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TEISI IN Ab

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Abstract

Teisi are certain higher-dimensional categorical structures proposed for doing non-abelian homotopical and homological algebra. I give a partial justification for this proposal by showing that in the abelian case teisi indeed reduce to chain complexes. I also show that the result holds for a much wider class of weaker higher-dimensional categorical structures, which however need to have strict identities. The main step in the proof is an elegant Eckmann-Hilton type argument.

1. Introduction

In 1987 Street, following Roberts, suggested that that n-cohomology should be developed using (weak) n-categories as coefficient objects [**19, 18**]. Part of this development will be by clarifying the notion of n-stack [**12, 14, 3, 4**]. In 1995, Gordon, Power and Street made significant progress by proving a coherence theorem for tricategories, showing that they are triequivalent to **Gray**-categories [**13**]. For higher dimensions, I introduced *teisi* and I conjectured that weak 4-categories are weak equivalent to 4-dimensional teisi [**10**].

Teisi also come into the picture on the homotopical side. In 1977 Brown and Higgins introduced crossed complexes, generalizing crossed modules, and proved a generalized Van Kampen theorem for them [**5, 6**]. Crossed complexes don't model all homotopy types, but Carrasco and Cagarra's hypercrossed complexes, from 1991, do [**8**]. Now teisi are to hypercrossed complexes what categories are to groupoids: the former are the directed version of the latter. Alternatively, teisi are to (strict) ω-categories what hypercrossed complexes are to crossed complexes: the former allow non-trivial Whitehead products whereas the latter don't, but nothing more.

In 1998, during one of my talks in Sydney [**11**], George Janelidze asked what teisi are in the category of abelian groups. This paper answers that question by showing that in Ab, teisi reduce to ω-groupoids, and hence to chain complexes. This gives a useful and practical gauge point for future developments in non-abelian homological and homotopical algebra: one can now reasonably expect the abelian version of non-abelian methods to reduce to known methods, such as derived functors, resolutions,

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spectral sequences, etc., or to trivialize, if they deal with inherently non-abelian issues. I think that the third alternative, that of a non-abelian method which gives something new even in the abelian case, is highly unlikely, though not entirely impossible.

The main step in the proof that teisi in Ab are ω-groupoids is an Eckmann-Hilton type argument. This is not surprising as teisi ultimately generalize groups, although the argument is a little bit more involved because not all identities are units. The argument expresses composition in terms of the group operations, with the consequence that composition must be dimension preserving. In fact, abelianness of the group operations is only needed to prove the interchange axiom, so teisi in the category of groups reduce to *no interchange ω-groupoids*. Also, the proof only needs the elementary axioms for teisi: that n-source and n-target of an n-composite are the n-source of the first one and n-target of the second one respectively, which makes teisi a categorical structure, and that composing something with an identity results in (the identity on) the same thing. In this paper I call structures satisfying these elementary axioms *pre-teisi*. Thus, pre-teisi in Ab also reduce to ω-groupoids.

In order to formulate a generalization of the result to weaker structures, I introduce what I call *diagrammatic globular sets*. I show that they are precisely well-formed loop-free pasting schemes [**16**] in which every cell is a globe, which makes that I can talk about composites of *globular diagrams*, and I show that diagrammatic globular sets are precisely globular cardinals [**20**], which means that they occur as arities of operations in weak ω-categories [**2**]. Another advantage of diagrammatic globular sets is that the conditions are more efficient than for globular cardinals or for simple ω-graphs [**17**], that is, it costs less time to check them in practical examples.

I show that for any ω-groupoid in Ab an extra operation of arity a diagrammatic globular set A is trivial if it satisfies the following condition: if all cells except one of A are realized by identities then the result of the operation is equal to (the identity on) the realization of that single cell. The main step in the proof is again an Eckmann-Hilton type argument, using the above condition to express the new operation in terms of the group operations. The previous result is then used to relate it to composition. Calling a weak ω-category *normalized* if all its operations satisfy this condition, the conclusion is that normalized weak ω-categories in Ab also reduce to ω-groupoids.

Interestingly, the condition of having strict identities is also what makes the many-sorted and one-sorted definitions of ω-categories equivalent. This suggests that it could be relevant for a higher-dimensional structure's usefulness for non-abelian homotopical and homological algebra to be definable one-sorted. In particular, it might be useful to start considering a one-sorted version of weak ω-categories.

2. Preliminaries

Definition 2.1 (Brown-Higgings [7, p. 372–373]) An *ω-category* consists of a set C together with operations $s_k, t_k : C \to C$, also written d_k^-, d_k^+, for $0 \leq k < \omega$ and $*_m : C \;{}_{s_m}\!\times_{t_m} C \to C$ for $0 \leq m < \omega$, such that:

(i) $d_k^\alpha d_\ell^\beta = \begin{cases} d_k^\alpha & \text{if } k < \ell \\ d_\ell^\beta & \text{if } k \geq \ell \end{cases}$
for $\alpha, \beta = \pm$ and all k, ℓ,

(ii) $d_k^\alpha(c' *_m c) = \begin{cases} d_k^-(c) & \text{if } k = m \text{ and } \alpha = - \\ d_k^+(c') & \text{if } k = m \text{ and } \alpha = + \\ d_k^\alpha(c') *_m d_k^\alpha(c) & \text{if } k \neq m \end{cases}$
for all $c, c' \in C$, $\alpha = \pm$ and all k, m,

(iii) $d_m^+(c) *_m c = c = c *_m d_m^-(c)$ for all $c \in C$, all m,

(iv) (associativity) $c'' *_m (c' *_m c) = (c'' *_m c') *_m c$ for all $c, c', c'' \in C$, all m,

(v) (interchange) $(d' *_n d) *_m (c' *_n c) = (d' *_m c') *_n (d *_m c)$ for all $c, c', d, d' \in C$, all $m < n$,

(vi) there exists k such that $d_k^-(c) = c = d_k^+(c)$ for all $c \in C$. $\diamond$

Street omits condition (vi) [**19**, p. 305], and Brown and Higgins use '∞-category' having used 'ω-category' for something else; categorical usage has now firmly settled on the terminology given here.

Lemma 2.2 (Brown-Higgings [7, p. 376–377]) *An ω-category is a reflexive globular set $((C_i)_{i\in\omega}, \{d_k^-, d_k^+ : C_i \to C_k\}_{k<i}, \{\mathrm{id}_- : C_i \to C_{i+1}\}_i)$ together with operations $\#_m : C_i \;{}_{s_m}\!\times_{t_m} C_i \to C_i$ for $m < i$, such that:*

(i) $d_k^\alpha(c' \#_m c) = \begin{cases} d_k^-(c) & \textit{if } k = m \textit{ and } \alpha = - \\ d_k^+(c') & \textit{if } k = m \textit{ and } \alpha = + \\ d_k^\alpha(c') \#_m d_k^\alpha(c) & \textit{if } k > m \end{cases}$
for all $c, c' \in C$, $\alpha = \pm$ and all k, m,

(ii) $\mathrm{id}^{i-m}_{d_m^+(c)} \#_m c = c = c \#_m \mathrm{id}^{i-m}_{d_m^-(c)}$ *for all $c \in C_i$, all m,*

(iii) $\mathrm{id}_{c' \#_m c} = \mathrm{id}_{c'} \#_m \mathrm{id}_c$ *for all $c, c' \in C$,*

(iv) *(associativity) $c'' \#_m (c' \#_m c) = (c'' \#_m c') \#_m c$ for all $c, c', c'' \in C$, all m,*

(v) *(interchange) $(d' \#_n d) \#_m (c' \#_n c) = (d' \#_m c') \#_n (d \#_m c)$ for all $c, c', d, d' \in C$, all $m < n$.* $\square$

The relation with the one-sorted definition is obtained by identifying elements with their identities.

In an ω-category it is possible to define composition of elements of different dimension by taking the appropriate identity on one of them. However, it is better to see this 'whiskering' as basic operation, and to introduce an extra axiom saying it can be obtained in the way just described.

Lemma 2.3 *An ω-category is a reflexive globular set $((C_i)_{i\in\omega}, \{d_k^-, d_k^+ : C_i \to C_k\}_{k<i}, \{\mathrm{id}_- : C_i \to C_{i+1}\}_i)$ together with operations $\#_m : C_q \;{}_{s_m}\!\times_{t_m} C_p \to C_{\max\{p,q\}}$ for $m < \min\{p, q\}$, such that:*

(i) $d_k^\alpha(c' \#_m c) = \begin{cases} d_k^-(c) & \textit{if } k = m \textit{ and } \alpha = - \\ d_k^+(c') & \textit{if } k = m \textit{ and } \alpha = + \\ d_k^\alpha(c') \#_m d_k^\alpha(c) & \textit{if } \min\{p,q\} > k > m \\ c' \#_m d_k^\alpha(c) & \textit{if } p > k \geq q \\ d_k^\alpha(c') \#_m c & \textit{if } q > k \geq p \end{cases}$
for all $c \in C_p$, $c' \in C_q$, $\alpha = \pm$ *and all* k, m, p, q,

(ii) $\mathrm{id}_{d_m^+(c)} \#_m c = c = c \#_m \mathrm{id}_{d_m^-(c)}$ *for all* $c \in C$, *all* m,

(iii) $c' \#_m \mathrm{id}_c = \mathrm{id}_{c' \#_m c}$ *for all* $c, c' \in C$ *and*
$\mathrm{id}_{c'} \#_m c = \mathrm{id}_{c' \#_m c}$ *for all* $c, c' \in C$,

(iv) *(associativity)* $c'' \#_m (c' \#_m c) = (c'' \#_m c') \#_m c$ *for all* $c, c', c'' \in C$, *all* m,

(v) *(interchange)* $(d' \#_n d) \#_m (c' \#_n c) = (d' \#_m c') \#_n (d \#_m c)$ *for all* $c, c', d, d' \in C$, *all* $m < n$. □

Thus, there are more identity axioms, but they do have a simpler form.

Definition 2.4 An *ω-groupoid* is an ω-category $\mathbb{C}$ such that:
for every $c \in C_i$ there is a $c^* \in C_i$ satisfying $c^* \#_{i-1} c = \mathrm{id}_{s_{i-1}(c)}$ and $c \#_i c^* = \mathrm{id}_{t_{i-1}(c)}$. ◇

For the purposes of this paper,

Definition 2.5 A *pre-tas* consists of a reflexive globular set $((C_i)_{i\in\omega}, \{d_k^-, d_k^+ : C_i \to C_k\}_{k<i}, \{\mathrm{id}_- : C_i \to C_{i+1}\}_i)$ together with operations $\#_m : C_q \,{}_{s_m}\!\times_{t_m} C_p \to C_{p+q-m-1}$ for $m < \min\{p, q\}$, such that:

(i) $d_m^\alpha(c' \#_m c) = \begin{cases} d_m^-(c) & \text{if } \alpha = - \\ d_m^+(c') & \text{if } \alpha = + \end{cases}$
for all $c \in C_p$, $c' \in C_q$, $\alpha = \pm$ and all m, p, q,

(ii) $\mathrm{id}_{d_m^+(c)} \#_m c = c = c \#_m \mathrm{id}_{d_m^-(c)}$ for all $c \in C_i$, all m,

(iii) $c' \#_m \mathrm{id}_c = \mathrm{id}_{c' \#_m c}$ for all $c, c' \in C$ and $\mathrm{id}_{c'} \#_m c = \mathrm{id}_{c' \#_m c}$ for all $c, c' \in C$. ◇

A *tas* furthermore has a myriad of axioms, like *naturality*, *functoriality*, *associativity*, and more. A complete list of axioms has been given up to dimension 4 and partially up to dimension 6 [**10**], but not yet for general n. This does not matter for the purposes of this paper, though, as the result is true no matter what the precise axioms will turn out to be, and also for many higher-dimensional categorical structures that are not teisi.

Weak(er) n-categories and weak(er) teisi have some or all of the axioms replaced by coherence constraints, with further coherence constraints relating them, etcetera.

3. The main argument

For $\mathbb{C}$ to be a pre-tas in $\underline{\mathrm{Ab}}$ means that all C_i are abelian groups and all sources, targets, identities and compositions are group homomorphisms. Thus, the source (target) of a sum is the sum of the respective sources (targets), and the following diagrams commute:

$$
\begin{tikzcd}
C_i \times C_i \arrow[r, "\mathrm{id}_- \times \mathrm{id}_-"] \arrow[d, "+"'] & C_{i+1} \times C_{i+1} \arrow[d, "+"] \\
C_i \arrow[r, "\mathrm{id}_-"'] & C_{i+1}
\end{tikzcd}
\tag{3.1}
$$

$$
\begin{tikzcd}
(C_q\,{}_{s_m}\!\times_{t_m} C_p) \times (C_q\,{}_{s_m}\!\times_{t_m} C_p) \arrow[r, "\#_m \times \#_m"] \arrow[d, "\cong"'] & C_{p+q-m-1} \times C_{p+q-m-1} \arrow[dd, "+"] \\
(C_q \times C_q)\,{}_{s_m \times s_m}\!\times_{t_m \times t_m} (C_p \times C_p) \arrow[d, "+\times+"'] & \\
C_q\,{}_{s_m}\!\times_{t_m} C_p \arrow[r, "\#_m"'] & C_{p+q-m-1}
\end{tikzcd}
\tag{3.2}
$$

In (3.2), the isomorphism together with the arrow on the left sends two m-composable pairs to the pair of their sums, which is again m-composable by the condition on source and target above.

It follows that the source (target) of an additive inverse is the additive inverse of the source (target), that the source (target) of a unit is a unit, and that the following diagrams commute:

$$
\begin{tikzcd}
C_i \arrow[r, "\mathrm{id}_-"] \arrow[d, "-"'] & C_{i+1} \arrow[d, "-"] \\
C_i \arrow[r, "\mathrm{id}_-"'] & C_{i+1}
\end{tikzcd}
\tag{3.3}
$$

$$
\begin{tikzcd}
C_q\,{}_{s_m}\!\times_{t_m} C_p \arrow[r, "\#_m"] \arrow[d, "-\times-"'] & C_{p+q-m-1} \arrow[d, "-"] \\
C_q\,{}_{s_m}\!\times_{t_m} C_p \arrow[r, "\#_m"'] & C_{p+q-m-1}
\end{tikzcd}
\tag{3.4}
$$

$$
\begin{tikzcd}
* \arrow[r, equal] \arrow[d, "e"'] & * \arrow[d, "e"] \\
C_i \arrow[r, "\mathrm{id}_-"'] & C_{i+1}
\end{tikzcd}
\tag{3.5}
$$

$$
\begin{tikzcd}
* \arrow[r, equal] \arrow[d, "e"'] & * \arrow[d, "e"] \\
C_q\,{}_{s_m}\!\times_{t_m} C_p \arrow[r, "\#_m"'] & C_{p+q-m-1}
\end{tikzcd}
\tag{3.6}
$$

In (3.6), the unit of the group $C_{q\ s_m} \times_{t_m} C_p$ is the pair (e, e), which is indeed a composable pair, again by the conditions on the source and target.

I will now investigate the composite of two elements in a pre-tas in Ab. For convenience, I will assume that the dimension of intersection is 0. This is without loss of generality because one can always achieve this by renumbering the dimensions, noting that the lower dimensions will take care of themselves. Also without loss of generality is to assume that both elements are of the same dimension. The general case would amount to almost the same calculations, but these assumptions avoid the necessity to consider various cases, and make keeping track of the dimensions somewhat simpler.

So consider an i-arrow f and an i-arrow g with $s_0(f) = A$, $t_0(f) = B = s_0(g)$, $t_0(g) = C$. I will picture this information as

$$A \xrightarrow{f} B \xrightarrow{g} C \quad ,$$

remembering that f and g need not be 1-dimensional. Now $g \#_0 f$ is equal to

$$A - B + B \xrightarrow{\quad f - \mathrm{id}^i_B + \mathrm{id}^i_B \quad} e + B \xrightarrow{\qquad e+g \qquad} e + C$$

where $e \in C_i$ is equal to id^i_e by repeated application of diagram (3.5). By (3.2), this is equal to

$$\begin{array}{c} A - B \xrightarrow{f - \mathrm{id}^i_B} e \xrightarrow{\ e\ } e \\ + \\ B \xrightarrow[\mathrm{id}^i_B]{} B \xrightarrow[g]{} C \end{array} \quad .$$

Carrying out these composites, this gives

$$\begin{array}{c} A - B \xrightarrow{\quad \mathrm{id}^{i-1}_f - \mathrm{id}^{i+i-1}_B \quad} e \\ + \\ B \xrightarrow[\quad \mathrm{id}^{i-1}_g \quad]{} C \end{array}$$

because

$$\begin{aligned} &\mathrm{id}^i_e \#_0 (f - \mathrm{id}^i_B) = \\ &\quad = \mathrm{id}^{i-1}_{\mathrm{id}_e \#_0 (f - \mathrm{id}^i_B)} && \text{by repeated application of axiom (iii)} \\ &\quad = \mathrm{id}^{i-1}_{f - \mathrm{id}^i_B} && \text{by axiom (ii)} \\ &\quad = \mathrm{id}^{i-1}_f - \mathrm{id}^{i+i-1}_B && \text{by repeated application of diagram (3.1)} \\ &\quad = \mathrm{id}^{i-1}_f - \mathrm{id}^{i-1}_{-\mathrm{id}^i_B} && \text{by repeated application of diagram (3.3),} \end{aligned}$$

and a similar but simpler calculation for g. So

$$g \#_0 f = \mathrm{id}^{i-1}_f - \mathrm{id}^{i+i-1}_B + \mathrm{id}^{i-1}_g \, .$$

Similarly, putting g first one also gets that $g \#_0 f = \mathrm{id}_g^{i-1} - \mathrm{id}_B^{i+i-1} + \mathrm{id}_f^{i-1}$.

So, although horizontal composition lands, by definition, in C_{i+i-1}, the result is an $(i-1)$-identity, i.e., composition factors through $\mathrm{id}_{-}^{i-1} : C_i \to C_{i+i-1}$, and so one can say that *composition in a pre-tas in* $\underline{\mathrm{Ab}}$ *is dimension preserving.*

It remains to be shown that this does give an ω-groupoid. More precisely, given a pre-tas in $\underline{\mathrm{Ab}}$, take the same reflexive globular set but with new composition operations $C_i\ {}_{s_m}\!\times_{t_m} C_i \to C_i$, given by $g \#_m f = f - \mathrm{id}_B^i + g$. It is obvious that this satisfies the axioms for sources and targets.

For three 0-composable i-arrows,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

say, either way of composing these gives $f - \mathrm{id}_B^i + g - \mathrm{id}_C^i + h$, proving associativity.

Consider now four elements

$$\begin{tikzcd} A \arrow[r, bend left, ""] \arrow[r, "b"] \arrow[r, bend right, ""] & B \arrow[r, bend left, ""] \arrow[r, "b'"] \arrow[r, bend right, ""] & C \end{tikzcd} \quad \Downarrow f,\ \Downarrow f',\ \Downarrow g,\ \Downarrow g'$$

where A, B, C are 0-dimensional, f, f', g, g' are i-dimensional, and b, b' are j-dimensional (which is general enough). Then obviously $g' \#_j g$ and $f' \#_j f$ are 0-composable, and

$$(g' \#_j g) \#_0 (f' \#_j f) = f - \mathrm{id}_b^{i-j} + f' - \mathrm{id}_B^i + g - \mathrm{id}_{b'}^{i-j} + g'.$$

On the other hand, obviously $g' \#_0 f'$ and $g \#_0 f$ are j-composable, and

$$\begin{aligned} &(g' \#_0 f') \#_j (g \#_0 f) = \\ &= \quad f - \mathrm{id}_B^i + g - \mathrm{id}_{b-\mathrm{id}_B^j + b'}^{i-j} + f' - \mathrm{id}_B^i + g' \\ &= \quad f - \mathrm{id}_B^i + g - \mathrm{id}_b^{i-j} + \mathrm{id}_B^i - \mathrm{id}_{b'}^{i-j} + f' - \mathrm{id}_B^i + g' \quad \text{by (3.1) and (3.3),} \end{aligned}$$

which is quickly seen to give the same as above by abelianness of C_i.

Given

$$A \xrightarrow{f} B \quad ,$$

define $f^* = \mathrm{id}_B^i - f + \mathrm{id}_A^i$, which has 0-source $B - A + A = B$ and 0-target $B - B + A = A$. Then $f^* \#_0 f = f - \mathrm{id}_B^i + \mathrm{id}_B^i - f + \mathrm{id}_A^i = \mathrm{id}_A^i$, and similarly $f \#_0 f^* = \mathrm{id}_B^i$.

All this proves

Theorem 3.1 *A pre-tas in* $\underline{\mathrm{Ab}}$*, and so a fortiori a tas in* $\underline{\mathrm{Ab}}$*, is an* ω*-groupoid.* $\square$

4. Remarks

The proof does not work for monoids nor for semigroups with unit — trivially so, because it uses substraction.

I have not used (3.4) and (3.6), but they come for free with (3.2) anyway. In fact, this uses associativity of the group operation, which is needed at several other points in the proof too.

The only place I do need abelianness is in interchange. Not even in that it is an ω-groupoid! So pre-teisi in $\underline{\mathrm{Gp}}$ are *no interchange ω-groupoids*, which are to ω-groupoids what sesqui-categories are to 2-categories.

The crucial thing really is that composing something with an identity results in (the identity on) the same thing, which follows from (ii) and (iii). In $\underline{\mathrm{Ab}}$, this extended form of (ii) implies (iii) because all compositions are addition, but in general there need not be relations between the compositions and identities.

With this modified (ii), the argument above shows that for any operation $\# : C_q {}_{s_m}\times_{t_m} C_p \to C_{\vartheta(m,p,q)}$ for $m < \min\{p, q\}$ with $\vartheta(m,p,q) \geq \max\{p,q\}$, one has that for $f \in C_p$ and $g \in C_q$ with $t_m(f) = B = s_m(g)$ it is the case that $g \# f = \mathrm{id}_f^{\vartheta(m,p,q)-p} - \mathrm{id}_B^{\vartheta(m,p,q)-m} + \mathrm{id}_g^{\vartheta(m,p,q)-q}$. I will extend this further, but for that I need to do something else first.

5. Intermezzo

Before extending theorem 3.1 to other higher-dimensional categorical structures, I need two results about certain globular sets.

Let A be a globular set. Write s, t for all the maps $d_{k-1}^-, d_{k-1}^+ : A_k \to A_{k-1}$ respectively. Define s and t also for $k = 0$ by defining, for $x \in A_0$, $s(x) = t(x) = *$.

For $a, b \in A_k$, say that $a < b$ if $t(a) = s(b)$. Let $\lhd$ be the transitive closure of $<$ on A.

Definition 5.1 A globular set A is *diagrammatic* if

(i) A is finite,

(ii) (*no direct loops*) there is no $x \in A$ with $x \lhd x$,

(iii) (*well formed*) if $s(x) = s(y)$ or $t(x) = t(y)$ then $x = y$ or there exist w such that $t(w) = x$ or $t(w) = y$. $\diamond$

Proposition 5.2 *A diagrammatic globular set is precisely a well-formed loop-free pasting scheme [**16**] in which for every n any n-cell begins and ends in only one $(n-1)$-cell respectively.* $\square$

Proof. Most of this is straightforward; the only interest lies in the dual form of well-formedness: if $s(x) = s(y)$ or $t(x) = t(y)$ and $x \neq y$ then there also exists a w' with $s(w') = x$ or $s(w') = y$.

If $s(x) = s(y)$ and $x \neq y$ then well-formedness gives an a with $t(a) = x$ say. Repeating for $s(a)$ and y and so on gives $a' \lhd a$ and possibly also $b' \lhd b$ with $t(b) = y$. Because A is finite and has no direct loops this must stop, by having $s(a') = s(b') = v$ say. By globularity, then $t(x) = t(v) = t(y)$. Take a' and b' the first such, and apply the same process to a' and b', which have $s(a') = s(b')$, and conclude that $t(a') = t(b')$. But then the process for x and y stopped one step earlier, contradicting minimality of a' and b', which can only be resolved when $s(a') = y$ say, i.e., when y is a source. $\square$

Definition 5.3 A *globular diagram* in an ω-category $\mathbb{C}$ is a diagrammatic globular set A together with a morphism of globular sets $f : A \to \mathbb{C}$. $\diamond$

Say that for a globular diagram (A, f) in $\mathbb{C}$, it has *shape* A. Denote the collection of globular diagrams in $\mathbb{C}$ with shape A by $\mathbb{C}^A$.

Corollary 5.4 *A globular diagram in an ω-category $\mathbb{C}$ determines a unique element of $\mathbb{C}$, the* composite *of the diagram.*

Proof. This is just a special case of Johnson's pasting theorem [**16**], with the morphism giving an appropriate realization of the pasting scheme A. $\square$

The composite of a globular diagram (A, f) will be denoted by $f(A)$.

For $a \in A_k$, $b \in A_{k-1}$, say that $a \prec b$ if $a = s(b)$ or $b = t(a)$. Let $\blacktriangleleft$ be the transitive closure of $\prec$ on A.

Definition 5.5 (Street [20]) A *globular cardinal* is a finite globular set for which $\blacktriangleleft$ is a total order. $\diamond$

'Globular' has been used for pasting schemes in a slightly different meaning [**9**, Section 2-8].

Proposition 5.6 *A diagrammatic globular set is precisely a globular cardinal.*

Proof. Given a diagrammatic globular set A, the same argument as in the proof of proposition 5.2 applied to 0-cells totally $\blacktriangleleft$-orders them. Then, inductively, apply it to the collection of i-cells with same source, which, again by the same argument, also have the same target. This gives a total order because if $x \blacktriangleleft x$ via lowest dimension j say then if x has dimension higher than j then $s_{j+1}(x) \triangleleft s_{j+1}(x)$ and if x has dimension lower than j there is a w with $t(w) = x$ and for which $w \triangleleft w$, both cases contradicting A having no direct loops.

For the converse, $x \triangleleft x$ implies $x \blacktriangleleft x$, and if $s(x) = s(y)$ then $\blacktriangleleft$-between x and y there are no elements of lower dimension, so one of x and y must be a target. $\square$

6. Extensions

Weak n- or ω-categories, of whatever kind, are not pre-teisi as they do not have strict identities. For kinds of weak n-categories which are not defined algebraically [**1, 15**] it even makes no sense to ask for strict identities; for Batanin's weak ω-categories, which are defined using operads [**2**], it does.

Say that an element of a diagrammatic globular set A is *high* if y is not a source nor a target: there is no z with $y = s(z)$ or $y = t(z)$. Say it is *low* if it is both a source and a target: there are z, z' with $s(z') = y = t(z)$.

Definition 6.1 Let A be an n-dimensional diagrammatic globular set. An *A-ω-category* consists of an ω-category $\mathbb{C}$ together with an operation $\#_A : \mathbb{C}^A \to C_j$ with $j \geq n$ such that:

(i) if (A, f) is a globular diagram in $\mathbb{C}$ and there is one high cell $y \in A_i$ such that for any $x \in A_k$ with $k > 0$ and with $x \neq s_k(y)$ and $x \neq t_k(y)$ and there exist no $u \trianglelefteq v$ with $s(u) = x$ and $t(v) = s_k(y)$ or with $s(u) = t_k(y)$ and $t(v) = x$ one has $f(x) = \mathrm{id}_{s(f(x))}$, then $\#_A(f) = \mathrm{id}^{j-i}_{f(y)}$. $\diamond$

This condition extends the identity axioms for a pre-tas to compositions of other arities. It basically says that if all cells except one are realized by identities then the A-composite is equal to (the identity on) the realization of that single cell. The condition together with globularity determines f from $f(y)$: if $x = s_k(y)$ then $f(x) = s_k(f(y))$ and if there exist $u \trianglelefteq v$ with $s(u) = x$ and $t(v) = s_k(y)$ then also it must be that $f(x) = s_k(f(y))$ because $u, \ldots, v$ are identities, and similarly on the other side of y. In an A-ω-category condition (i) holds also for non-high y: if y is not high it is the face of a high cell which must be realized by the identity on y, and then the A-composite of this realization must be the identity on that, so the identity on y.

I will want to relate the A-composite of a globular diagram of shape A in an A-ω-category in $\underline{\mathrm{Ab}}$ to its composite, so I first calculate the latter.

Lemma 6.2 *Let $\mathbb{C}$ be an ω-category in $\underline{\mathrm{Ab}}$, and let (A, f) be an n-dimensional globular diagram in $\mathbb{C}$. Then*

$$f(A) = \sum_{y \in A,\ y\ high} \mathrm{id}^{n-\dim(y)}_{f(y)} - \sum_{y \in A,\ y\ low} \mathrm{id}^{n-\dim(y)}_{f(y)}.$$

Proof. By corollary 5.4, I can calculate the composite of (A, f) in any way I like. Take a lowest-dimensional low element u of A, then the collection A_ℓ of elements of A $\blacktriangleleft$-before u together with u is again a globular cardinal hence a diagrammatic globular set by proposition 5.6, and similarly the collection A_r of elements of A $\blacktriangleleft$-after u together with u. The statement follows by induction, from the formula $f(A) = \mathrm{id}^{n-\dim A_\ell}_{f(A_\ell)} - \mathrm{id}^{n-\dim u}_{u} + \mathrm{id}^{n-\dim A_r}_{f(A_r)}$, and the fact that if A does not contain low elements it contains a unique high element v and $f(A) = f(v)$. $\square$

In order to calculate the A-composite of a globular diagram of shape A in an A-ω-category in $\underline{\mathrm{Ab}}$ I will follow the main argument above closely.

For $\mathbb{C}$ to be an A-ω-category in $\underline{\mathrm{Ab}}$ means that A-composition is a group homomorphism too. Thus, $\#_A(f + g) = \#_A(f) + \#_A(g)$ for all $f, g : A \to \mathbb{C}$, where $(f + g)(a) = f(a) + g(a)$, this again being a morphism of globular sets because source and target are group homomorphisms.

Consider a morphism of globular sets $f : A \to \mathbb{C}$. The first step is to write $f(x)$ as a sum involving identities. I will use the convention that $\mathrm{id}^k_{f(*)} = e$.

For high y, define $m(y)$ to be the greatest m for which $t_m(y)$ is a source, if such m exists, and -1 otherwise. Obviously, there is only one high y with $m(y) = -1$, namely the $\blacktriangleleft$-last one. For $x \in A_k$ and high y for which $x \neq s_k(y)$ and $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft y$, define $m'(x, y)$ to be the greatest m' for which $s_{m'-1}(x) = s_{m'-1}(y)$. Obviously, $m'(x, y) \geq 0$, and $k \geq m'(x, y) \geq m(y)$.

For $x \in A_k$ and high y, define

$$f_y(x) = \begin{cases} f(x) - \mathrm{id}^{k-m(y)}_{t_{m(y)}(f(y))} & \text{if } k > m(y) \text{ and } x = t_k(y) \text{ or} \\ & k \geq m(y) \text{ and } x = s_k(y) \\ \mathrm{id}^{k-m'(x,y)}_{s_{m'(x,y)}(f(y))} - \mathrm{id}^{k-m(y)}_{t_{m(y)}(f(y))} & \text{if } x \neq s_k(y) \text{ and} \\ & m(y) = -1 \text{ or } s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft y \\ e & \text{otherwise.} \end{cases}$$

Lemma 6.3 $f(x) = \sum_{y \in A,\ y\ high} f_y(x)$.

Proof. The only y's that contribute to the sum in the statement are those for which $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft t_{m(y)}(y)$. There is only one such for which $x = s_k(y)$ or $x = t_k(y)$, namely the $\blacktriangleleft$-first one. Indeed, let $y \blacktriangleleft y'$ both contributing, then $x = t_k(y')$ contradicts $x \blacktriangleleft t_{m(y)}(y)$, and $x = s_k(y')$ implies $s_{m(y)-1}(y) \blacktriangleleft s_k(y') \blacktriangleleft t_{m(y)}(y) \blacktriangleleft y'$, again a contradiction. So the $\blacktriangleleft$-first y takes care of $f(x)$.

For any high y with $s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft t_{m(y)}(y)$ and $x \neq s_k(y)$ and $x \neq t_k(y)$, one has that $x \blacktriangleleft s_{m'(x,y)}(y)$ because $s_{m'(x,y)}(x) \neq s_{m'(x,y)}(y)$. Also, $s_{m'(x,y)-1}(y) = s_{m'(x,y)-1}(x) \blacktriangleleft x$, so $s_{m'(x,y)}(y)$ is a source, and I can consider the $\blacktriangleleft$-last high y' before $s_{m'(x,y)}(y)$. For this y', $m(y') = m'(x,y)$ and $t_{m(y')}(y') = s_{m'(x,y)}(y)$, and hence also $s_{m(y')-1}(y') = s_{m(y')-1}(x) \blacktriangleleft x \blacktriangleleft t_{m(y')}(y')$. Moreover, y' is the $\blacktriangleleft$-last high y' before y having this property because any $y'' \blacktriangleleft y$ with $s_{m'(x,y)}(y) \blacktriangleleft y''$ has $m(y'') > m'(x,y)$ by definition of $m'(x,y)$, and hence $x \blacktriangleleft s_{m(y'')-1}(y'')$. So for every contributing y, the $\blacktriangleleft$-next contributing one cancels $\mathrm{id}^{k-m(y)}_{t_{m(y)}(f(y))}$.

The $\blacktriangleleft$-last contributing y is the one with $m(y) = -1$; and for this one has $\mathrm{id}^{k-m(y)}_{t_{m(y)}(f(y))} = e$. □

The second step is to split up f using the sum just obtained.

Lemma 6.4 *For high y, f_y is a globular diagram of shape A.*

Proof. If $x = t_k(y)$ or $x = s_k(y)$ then the calculations are easy.

If $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft y$ and $k > m'(x,y)$ then also $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft s(x) \blacktriangleleft y$ with $m'(s(x),y) = m'(x,y)$, and $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft t(x) \blacktriangleleft y$ with $m'(t(x),y) = m'(x,y)$, except when $k-1 = m'(x,y)$ and $t(x) = s_{m'(x,y)}(y)$, in which case $f(t(x)) = s_{m'(x,y)}(f(y))$, as required.

If $m(y) = -1$ or $s_{m(y)-1}(y) \blacktriangleleft x \blacktriangleleft y$ and $k = m'(x,y)$ then $s(x) = s_{m'(x,y)-1}(y)$ which presents no problem, and $t(x) = t_{k-1}(y)$; when $m'(x,y) = m(y)$ the calculation is easy, and when $m'(x,y) > m(y)$ then $f(t(x)) = t(s_{m'(x,y)}(f(y)))$, as required. □

The third step is to calculate the A-composites of the f_y's.

Lemma 6.5 *For high $y \in A_i$, $\#_A(f_y) = \mathrm{id}^{j-i}_{f(y)} - \mathrm{id}^{j-m(y)}_{t_{m(y)}(f(y))}$.*

Proof. f_y satisfies condition (i) of definition 6.1, and $f_y(y) = f(y) - \mathrm{id}^{i-m(y)}_{t_{m(y)}(f(y))}$. □

Now there is one more high element than there are low elements, so the low elements are precisely the $t_{m(y)}(f(y))$ for the high y with $m(y) \neq -1$. So

$$\begin{array}{rcll}
\#_A(f) & = & \sum_{y\in A,\ y \text{ high}} \#_A(f_y) & \text{by lemma 6.3} \\
& = & \sum_{y\in A,\ y \text{ high}} (\mathrm{id}^{j-\dim(y)}_{f(y)} - \mathrm{id}^{j-m(y)}_{t_{m(y)}(f(y))}) & \text{by lemma 6.5} \\
& = & \mathrm{id}^{j-n}_{\sum_{y\in A,\ y \text{ high}} \mathrm{id}^{n-\dim(y)}_{f(y)} - \sum_{y\in A,\ y \text{ low}} \mathrm{id}^{n-\dim(y)}_{f(y)}} & \\
& = & \mathrm{id}^{j-n}_{f(A)} & \text{by lemma 6.2.}
\end{array}$$

This proves

Theorem 6.6 *Let A be a diagrammatic globular set. An A-ω-category in* $\underline{\mathrm{Ab}}$ *is an ω-groupoid.* □

Say that a weak ω-category [**2**] is *normalized* if all its operations satisfy condition (i) of definition 6.1.

Corollary 6.7 *A normalized weak ω-category in* $\underline{\mathrm{Ab}}$ *is an ω-groupoid.*

Proof. First, the main argument above applies to any operation $C_q\,{}_{s_m}\!\times_{t_m} C_p \to C_{\alpha(p,q)}$ with $\alpha(p,q) \geq \max(p,q)$, which takes care of the binary operations in a normalized weak ω-category. Condition (i) of definition 6.1 takes care of unary operations, and the theorem shows that operations of higher arity are in fact trivial. □

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