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CORES OF SPACES, SPECTRA, AND E_∞ RING SPECTRA

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(*communicated by Gunnar Carlsson*)

Abstract

In a paper that has attracted little notice, Priddy showed that the Brown-Peterson spectrum at a prime p can be constructed from the p-local sphere spectrum S by successively killing its odd dimensional homotopy groups. This seems to be an isolated curiosity, but it is not. For any space or spectrum Y that is p-local and (n_0-1)-connected and has $\pi_{n_0}(Y)$ cyclic, there is a p-local, (n_0-1)-connected "nuclear" CW complex or CW spectrum X and a map $f : X \to Y$ that induces an isomorphism on π_{n_0} and a monomorphism on all homotopy groups. Nuclear complexes are atomic: a self-map that induces an isomorphism on π_{n_0} must be an equivalence. The construction of X from Y is neither functorial nor even unique up to equivalence, but it is there. Applied to the localization of MU at p, the construction yields BP.

In 1999, the third author gave an April Fool's talk on how to prove that BP is an E_∞ ring spectrum or, in modern language, a commutative S-algebra. As explained in [**20**], he gave a quite different April Fool's talk on the same subject two years earlier. His new idea was to exploit the remarkable paper of Stewart Priddy [**23**], in which Priddy constructed BP by killing the odd degree homotopy groups of the sphere spectrum. The hope was that by mimicking Priddy's construction in the category of commutative S-algebras, one might arrive at a construction of BP as a commutative S-algebra. As the first two authors discovered, that argument fails. However, the ideas are still interesting. As we shall explain, Priddy's construction of BP is not an accidental fluke but rather a special case of a very general construction. The elementary space and spectrum level construction is given in Section 1. The more sophisticated E_∞ ring spectrum analogue and its specialization to MU are discussed in Section 2.

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1. Cores of spaces and spectra

To set context, we begin by recalling some standard properties of spaces and spectra that still have not been fully explored. The following successively stronger conditions are studied in Wilkerson [**24**]. We assume that all spaces and spectra are p-local for a fixed prime p and of the homotopy types of p-local CW objects of finite type. Thus we require that each $\pi_n(Y)$ be a finitely generated $\mathbb{Z}_{(p)}$-module. Spaces are to be based and simply connected and spectra are to be bounded below. Unless otherwise specified, cohomology is to be taken with mod p coefficients.

Definition 1.1. Let X be a space or a spectrum, as above.

(i) X is *indecomposable* if X admits no non-trivial product decomposition.

(ii) X is *irreducible* if it admits no non-trivial retracts.

(iii) X is H^**-prime* if, for a map $f : X \longrightarrow X$, either $f^* : H^*(X) \longrightarrow H^*(X)$ is an isomorphism or $f^* : H^q(X) \longrightarrow H^q(X)$ is nilpotent for each $q > 0$.

The term "indecomposable" is suggested by the analogy with module theory. Clearly an H^*-prime space is irreducible and an irreducible space is indecomposable. Additional hypotheses on X which ensure that irreducible implies H^*-prime are given in [**24**, 3.4, 3.5]. Since retracts of spectra split in the stable category, where finite wedges are equivalent to finite products, indecomposable spectra are irreducible. An elementary space level analogue is that if A is a retract of a co-H-space X, with $A \longrightarrow X$ a cofibration, then X is equivalent to $A \vee X/A$.

Definition 1.2. Assume further that X is $(n_0 - 1)$-connected, where we assume henceforward that $n_0 \geqslant 2$ in the case of spaces, and that $\pi_{n_0}(X)$ is a cyclic module over $\mathbb{Z}_{(p)}$.

(i) X is *atomic* if a map $f : X \longrightarrow X$ that induces an isomorphism on $\pi_{n_0}(X)$ is an equivalence.

(ii) X is H^**-monogenic* if $H^*(X)$ is a cyclic algebra (in the case of spaces) or module (in the case of spectra) over the Steenrod algebra.

If X is H^*-monogenic, then X is atomic. If X is atomic, then X is indecomposable. If the entire image of the Hurewicz homomorphism of X is concentrated in the Hurewicz dimension, then X is atomic. We are interested in an especially rigid type of atomic space or spectrum. In general, when studying atomic spaces or spectra, it seems most natural to work with p-complete rather than just p-local objects, but this paper is concerned with cellular constructions, which work more naturally in the p-local context. Thus we consider p-local CW objects (spaces or spectra) of finite type, and we agree to call such CW objects "complexes" throughout. Let S^n denote a p-local n-sphere. Such spheres are the domains of the attaching maps of our complexes. In the case of spaces, we require attaching maps to be based. Thus, if X is a complex, then its $(n+1)$-skeleton is the cofiber of a map $j_n : J_n \longrightarrow X_n$, where J_n is a wedge of finitely many copies of S^n.

Priddy used the term "irreducible" for a version of the following concept, but it seems more sensible to reserve that term for the standard notion defined above.

Definition 1.3. A complex is *nuclear* of Hurewicz dimension n_0 if its $(n_0 - 1)$-skeleton is trivial, its n_0-skeleton is S^{n_0}, it has finitely many n-cells for each $n > n_0$, and

$$\operatorname{Ker}(j_{n*} : \pi_n(J_n) \longrightarrow \pi_n(X_n)) \subset p \cdot \pi_n(J_n) \tag{1.4}$$

for each $n \geqslant n_0$. When $n = n_0$, this implies that J_{n_0} is either $*$ or S^{n_0}. Thus the attaching maps of X are detected by mod p homotopy. If $\eta : S^{n_0} \longrightarrow X$ is the inclusion, it induces an epimorphism $\eta_* : \pi_{n_0}(S^{n_0}) \longrightarrow \pi_{n_0}(X)$.

The following result is based on a proposition of Priddy [**23**, §1].

Proposition 1.5. *A nuclear complex is atomic.*

Proof. Let X be nuclear and let $f : X \longrightarrow X$ be a map that induces an isomorphism on π_{n_0}. We must prove that f is a homotopy equivalence. We may assume that f is cellular, and we prove that f restricts to a homotopy equivalence $X_n \longrightarrow X_n$ for all n. Thus assume inductively that $f : X_n \longrightarrow X_n$ is a homotopy equivalence. This holds trivially if $n < n_0$ and is easily checked if $n = n_0$. We deduce that $f : X_{n+1} \longrightarrow X_{n+1}$ is a homotopy equivalence. Take homology with coefficients in $\mathbb{Z}_{(p)}$. It suffices to prove that

$$f_* : H_q(X_{n+1}) \longrightarrow H_q(X_{n+1})$$

is an isomorphism for $q = n$ and $q = n+1$. It is easy to check (using the Freudenthal suspension theorem) that f induces a map from the cofiber sequence

$$J^n \longrightarrow X_n \longrightarrow X_{n+1}$$

to itself. There results a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(X_{n+1}) \arrow[r] \arrow[d, "f_*"] & H_n(J_n) \arrow[r] \arrow[d, "f_*"] & H_n(X_n) \arrow[r] \arrow[d, "f_*", "\cong"'] & H_n(X_{n+1}) \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & H_{n+1}(X_{n+1}) \arrow[r] & H_n(J_n) \arrow[r] & H_n(X_n) \arrow[r] & H_n(X_{n+1}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By the five lemma, it suffices to prove that

$$f_* : H_n(J_n) \longrightarrow H_n(J_n)$$

is an isomorphism. By the Hurewicz theorem, it suffices to prove that

$$f_* : \pi_n(J_n) \longrightarrow \pi_n(J_n)$$

is an isomorphism. We have a commutative diagram with exact rows

$$\begin{tikzcd}
\pi_n(J_n) \arrow[r] \arrow[d, "f_*"] & \pi_n(X_n) \arrow[r] \arrow[d, "f_*", "\cong"'] & \pi_n(X_{n+1}) \arrow[r] \arrow[d, "f_*"] & 0 \\
\pi_n(J_n) \arrow[r] & \pi_n(X_n) \arrow[r] & \pi_n(X_{n+1}) \arrow[r] & 0.
\end{tikzcd}$$

The right arrow f_* is an epimorphism by the diagram and is therefore an isomorphism since any epimorphic endomorphism of a finitely generated module over a

PID is an isomorphism. It follows that the right arrow is an isomorphism in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker j_{n*} \arrow[r, "i"] \arrow[d] & \pi_n(J_n) \arrow[r] \arrow[d, "f_*"] & \operatorname{im} j_{n*} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \ker j_{n*} \arrow[r, "i"] & \pi_n(J_n) \arrow[r] & \operatorname{im} j_{n*} \arrow[r] & 0
\end{tikzcd}$$

In view of (1.4), the inclusion i becomes 0 after tensoring over $\mathbb{Z}/p$. Therefore $f_* \otimes \mathbb{Z}/p$ is an isomorphism. This implies that f_* is an isomorphism. □

The following construction is a generalization of Priddy's construction [**23**] of the Brown-Peterson spectrum by killing the odd dimensional homotopy groups of the (p-local) sphere spectrum. He was motivated by the fact that the homotopy groups of MU are concentrated in even degrees, but MU played no role in his actual construction. We change the point of view. We consider a preassigned space or spectrum Y under a sphere S^{n_0}, and we kill the homotopy groups of the kernel of the given map $S^{n_0} \longrightarrow Y$.

Construction 1.6. Let Y be $n_0 - 1$ connected with a given map $\eta : S^{n_0} \longrightarrow Y$. We construct a nuclear complex X together with a map $f : X \longrightarrow Y$ under S^{n_0} that induces a monomorphism on all homotopy groups. We start with $X_{n_0} = S^{n_0}$ and $f_{n_0} = \eta : X_{n_0} \longrightarrow Y$. Assume inductively that we have constructed X_n and a map $f_n : X_n \longrightarrow Y$ that induces a monomorphism on homotopy groups in dimension less than n. Choose a minimal (finite) set of generators for the kernel of $f_{n*} : \pi_n(X_n) \longrightarrow \pi_n(Y)$, let J_n be the wedge of a copy of S^n for each chosen generator, and let $j_n : J_n \longrightarrow X_n$ represent the chosen generators. Define X_{n+1} to be the cofiber of j_n. Choose a null homotopy h_n of the composite $f_n \circ j_n$ and use it to extend f_n to a map $f_{n+1} : X_{n+1} \longrightarrow Y$. The cofibration $X_n \longrightarrow X_{n+1}$ induces an isomorphism on π_i for $i < n$ and an epimorphism on π_n, and f_{n+1} induces a monomorphism on π_i for $i \leqslant n$. On passage to colimits, we obtain $f : X \longrightarrow Y$ that induces a monomorphism on all homotopy groups. The minimality of our chosen sets of generators ensures that (1.4) holds.

Note that spheres are obviously nuclear and a two cell complex (with cells in different dimensions) is either nuclear or a wedge (trivial attaching map). The obvious mechanism for a finite complex not to be nuclear is to have at least one cell with a trivial attaching map. Some later cell might attach both to this one and to another cell lower down. Construction 1.6 then gives a space with smaller homotopic groups and no non-trivial attaching maps. For example, with $Y = S^{n_0} \times S^n$, $n > n_0$, the construction just gives S^{n_0}.

The construction is most interesting when $\pi_{n_0}(Y)$ is cyclic. Here it shows that Y has a core, in the sense of the following definition.

Definition 1.7. Let Y be $n_0 - 1$ connected with $\pi_{n_0}(Y)$ cyclic. A *core of* Y *is a nuclear complex* X *together with a map* $f : X \longrightarrow Y$ *that induces an isomorphism on* π_{n_0} *and a monomorphism on all homotopy groups.*

The homotopy groups of the fiber Ff are then $\pi_q(Ff) \cong \pi_{q+1}(Y)/f_*\pi_{q+1}(X)$, and the Hurewicz dimension of Ff is at least n_0. However, Ff need not have a cyclic bottom homotopy group. For spectra, we can use cofibers rather than fibers, and this leads to the following inductive construction.

Construction 1.8. Let $Y = Y_0$ be an (n_0-1)-connected spectrum. We construct a *nuclear decomposition* of Y. Choose a (finite) minimal set of generators for $\pi_{n_0}(Y)$. Construct a nuclear complex X_0 and a map $f_0 : X_0 \longrightarrow Y_0$ from a representative map $\eta : S^{n_0} \longrightarrow Y_0$ for one of these generators. Let Y_1 be the cofiber of f_0. The remaining generators of $\pi_{n_0}(Y)$ give generators of $\pi_{n_0}(Y_1)$, and we can repeat the construction starting with Y_1. Iterating, we kill π_{n_0} after finitely many steps, and we then continue by killing π_{n_0+1} similarly. Iterating, we obtain nuclear complexes X_i and maps $f_i : X_i \longrightarrow Y_i$ with cofibers Y_{i+1} such that each f_i induces a monomorphism on all homotopy groups. The homotopy groups of Y_{i+1} are the quotients of the homotopy groups of Y_i by the images of the homotopy groups of X_i, and the colimit of the sequence of cofibrations $Y_i \longrightarrow Y_{i+1}$ is trivial.

These constructions are related to early work of Freyd, Margolis, and Wilkerson [**16, 19, 24**] on cancellation and unique decomposition of finite p-local or p-complete spectra and spaces. In the case of spaces, little seems to be known about which spaces admit factorizations as products of indecomposable spaces. The notion of an atomic space, which is due to Cohen, arose in connection with the work of Selick, Cohen, Neisendorfer, and Moore on exponents of homotopy groups. Some of the relevant papers are [**1, 6, 7, 8, 9, 11, 12, 13, 14, 25, 26**]. These papers identify many particular spaces that arise in applications as being atomic, examine the relationship between atomicity of spaces and atomicity of their loop spaces, and study the relationship of atomicity of spaces to the structure of their monoids of self-maps. There are many open questions about these concepts. For example, Cohen points out that it is not known whether or not the suspension spectra of $K(\mathbb{Z}/2, n)$ or of the $p-1$ wedge summands of $\Sigma K(\mathbb{Z}/p, n)$, p odd, are atomic. Our constructions raise many new questions. Here are a few general ones.

Questions 1.9. *Assume that Y is (n_0-1)-conected and $\pi_{n_0}(Y)$ is cyclic.*

(i) For which Y is the core of Y unique?

(ii) Can one classify the cores of Y?

(ii) Can one explicitly identify cores of some interesting spaces?

(iii) Is every atomic space equivalent to a nuclear complex?

The ideas so far are due to the third author, who tried hard to prove that the core of Y is unique. The first and second authors provided a spectrum level counterexample: see Example 1.17 below. The point is that, in Construction 1.6, there are many choices for the homotopy class of f_{n+1}, which differ by elements in $\mathrm{Im}([\Sigma J_n, Y] \longrightarrow [X_{n+1}, Y])$. Changing the choice can change the kernel that one is killing at the next stage, and different choices can lead to very different cores.

We also expect the answer to question (iii) to be no: it seems likely that nuclear complexes give a special class of atomic spaces. This is strongly suggested by the following example, which is due to Fred Cohen.

Example 1.10. A map $f : X \longrightarrow Y$ between atomic spaces that induces an isomorphism on the bottom homotopy group and a monomorphism on all homotopy groups need not be an equivalence. For a counterexample, take $Y = \Omega^2 S^5$ at the prime 2. Clearly $\pi_6(Y) = \pi_8(S^5) \cong \mathbb{Z}/8$ with generator ν. The class 4ν is detected in cohomology by a map $g : Y \longrightarrow K(\mathbb{Z}/2, 6)$, so that g induces an epimorphism on homotopy groups. Let $f : X \longrightarrow Y$ be the homotopy fiber of g. Then f induces an isomorphism on π_3 and a monomorphism on all homotopy groups. The space Y is atomic by [**11**]. Cohen [unpublished] has shown that X is also atomic.

Definition 1.11. Say that an atomic space Y is *minimal* if a map $f : X \longrightarrow Y$ from an atomic space X into Y that induces an isomorphism on the bottom homotopy group and a monomorphism on all homotopy groups is necessarily an equivalence.

Clearly, if Y is a minimal atomic space, then a core $f : X \longrightarrow Y$ is an equivalence. Thus minimal atomic spaces are equivalent to nuclear complexes.

Question 1.12. *Is every nuclear complex a minimal atomic space?*

Assume that Y is $(n_0 - 1)$-connected and that $\pi_{n_0}(Y)$ is cyclic in the following two lemmas. The first is immediate from the definitions. The second shows that the core of Y is unique under very restrictive hypotheses on Y.

Lemma 1.13. *If $g : Y \longrightarrow Z$ is a map that induces an isomorphism on π_{n_0} and a monomorphism on all homotopy groups and if $f : X \longrightarrow Y$ is a core of Y, then $g \circ f : X \longrightarrow Z$ is a core of Z.*

Lemma 1.14. *If the homotopy groups and p-local cohomology groups of Y are concentrated in even degrees, then the core of Y is a retract of Y and is unique.*

Proof. Let $f : X \longrightarrow Y$ be a core. Then $H^{n+1}(Y; \pi_n(X)) = 0$ for all n since the homotopy groups of X and the cohomology groups of Y are both concentrated in even degrees. By obstruction theory, there is a map $g : Y \longrightarrow X$ under S^{n_0}. The composite $g \circ f : X \longrightarrow X$ is an equivalence by Proposition 1.5. Thus X is a retract of Y, hence also has p-local cohomology groups concentrated in even degrees. If $f' : X' \longrightarrow Y$ is another core, there are no obstructions to constructing maps $i : X \longrightarrow X'$ and $j : X' \longrightarrow X$ under S^{n_0}. The composites $j \circ i$ and $i \circ j$ are equivalences by Proposition 1.5, hence X and X' are equivalent. $\square$

Since BP is irreducible, these lemmas imply a version of Priddy's result [**23**].

Proposition 1.15. *BP is the core of MU.*

Remark 1.16. In Lemma 1.14, we are only claiming that X is unique up to equivalence, not that a map $f : X \longrightarrow Y$ that identifies X as the core of Y is unique. For example, BP is the core of $BP \wedge BP$, as is displayed by both the left and the right unit maps $BP \longrightarrow BP \wedge BP$.

The conclusion of Lemma 1.14 fails if we drop the hypothesis about cohomology.

Example 1.17. The units of BP and $H\mathbb{Z}_{(p)}$ induce maps

$$BP \longrightarrow BP \wedge H\mathbb{Z}_{(p)} \longleftarrow H\mathbb{Z}_{(p)},$$

both of which induce monomorphisms on all homotopy groups and an isomorphism on π_0. Since BP and $H\mathbb{Z}_{(p)}$ are each their own cores, it follows from Lemma 1.13 that both are cores of $BP \wedge H\mathbb{Z}_{(p)}$.

The following analogue of Proposition 1.15 must be true, but even this does not seem to be quite trivial.

Conjecture 1.18. *For p odd, $BP\langle 1\rangle$ is the core of ku.*

Certainly the first non-zero positive dimensional homotopy group of a core X is $\pi_{2p-2}(X)$, since the first cell that we attach to S kills $\pi_{2p-3}(S)$.

Warning 1.19. We can construct cores similarly for integral or rational spaces or spectra, rather than just for p-local ones. However, these constuctions will not be compatible with the p-local construction. For example, the rational core of MU is S rather than BP, and it is unclear what the integral core of MU is. Whatever it is, it is unique by the proof of Lemma 1.14. Comparison with the versions of BP and MU in Boardman's papers [**2, 3**] may be of interest.

2. Cores of E_∞ ring spectra

The ideas of the previous section can be adapted to a variety of frameworks in which one has a notion of CW objects. We shall illustrate this by presenting the construction used in the third author's failed attempt to prove that BP is an E_∞ ring spectrum. The construction surely gives rise to new E_∞ ring spectra, but it is hard to analyze what they look like. We work in the context of [**15**], replacing E_∞ ring spectra by weakly equivalent commutative S-algebras. Again, we work p-locally. Recall from [**15**, VIII.2.2] that localizations of commutative S-algebras are commutative algebras over the p-local sphere spectrum, which we denote henceforward by S. We let S^n be a p-local cofibrant n-sphere in the category of S-modules.

Everything done for spectra in §1 could equally well and perhaps more sensibly have been done in the category $\mathscr{M}$ of S-modules. We let $\mathscr{C}$ denote the category of commutative S-algebras; our constructions below have evident analogues for non-commutative S-algebras (or A_∞-ring spectra). Note that S is cofibrant in $\mathscr{C}$ but not in $\mathscr{M}$; [**15**] explains how to deal with such homotopical details. We have a forgetful functor $\mathscr{C} \longrightarrow \mathscr{M}$ with left adjoint free functor $\mathbb{P} : \mathscr{M} \longrightarrow \mathscr{C}$. Let $\nu : X \longrightarrow \mathbb{P}X$ denote the unit of the adjunction. Let CX denote the cone on an S-module X and let $\iota : X \longrightarrow CX$ be the canonical inclusion.

Construction 2.1. Let R be a connective cofibrant commutative S-algebra whose unit $S \longrightarrow R$ induces an isomorphism on π_0 and whose homotopy groups are finitely generated $\mathbb{Z}_{(p)}$-modules. We construct a map of S-algebras $g : Q \longrightarrow R$, which we call a *core of R as a commutative S-algebra*, by inductively killing homotopy groups. Let $Q_0 = S$ and let $g_0 : Q_0 \longrightarrow R$ be the unit of R. Assume that we have constructed an S-algebra Q_n and a map of S-algebras $g_n : Q_n \longrightarrow R$. Let K_n be the wedge of one copy of S^n for each element in a chosen (finite) minimal set of generators for the kernel of $g_{n*} : \pi_n(Q_n) \longrightarrow \pi_n(R)$. Let $k_n : K_n \longrightarrow Q_n$ be a map of S-modules that realizes these generators. By minimality,

$$\mathrm{Ker}(k_{n*} : \pi_n(K_n) \longrightarrow \pi_n(Q^n)) \subset p \cdot \pi_n(K_n). \tag{2.2}$$

The induced map $\tilde{k}_n : \mathbb{P}K_n \longrightarrow Q_n$ of S-algebras gives Q_n a structure of $\mathbb{P}K_n$-algebra. Define

$$Q_{n+1} = \mathbb{P}CK_n \wedge_{\mathbb{P}K_n} Q_n.$$

Thus, as in [**15**, II.3.7], Q_{n+1} is the pushout of the diagram

$$\mathbb{P}CK_n \xleftarrow{\mathbb{P}\iota} \mathbb{P}K_n \xrightarrow{\tilde{k}_n} Q_n \tag{2.3}$$

in the category of commutative S-algebras. By construction, $g_n \circ k_n$ is null homotopic. Choose a null homotopy $h_n : CK_n \longrightarrow R$ and let $\tilde{h}_n : \mathbb{P}CK_n \longrightarrow R$ be the induced map of S-algebras. By the universal property of pushouts, there results a map $g_{n+1} : Q_{n+1} \longrightarrow R$ of S-algebras that restricts to g_n on Q_n. Define $Q = \mathrm{colim}\, Q_n$ and let $g : Q \longrightarrow R$ be the map of S-algebras obtained by passage to colimits from the g_n. By construction, the induced map of homotopy groups

$$g_* : \pi_*(Q) \longrightarrow \pi_*(R)$$

is a monomorphism of $\mathbb{Z}_{(p)}$-algebras.

Example 2.4. Arguing exactly as in Example 1.17, but with BP there replaced by MU since we do not know that BP is a commutative S-algebra, we obtain an explicit counterexample that shows that cores of commutative S-algebras are not unique. At $p = 2$, we will identify a core of MU in Theorem 2.12 below.

Recall that colimits of sequences of S-algebras are computed as the colimits of the underlying sequences of S-modules [**15**, VII.3.10]. By construction, Q is a cell S-algebra and is thus cofibrant. The analogue of Lemma 1.13 is obvious.

Lemma 2.5. *Let $h : R \longrightarrow T$ be a map of commutative S-algebras such that $h_* : \pi_*(R) \longrightarrow \pi_*(T)$ is a monomorphism. If $g : Q \longrightarrow R$ is a core of R, then $h \circ g$ is a core of T.*

If the homotopy groups of R are concentrated in even degrees, then so are the homotopy groups of Q. In particular, we are then killing all of the odd dimensional homotopy groups of S in our inductive construction of Q. In a small range of dimensions the homotopy groups of Q_{n+1} agree with those of the cofiber of $\tilde{k}_n$, as we see, for example, from the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*(\mathbb{P}K_n)}_{p,q}(\pi_*(S), \pi_*(Q_n)) \Longrightarrow \pi_{p+q}(Q_{n+1}) \tag{2.6}$$

of [**15**, IV.4.1]. Cores of commutative S-algebras are nuclear, in the following sense.

Definition 2.7. A commutative S-algebra Q is *nuclear* if $Q = \mathrm{colim}\, Q_n$ where $Q_0 = S$ and, inductively, Q_{n+1} is the pushout of a diagram of the form (2.3), where K_n is a wedge of finitely many copies of S^n and $k_n : K_n \longrightarrow Q_n$ is a map of S-modules that satisfies (2.2).

Definition 2.8. A connective commutative S-algebra Q whose unit induces an isomorphism on π_0 is *atomic* if any map of S-algebras $f : Q \longrightarrow Q$ is a weak equivalence.

It is plausible but not obvious that the analogue of Proposition 1.5 holds.

Conjecture 2.9. *A nuclear commutative S-algebra is atomic.*

It might also seem plausible that a core of an S-algebra R is also a core of its underlying S-module, but we shall see that that is false. We do have the following comparison, which is what remains of the third author's original program.

Proposition 2.10. *For any core $g : Q \longrightarrow R$ of commutative S-algebras, there exists a core $f : X \longrightarrow R$ of S-modules and a map $\xi : X \longrightarrow Q$ of S-modules such that $f = g \circ \xi$. In particular, ξ induces a monomorphism on homotopy groups.*

Proof. We construct a commutative diagram

$$\begin{tikzcd}
J_n \arrow[rr, "j_n"] \arrow[dr, "\mu_n"] \arrow[dd, "\iota"'] & & X_n \arrow[dr, "\xi_n"] \arrow[dd] & & \\
& \mathbb{P}K_n \arrow[rr, "\tilde{k}_n"] \arrow[dd, "\mathbb{P}\iota"] & & Q_n \arrow[dd] \arrow[dddr, "g_n"] & \\
CJ_n \arrow[rr] \arrow[dr, "\nu_n"'] & & X_{n+1} \arrow[dr, "\xi_{n+1}"] & & \\
& \mathbb{P}CK_n \arrow[rr] \arrow[drrr, "\tilde{h}_n"'] & & Q_{n+1} \arrow[dr, "g_{n+1}"] & \\
& & & & R.
\end{tikzcd}$$

The front part of the diagram displays underlying S-modules in our construction of the S-algebra core $g : Q \longrightarrow R$. The back square of the diagram is a pushout in $\mathscr{M}$ that will display the inductive step of a construction of an S-module core $f : X \longrightarrow R$ such that $f_n = g_n \circ \xi_n$ for a map $\xi_n : X_n \longrightarrow Q_n$ of S-modules that induces a monomorphism on π_q for $q \leqslant n$. We have $X_0 = S^0$ and $Q_0 = S$, and we let $\xi_0 : X_0 \longrightarrow Q_0$ be a weak equivalence of S-modules (a cofibrant approximation). We let $f_0 = \xi_0 \circ g_0$, where g_0 is the unit of R. Assume inductively that we have constructed $\xi_n : X_n \longrightarrow Q_n$ and let $f_n = g_n \circ \xi_n$. Let J_n be a wedge of copies of S^n, one for each element in a chosen minimal set of generators for the kernel of $(f_n)_* : \pi_n(X_n) \longrightarrow \pi_n(R)$, and let $j_n : J_n \longrightarrow X_n$ represent these generators. Recall that K_n is a wedge of copies of S^n, one for each element in a chosen minimal set of generators for the kernel of $(g_n)_* : \pi_n(Q_n) \longrightarrow \pi_n(R)$, and that $k_n : K_n \longrightarrow Q_n$ represents these generators. The cofiber Ck_n is the pushout in $\mathscr{M}$ of the diagram

$$CK_n \xleftarrow{\ \iota\ } K_n \xrightarrow{\ k_n\ } Q_n,$$

and the universal property of pushouts in $\mathscr{M}$ gives a canonical map

$$\rho : Ck_n \longrightarrow Q_{n+1},$$

which is a $(2n-1)$-equivalence by inspection or use of (2.6). Since $g_n \circ \xi_n = f_n$, $(\xi_n)_* : \pi_n(X_n) \longrightarrow \pi_n(Q_n)$ restricts to a homomorphism $\mathrm{Ker}(f_n)_* \longrightarrow \mathrm{Ker}(g_n)_*$. Choosing preimages in $\pi_n(K_n)$ of the images under $(\xi_n)_* \circ (j_n)_*$ of the generators of $\pi_n(J_n)$, we obtain a homomorphism $\pi_n(J_n) \longrightarrow \pi_n(K_n)$. We can realize this homomorphism by a map $\gamma_n : J_n \longrightarrow K_n$ such that the left square commutes up to homotopy in the diagram

$$\begin{tikzcd}
J_n \arrow[r, "j_n"] \arrow[d, "\gamma_n"'] & X_n \arrow[r] \arrow[d, "\xi_n"] & X_{n+1} \arrow[r] \arrow[d, "\xi'_{n+1}"] & \Sigma J_n \arrow[d, "\Sigma\gamma_n"] \\
K_n \arrow[r, "k_n"'] & Q_n \arrow[r] & Ck_n \arrow[r] & \Sigma K_n
\end{tikzcd}$$

A standard comparison of cofibration sequences argument in $\mathscr{M}$ gives a map ξ'_{n+1} such that the middle square commutes and the right square commutes up to homotopy. Moreover, ξ'_{n+1} induces an monomorphism on π_q for $q \leqslant n$, since in degree n it induces the inclusion $\mathrm{Im}(f_n)_* \longrightarrow \mathrm{Im}(g_n)_*$ (up to isomorphism). Define

$$\xi_{n+1} = \rho \circ \xi'_{n+1} : X_{n+1} \longrightarrow Q_{n+1}$$

and define μ_n and ν_n to be the evident composites

$$J_n \xrightarrow{\gamma_n} K_n \xrightarrow{\nu} \mathbb{P}K_n \quad \textit{and} \quad CJ_n \xrightarrow{C\gamma_n} CK_n \xrightarrow{\nu} \mathbb{P}CK_n.$$

Then the cube in our main diagram is a commutative diagram in $\mathscr{M}$. The composite $\tilde{h}_n \circ \nu_n$ in the diagram coincides with the composite $h_n \circ C\gamma_n$, which is a null homotopy of $f_n \circ j_n$. The map $f_{n+1} = g_{n+1} \circ \xi_{n+1}$ is induced by this null homotopy, in agreement with our inductive prescription of a core of the S-module R. Passing to colimits, we obtain the maps ξ and f of the conclusion. $\square$

Thus an S-algebra core has larger homotopy groups than the corresponding S-module core. In particular, with $R = MU$, for any S-algebra core $g : Q \longrightarrow MU$, we have a factorization of an S-module core $f : BP \longrightarrow MU$ as $g \circ \xi$ for a map $\xi : BP \longrightarrow Q$ of S-modules that induces a monomorphism on homotopy groups. It is easy to see that the lowest positive degree homotopy group of Q must be $2p-2$, so it seems reasonable to hope that ξ is an equivalence. However, that is false: ξ cannot be an equivalence, since that would contradict the following observation of the first two authors.

Proposition 2.11. *There is no map $g : BP \longrightarrow MU$ of commutative S-algebras.*

Proof. If there were such a map g, it would commute with units and so induce an isomorphism on π_0. Therefore, since BP is atomic, the composite of g and a splitting map $MU \longrightarrow BP$ would be a self-equivalence of BP, so that g would be the inclusion of a retract. The map $g_* : H_*(BP) \longrightarrow H_*(MU)$ on mod p homology would be a monomorphism that commutes with Dyer-Lashof operations. The Thom isomorphism $\theta : H_*(MU) \longrightarrow H_*(BU)$ commutes with Dyer-Lashof operations by a result of Lewis [**18**, IX.7.4]. Kochman [**17**] and Priddy [**22**] have computed the Dyer-Lashof operations in $H_*(BU)$ and thus on $H_*(MU)$. Write $H_*(MU) = P[a_i | \deg a_i = 2i]$, where a_i is the standard generator coming from $H_*(BU(1))$. If $p = 2$, then $Q^5(a_1) \equiv a_5$ mod decomposables, and, if $p > 2$, then $Q^p(a_{p-1}) \equiv a_{(p+1)(p-1)}$ mod decomposables, by [**22**] or [**10**, II.8.1]. Here a_{p-1} is in the image of $H_*(BP)$, but $H_*(BP)$ has no indecomposable elements in degree 10 if $p = 2$ or in degree $2(p+1)(p-1)$ if $p > 2$. □

So, if BP is not a core of MU, what is? The first two authors succeeded in answering this question when $p = 2$.

Theorem 2.12. *If $p = 2$, then $M(Sp/U)$ is a core of MU, regarded as a commutative S-algebra.*

Proof. We have an infinite loop map $Sp/U \longrightarrow BU$, and work of Lewis [**18**, IX] gives an E_∞-ring Thom spectrum $M(Sp/U)$, a map $\alpha : M(Sp/U) \longrightarrow MU$ of E_∞-ring spectra, and compatible Thom isomorphisms that commute with Dyer-lashof operations. It is standard that Sp/U has no 2-torsion and that $H_*(Sp/U)$, with coefficients in $\mathbb{Z}_{(2)}$ or $\mathbb{Z}/2$, is a polynomial algebra on generators of degrees $4r + 2$, $r \geqslant 0$, that map to generators of these degrees in $H_*(BU)$. By [**22**] or [**10**, II.8.1], $Q^{4r}(a_1) \equiv a_{2r+1}$ mod decomposables for $r \geqslant 1$ in $H_*(MU)$. It follows that Q^{4r} maps the generator of degree 2 in $H_*(M(Sp/U))$ to a generator in degree $4r + 2$. Thus, intuitively, the image of $H_*(M(Sp/U))$ is the smallest possible subalgebra of $H_*(MU)$ that contains a_1 and is closed under the Dyer-Lashof operations.

The composite of α with the canonical map $MU \longrightarrow BP$ induces an epimorphism on homology, hence a monomorphism on cohomology. A theorem of Milnor and Moore [**21**, 4.4] shows that $H^*(M(Sp/U))$ is a free $A/(\beta)$-module, and a theorem of Brown and Peterson [**4**] shows that $M(Sp/U)$ is a wedge of suspensions of copies of BP. Therefore α induces a monomorphism on homotopy groups since it induces a monomorphism on homology groups. By Lemma 2.5, the composite of α and a core $g : Q \longrightarrow M(Sp/U)$ of the commutative S-algebra $M(Sp/U)$ is a core of MU. It is clear by construction that b_1, 2-locally, must be in the image of g_*, and it follows by consideration of Dyer-Lashof operations that g induces an epimorphism on mod 2 homology and therefore on 2-local homology. Moreover, Lemmas 1.13 and 1.14 imply that BP is the core of $M(Sp/U)$ as an S-module. By Proposition 2.10, we can factor a core $BP \longrightarrow M(Sp/U)$ through g, giving a composite map, necessarily an equivalence

$$BP \longrightarrow Q \xrightarrow{g} M(Sp/U) \longrightarrow MU \longrightarrow BP.$$

Now BP is complex oriented, and the image of its orientation gives Q a complex orientation. Since Q is a 2-local commutative and associative ring spectrum, the complex orientation can be modified if necessary to give a 2-typical formal group law, and then there is a map of ring spectra $BP \longrightarrow Q$ that is compatible with the orientation. In particular, this gives Q a structure of BP-module spectrum. Enumerate the wedge summands $\Sigma^{n_i} BP$ of $M(Sp/U)$ so that $n_i \leqslant n_j$ if $i < j$, where $i \geqslant 1$. Via the unit of BP, each summand is determined by an element

$\mu_i \in \pi_{n_i}(M(Sp/U))$. We claim that the generators lift to elements $\nu_i \in \pi_{n_i}(Q)$. Using the BP-module structure, the ν_i determine maps $\Sigma^{n_i}BP \longrightarrow Q$ that together give a map $\nu : M(Sp/U) \longrightarrow Q$ such that $g \circ \nu \simeq \mathrm{id}$. This implies that g induces an epimorphism and therefore an isomorphism on homotopy groups.

Thus assume inductively that μ_i lifts to ν_i for $i < j$. Let N be the wedge of the $\Sigma^{n_i}BP$, $i < j$, choose a splitting map $\pi : M(Sp/U) \longrightarrow N$, and let L be its fiber. We may regard μ_i as an element of $\pi_{n_i}(L)$. Let K be the fiber of $\pi \circ g : Q \longrightarrow N$. Comparing fiber sequences, we obtain a map $f : K \longrightarrow L$. Inductively, g induces an isomorphism of homotopy groups in degrees less than n. Therefore K, like L, must be $(n_i - 1)$-connected. Now, since g induces an epimorphism on 2-local homology, so does f. Since the 2-local Hurewicz homomorphism for K and L is an isomorphism in degree n_j, we may lift μ_j to $\pi_{n_j}(K)$ and therefore to $\pi_{n_j}(Q)$. $\square$

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