ is an equivalence of categories and each component of each Y^H is a $K(\pi, 1)$.

The proof of Theorem 24.1 will be a direct application of two basic results about Moore loop categories. The first generalizes the classical nonequivariant weak equivalence $X \simeq B\Lambda X$ for connected spaces X [**21**, 14.3].

Proposition 24.5. *For G-spaces X, there is a natural weak G-equivalence between X and $B\Lambda_G X$. When X is a G-CW complex, there is a weak G-equivalence $\zeta : X \longrightarrow B\Lambda_G X$ that is natural up to G-homotopy. If G is a compact Lie group, ζ is a G-homotopy equivalence.*

Proof. Nonequivariantly, the Moore path space $PX = P(X, x)$ has a right action of the Moore loop space $\Lambda X = \Lambda(X, x)$ and, when X is path connected, there is a natural weak equivalence

$$X \xleftarrow{\varepsilon} B(PX, \Lambda X, *) \xrightarrow{q} B(*, \Lambda X, *) = B\Lambda X.$$

Here ε is induced by the endpoint evaluation map $p : PX \longrightarrow X$ and q is induced by the trivial map $PX \longrightarrow *$. The equivariant generalization is precisely parallel. There is a contravariant Moore path functor $P_G X : \Lambda_G X \longrightarrow \mathscr{U}$. For an object $x : G/H \longrightarrow X$ of $\Lambda_G X$, $P_G X(x)$ is the Moore path space $P(X^H, x)$, points of which may be viewed as G-maps $G/H \times [0, \infty] \longrightarrow X$. For a map $(\lambda, r, \alpha) : x \longrightarrow y$, $y : G/K \longrightarrow X$, and a path $(\mu, s) \in P(X^K, y)$,

$$P_G X(\lambda, r, \alpha)(\mu, s) = ((\mu \circ \alpha) \cdot \lambda, r + s) \in P(X^H, x).$$

We construct a natural weak G-equivalence

$$X \xleftarrow{\varepsilon} B(P_G X, \Lambda_G X) \xrightarrow{q} B(*, \Lambda_G X) = B\Lambda_G X.$$

The G-maps $P_G X(x) \times G/H \longrightarrow X$ given by evaluation of paths at cosets gH and end-point evaluation give rise to a G-map $P_G X \otimes_{\Lambda_G X} \iota \circ \pi \longrightarrow X$ that induces the G-map ε. The natural map from $P_G X$ to the trivial functor $*$ induces the G-map q. Since the map $P_G X \longrightarrow *$ is a spacewise equivalence, q is a weak G-equivalence. We must show that ε is a weak G-equivalence. This means that the H-fixed map

$$\varepsilon^H : B(P_G X, \Lambda_G X)^H \longrightarrow X^H$$

is a weak equivalence for any G-space X and any $H \subset G$. Recall that

$$B(P_G X, \Lambda_G X)^H = B(P_G X, \Lambda_G X, \mathscr{O}_G(G/H, \pi(-)))$$

and consider the following diagram:

$$\begin{tikzcd}
\coprod_{[x]} \Lambda(X^H, x) \arrow[r, equal] \arrow[d] & \coprod_{[x]} \Lambda(X^H, x) \arrow[d] \\
\coprod_{[x]} B(P_G X, \Lambda_G X, \Lambda_G X(x, -)) \arrow[r, "\simeq"] \arrow[d] & \coprod_{[x]} P(X^H, x) \arrow[d] \\
B(P_G X, \Lambda_G X, \mathscr{O}_G(G/H, \pi(-))) \arrow[r, "\epsilon^H"] & X^H.
\end{tikzcd}$$

The disjoint unions are taken over the isomorphism classes of objects in the fiber $(\Lambda_G X)_{G/H}$ over G/H, or equivalently over the components of X^H. It is standard that the right column is a fibration, and the left column is a quasifibration by the argument of [**21**, 7.6]. By another standard argument [**21**, 7.5], the space $B(P_G X, \Lambda_G X, \Lambda_G X(x, -))$ is contractible, and so is $P(X^H, x)$. This implies that ε^H is a weak equivalence.

When X is a G-CW complex, the composite of ε and a G-CW approximation $\gamma : \Gamma B(P_G X, \Lambda_G X) \longrightarrow B(P_G X, \Lambda_G X)$ is a G-homotopy equivalence. Choosing an inverse and composing with $q \circ \gamma$, we obtain the required weak G-equivalence $\zeta : X \longrightarrow B\Lambda_G X$. If G is a compact Lie group, results of Waner [**28**] imply that $B\Lambda_G X$ has the homotopy type of a G-CW complex, so that ζ is a G-homotopy equivalence by the G-Whitehead theorem. $\square$

The second basic result about the Moore loop category generalizes the discrete special case of the classical nonequivariant weak equivalence $G \simeq \Omega BG$ for topological groups G [**21**, 8.7].

Proposition 24.6. *Let $\pi : \mathscr{R} \longrightarrow \mathscr{O}_G$ be a groupoid over $\mathscr{O}_G$. Then there is an equivalence $\xi : \Pi_G B\mathscr{R} \longrightarrow \mathscr{R}$ of groupoids over $\mathscr{O}_G$, natural up to isomorphism over $\mathscr{O}_G$.*

Proof. We mimic the nonequivariant argument in [**21**, 8.7]. We define a natural equivalence $\mu\colon \mathscr{R} \to \Pi B\mathscr{R}$ over $\mathscr{O}_G$ and let ξ be any chosen inverse to μ. Note that we are working here with our original fundamental groupoid $\Pi_G B\mathscr{R}$ based on paths of length one. The G-space of zero simplices of $B\mathscr{R}$ is the disjoint union over objects $x \in \mathscr{R}$ of the orbit G-spaces G/H, where $\pi(x) = G/H$. We let $\mu(x)$ be the point $1_x \equiv eH$ in the orbit corresponding to x. Writing 1 for the unique point in the standard 0-simplex, we can write $\mu(x) = |*[\,]1_x;1|$. Write $(t, 1-t)$, $0 \leqslant t \leqslant 1$, for the points in the standard 1-simplex. If $\omega\colon x \longrightarrow y$ is a morphism of $\mathscr{R}$ with $\pi(\omega) = \alpha : G/H \longrightarrow G/K$, let $\mu(\omega)$ be the homotopy class of the path $t \mapsto |*[\omega]1_x;(t,1-t)|$ from $\mu(x)$ to $\mu(y)\circ\alpha$. This is a morphism in $\Pi_G B\mathscr{R}$. We see that μ is a functor by using the way in which 2-simplices in $B\mathscr{R}$ are attached to the 1-skeleton.

To show that μ is an equivalence of groupoids over $\mathscr{O}_G$, it suffices to show that

$$\mu\colon \mathscr{R}(x,y) \longrightarrow \Pi_G B\mathscr{R}(\mu(x),\mu(y)) \tag{24.7}$$

is a homeomorphism for each pair of objects x and y of $\mathscr{R}$. We believe that the following lemma holds in general, but our proof is only complete when G is discrete.

Lemma 24.8. *$\pi(\mathscr{R}(x,y)) = \pi(\Pi_G B\mathscr{R}(\mu(x),\mu(y)))$ in $\mathscr{O}_G(\pi(x),\pi(y))$.*

Proof. Since $\pi\circ\mu = \pi$, $\pi(\mathscr{R}(x,y)) \subset \pi(\Pi_G B\mathscr{R}(\mu(x),\mu(y)))$. Let $\pi(x) = G/H$ and fix $\alpha : \pi(x) \longrightarrow \pi(y)$. We must show that if there is a path in $(B\mathscr{R})^H$ that connects $\mu(x)$ to $\mu(y)\circ\alpha$, then there is a map $\omega : x \longrightarrow y$ in $\mathscr{R}$ such that $\pi(\omega) = \alpha$. Using a cellular approximation argument, if there is such a path, then it can be deformed to a path in the simplicial 1-skeleton of $(B\mathscr{R})^H$. If G is discrete, this skeleton is a graph, and paths in it are equivalent to reduced finite edge paths. The edges are of the form $\mu(\nu)$ or $\mu(\nu)^{-1}$ for morphisms ν of $\mathscr{R}$. Using source lifting and divisibility, we can deduce the result by induction on the number of edges. We believe that the covering property of Definition 5.1(i) can be used to adapt the argument to general topological groups G, but we have not worked out the details. □

In view of the lemma and Proposition 5.4, to prove that the map μ of (24.7) is a homeomorphism in general, it suffices to prove that, for each object x of $\mathscr{R}$, the restriction of μ to a map

$$\mathrm{Aut}_{\mathscr{R}}(x) \longrightarrow \mathrm{Aut}_{\Pi_G B\mathscr{R}}(\mu(x))$$

of discrete groups is a bijection and thus an isomorphism. Consider the diagram

$$\begin{tikzcd}
\mathrm{Aut}(x) \arrow[r, "\zeta"] \arrow[d] & \Omega B(*, \mathscr{R}, \pi\mathscr{R}(x,-)) \arrow[d] \\
B(*, \mathscr{R}, \mathscr{R}(x,-)) \arrow[r, "\nu"] \arrow[d] & PB(*, \mathscr{R}, \pi\mathscr{R}(x,-)) \arrow[d] \\
B(*, \mathscr{R}, \pi\mathscr{R}(x,-)) \arrow[r, equal] & B(*, \mathscr{R}, \pi\mathscr{R}(x,-)).
\end{tikzcd}$$

The (ordinary) loop and path spaces on the right use paths based at $\mu(x)$, and the group of components of the displayed loop space is $\mathrm{Aut}_{\Pi_G B\mathscr{R}}(\mu(x))$. The maps ζ and ν are defined by $\zeta(\omega)(t) = |*[\omega]1_{\omega(0)};(t,1-t)|$ and

$$\nu(|*[\omega_n,\ldots,\omega_1]\omega_0;u|)(t) = |*[\omega_n,\ldots,\omega_1,\omega_0]1_{\omega_0(0)};(tu,1-t)|,$$

where $u \in \Delta_n$. The right column of the diagram is a fibration, and the left column is a quasifibration, as in [**21**, 7.6]. Both total spaces are contractible, so ζ is a weak equivalence. This implies that $\mu = \pi_0(\zeta)\colon\ \mathrm{Aut}_{\mathscr{R}}(x) \to \mathrm{Aut}_{\Pi_G B}(\mu(x))$ is a bijection, as required. □

Proof of Theorem 24.1. We define inverse isomorphisms Ψ and Φ between the two functors of X in the statement. A G-map $f : X \longrightarrow B\mathscr{R}$ induces the representation $\Psi(f)$ specified as the composite

$$\Pi_G X \xrightarrow{f_*} \Pi_G B\mathscr{R} \xrightarrow{\xi} \mathscr{R}.$$

A representation $R : \Pi_G X \longrightarrow \mathscr{R}$ induces the G-map $\Phi(R)$ specified as the composite

$$X \xrightarrow{\zeta} B\Pi_G X \xrightarrow{BR} B\mathscr{R}.$$

By the definitions and the naturality of ξ,

$$\Psi\Phi(R) = \xi \circ BR_* \circ \zeta_* = R \circ \xi \circ \zeta_*.$$

This representation is isomorphic to R since $\xi \circ \zeta_* : \Pi_G X \longrightarrow \Pi_G X$ is isomorphic over $\mathscr{O}_G$ to the identity functor. Therefore $\Psi\Phi$ is the identity functor. Similarly, by the naturality of ζ,

$$\Phi\Psi(f) = B\xi \circ Bf_* \circ \zeta = B\xi \circ \zeta \circ f.$$

Since ξ is an isomorphism over $\mathscr{O}_G$, $B\xi$ is a weak G-equivalence, as is ζ. Therefore $B\xi \circ \zeta : B\mathscr{R} \longrightarrow B\mathscr{R}$ is a weak G-equivalence and the composite $\Phi\Psi$ is an automorphism. It follows formally that this automorphism must be the identity. □

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