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SEMI-ABELIAN EXACT COMPLETIONS

MARINO GRAN

(*communicated by Walter Tholen*)

Abstract

We characterize the categories which are projective covers of regular protomodular categories. Our result gives in particular a characterization of the categories with weak finite limits with the property that their exact completions are semi-abelian categories. As an application, we obtain a categorical proof of the recent characterization of semi-abelian varieties.

Introduction

The theory of protomodular categories provides a simple and general context in which the basic theorems needed in homological algebra of groups, rings, Lie algebras and other non-abelian structures can be proved [**2**] [**3**] [**4**] [**5**] [**6**] [**7**] [**9**] [**20**].

An interesting aspect of the theory comes from the fact that there is a natural intrinsic notion of normal monomorphism [**4**]. Since any internal reflexive relation in a protomodular category is an equivalence relation, protomodular categories also have all the nice properties of Maltsev categories [**12**], so that there is in particular a good theory of centrality of equivalence relations [**7**] [**8**] [**22**] [**25**].

In many respects protomodular and, more specifically, semi-abelian categories are in the same relationship with the varieties of groups, rings and other non-abelian varieties, as abelian categories are with the varieties of abelian groups and modules over a ring. Abelian and semi-abelian categories are related by the nice "equation"

$$(\mathrm{Semi-abelian}) + (\mathrm{Semi-abelian})^{op} = \mathrm{Abelian}$$

asserting that a category $\mathcal{C}$ is abelian if and only if both $\mathcal{C}$ and its dual category $\mathcal{C}^{op}$ are semi-abelian [**20**]. Semi-abelian categories appear then as a natural non-additive generalization of abelian categories. On the one hand, they are general enough to include also many important algebraic examples such as any variety of Ω-groups [**19**] and Heyting algebras; on the other hand, their axioms allow one to distinguish their exactness properties from the properties of the category of monoids, or from those of the category of sets.

In the present article we study protomodular and semi-abelian categories in relationship with the interesting construction of the free exact completion of a category

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with weak finite limits [**14**]. A motivation to do this also comes from the importance, in homological algebra, of abelian categories with enough projectives, so that it seems reasonable to expect further developments in the non-additive setting of semi-abelian categories with enough projectives (for instance by extending the theory developed in [**16**] and [**17**] for varieties of Ω-groups). Now, a classical result by Freyd [**15**] asserts that a category is equivalent to a projective cover of an abelian category if and only if it is preadditive and it has weak finite products and weak kernels; a first natural question then arises to determine which are the categories which occur as projective covers of exact protomodular or of semi-abelian categories. There are some technical difficulties in order to answer this question, essentially due to the fact that the protomodularity property is defined in terms of a pullback functor, and we can not expect to define such a functor in the projective cover P of a semi-abelian category, since P only has weak finite limits in general. However, it turns out that it is possible to solve this problem, and this is our main result, by using an equivalent formulation of the protomodularity property, which is probably also the most effective tool when dealing with protomodularity. We then characterize the categories $\mathcal{C}$ with the property that their exact completion $\mathcal{C}_{ex}$ is protomodular or semi-abelian. Our characterization of projective covers of exact protomodular categories applies at the same time to the free algebras of any protomodular variety, as well as to the projective objects in the dual category of any elementary topos, this latter being always an exact protomodular category with enough projectives. In the last section we show how this categorical characterization provides an alternative proof of the recent characterization of the varieties of universal algebras that are semi-abelian [**10**].

The results in the present paper follow the same line of research developed in various recent papers, where necessary and sufficient conditions on a category $\mathcal{C}$ with (weak) finite limits have been determined in order that $\mathcal{C}_{ex}$ is abelian [**11**], extensive [**18**], (locally) cartesian closed [**13**] [**23**], a topos [**21**], or a Maltsev category [**24**].

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1. Preliminaries

In this section we briefly recall some elementary categorical notions and two known results needed in the following.

A category with finite limits is regular if any kernel pair has a coequalizer and regular epimorphisms are stable under pullback. A regular category is Barr-exact [**1**] if any equivalence relation is a kernel pair. A functor $F: \mathcal{A} \to \mathcal{B}$ between regular or exact categories is *exact* if it preserves finite limits and regular epimorphisms.

By dropping the assumption of the uniqueness of the factorization in the definition of a limit, one obtains the definition of a weak limit. For brevity, we shall call *weakly lex* a category with weak finite limits. If $\mathcal{A}$ is a weakly lex category and $\mathcal{B}$ is a finitely complete category, a functor $F: \mathcal{A} \to \mathcal{B}$ is *left covering* if, for any finite diagram D in $\mathcal{A}$ and for each weak limit W of D, the comparison arrow from $F(W)$ to the limit of $F(D)$ is a strong epimorphism. Let us then recall that an object P

in a category is *(regular) projective* if, for any arrow $f: P \to X$ and for any regular epimorphism $g: Y \to X$ there exists an arrow $h: P \to Y$ such that $g \circ h = f$.

Following the terminology in [**14**] we say that a full subcategory $\mathcal{C}$ of $\mathcal{A}$ is a *projective cover* of $\mathcal{A}$ if two conditions are satisfied: 1) any object of $\mathcal{C}$ is regular projective in $\mathcal{A}$; 2) for any object X in $\mathcal{A}$ there exists a $\mathcal{C}$-cover of X, that is an object C in $\mathcal{C}$ and a regular epimorphism $C \to X$.

When $\mathcal{A}$ admits a projective cover, one says that $\mathcal{A}$ has *enough projectives*, that is any object in $\mathcal{A}$ is the codomain of a regular epimorphism whose domain is a projective object.

An important result in [**14**] asserts that exact categories with enough projectives are the exact completions of their full subcategories of projective objects. Let us then briefly recall the construction of the *free exact completion* $\mathcal{C}_{ex}$ of a weakly lex category $\mathcal{C}$ (see [**14**] for more details). An object in $\mathcal{C}_{ex}$ is a pseudo-equivalence in $\mathcal{C}$, which will be represented by a diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l] \end{tikzcd}.$$

A pseudo-equivalence relation as above can be thought as an equivalence relation except that one does not require that the pair of arrows d and c are jointly monic. A pre-arrow in $\mathcal{C}_{ex}$ is a pair of arrows (f_0, f_1) in $\mathcal{C}$

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] & Y_1 \arrow[d, shift right=3, "\delta"'] \arrow[d, shift left=3, "\gamma"] \\
X_0 \arrow[u] \arrow[r, "f_0"'] & Y_0 \arrow[u]
\end{tikzcd}$$

with $\delta \circ f_1 = f_0 \circ d$ and $\gamma \circ f_1 = f_0 \circ c$. An arrow in $\mathcal{C}_{ex}$ is an equivalence class $[f_0, f_1]$ of pre-arrows, where two parallel pre-arrows (f_0, f_1) and (g_0, g_1) are identified if there exists an arrow $h: X_0 \to Y_1$ such that $\delta \circ h = f_0$ and $\gamma \circ h = g_0$. The identities and the composition are the obvious ones.

The main facts concerning the free exact completion of a weakly lex category we shall need later on are recalled in the following

1.1. Theorem. *[14] Let $\mathcal{C}$ be a weakly lex category. Then there exists an exact category $\mathcal{C}_{ex}$ and a fully faithful functor $\Gamma: \mathcal{C} \to \mathcal{C}_{ex}$ with the following properties:*

1. *$\Gamma(\mathcal{C})$ is a projective cover of $\mathcal{C}_{ex}$*
2. *for any exact category $\mathcal{B}$, the composition $- \circ \Gamma$ with the functor Γ determines an equivalence of categories between the category of exact functors from $\mathcal{C}_{ex}$ to $\mathcal{B}$ and the category of left covering functors from $\mathcal{C}$ to $\mathcal{B}$:*

$$Ex[\mathcal{C}_{ex}, \mathcal{B}] \simeq Lco[\mathcal{C}, \mathcal{B}].$$

The second subject we shall be interested in is the notion of *protomodular category*. When $\mathcal{C}$ is a finitely complete category, we denote by $Pt(\mathcal{C})$ the category whose objects are the split epimorphisms with a given splitting and morphisms the commutative squares between these data. Let $\pi: Pt(\mathcal{C}) \to \mathcal{C}$ be the functor sending a split epi to its codomain. The existence of pullbacks implies that the functor π is a fibration, which is called the fibration of pointed objects. We denote by $Pt_B(\mathcal{C})$ the fibre of this fibration over a fixed object B; if $f: E \to B$ is an arrow in $\mathcal{C}$, then we denote by $f^* : Pt_B(\mathcal{C}) \to Pt_E(\mathcal{C})$ the associated change-of-base functor with respect to the fibration π. A protomodular category $\mathcal{C}$ is a left exact category with the property that every change-of-base functor with respect to the fibration π is conservative (i.e. it reflects isomorphisms) [**2**]. Any protomodular category is a Maltsev category [**3**], this meaning that any internal reflexive relation is an equivalence relation (for the notion of a Maltsev category see for instance [**12**], and references therein).

If $\mathcal{C}$ has a zero object, the protomodularity property is equivalent to the validity of the split short five lemma. A *semi-abelian* category is an exact protomodular category with a zero object and finite coproducts [**20**].

There are many interesting examples of semi-abelian categories: among these, let us recall the varieties of groups, rings, associative algebras, Lie algebras, crossed modules (more generally, any variety of Ω-groups [**19**]), and Heyting algebras. Any abelian category is semi-abelian, as well as the dual category of the category of pointed sets. If $\mathcal{C}$ is a category with finite limits then the category $Grp(\mathcal{C})$ of internal groups in $\mathcal{C}$ is protomodular, as are the fibres of the fibration $Grpd(\mathcal{C}) \to \mathcal{C}$ sending a groupoid in $\mathcal{C}$ to its object of objects. The dual category of any elementary topos is exact protomodular [**3**].

As mentioned by Bourn [**5**], the protomodularity property can be equivalently stated as follows:

1.2. Lemma. *Let $\mathcal{C}$ be a category with pullbacks. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is protomodular*
2. *in any pullback*

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] \\ C \arrow[r, "f"'] & B \arrow[u, shift right, "i"'] \end{tikzcd}$$

along an epimorphism $g: A \to B$ split by an arrow $i: B \to A$ ($g \circ i = 1_B$), the pair of arrows $\overline{f}$ and i is jointly strongly epimorphic.

Proof: Let $\mathcal{C}$ be a protomodular category and let us assume that $h: E \to A$ is a monomorphism with the property that the pullbacks h_1 and h_2 along $\overline{f}$ and i respectively are isomorphisms. Then the fact that h_2 is an iso implies that h is an arrow in $Pt_B(\mathcal{C})$. The fact that h_1 is an iso precisely means that $f^*(h)$ is an iso in $Pt_C(\mathcal{C})$, and then h itself is an iso by protomodularity.

Conversely, if $\mathcal{C}$ satisfies property 2, it is not difficult to prove that any change-of-base functor with respect to the fibration π is conservative on monomorphisms. Since any change-of-base functor preserves finite limits, and then in particular kernel pairs, this is sufficient to conclude that it reflects isomorphisms. Indeed, if $h: E \to A$ is an arrow in $Pt_B(\mathcal{C})$ such that $f^*(h)$ is an iso, then the diagonal $\Delta: E \to E \underset{A}{\times} E$ is a mono such that $f^*(\Delta)$ is an iso. Accordingly, Δ itself is an iso, and then h is a mono. Finally, the fact that f^* is conservative on monomorphisms implies that h is an iso. $\square$

1.3. Remark. If $\mathcal{C}$ has binary coproducts, the conditions in Lemma 1.2 are easily seen to be equivalent also to the following property: in any pullback as above the induced arrow $(\overline{f}, i): P + B \to A$ is a strong epimorphism.

2. Projective covers of protomodular categories

We are now going to present a characterization of the weakly lex categories $\mathcal{C}$ whose exact completion $\mathcal{C}_{ex}$ is protomodular. When $\mathcal{C}$ has finite coproducts and a zero object, this condition can be expressed in terms of weak kernels.

We begin with the following definition:

2.1. Definition. A weakly lex category $\mathcal{C}$ is *weak protomodular* if any weak pullback P in $\mathcal{C}$

$$\begin{tikzcd}
P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] \\
C \arrow[r, "f"'] & B \arrow[u, shift right, "i"']
\end{tikzcd}$$

along a split epimorphism $g: A \to B$ (with $g \circ i = 1_B$) has the following property: if $\overline{f}$ and i both factorize through an arrow $h: E \to A$ in $\mathcal{C}$

$$\begin{tikzcd}
 & & E \arrow[dl, dotted, "h"'] \\
P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] \arrow[urr, dotted, bend left] & A \arrow[d, shift right, "g"'] & \\
C \arrow[r, "f"'] & B \arrow[u, shift right, "i"'] \arrow[uur, dotted] &
\end{tikzcd}$$

then h is a split epimorphism.

2.2. Remark. If $\mathcal{C}$ is finitely complete and weak protomodular, then $\mathcal{C}$ is protomodular. Indeed, let h be a monomorphism with the property that the pullbacks h_1 and h_2 along $\overline{f}$ and i respectively are isomorphisms. This means that $\overline{f}$ and i both factorize through h, so that h is a split epi and then an iso.

On the other hand, there seems to be no reason for a protomodular category to be weak protomodular, in general. The property of *weak protomodularity* should be thought as a simple way of guaranteeing protomodularity when only weak finite limits exist.

2.3. Proposition. *Let $\mathcal{C}$ be a projective cover of a regular category $\mathcal{A}$. Then $\mathcal{A}$ is protomodular if and only if $\mathcal{C}$ is weak protomodular.*

Proof: Let $\mathcal{A}$ be a regular protomodular category. Consider a weak split pullback (i.e. a weak pullback along a split epimorphism) in $\mathcal{C}$:

$$\begin{tikzcd}
P \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\overline{f}"] \arrow[ddr, bend right, "\overline{g}"'] & & \\
& C \underset{B}{\times} A \arrow[r, "f_1"'] \arrow[d, "g_1"] & A \arrow[d, shift right, "g"'] \\
& C \arrow[r, "f"] & B. \arrow[u, shift right, "i"']
\end{tikzcd}$$

This weak pullback can be obtained by taking the actual pullback $(C \underset{B}{\times} A, g_1, f_1)$ of f along g in $\mathcal{A}$, and by then covering it with a regular epimorphism $\alpha: P \to C \underset{B}{\times} A$, where P is in $\mathcal{C}$. By Lemma 1.2 we know that the arrows f_1 and i are jointly strongly epimorphic, thus $\overline{f} = f_1 \circ \alpha$ and i are jointly strongly epimorphic since α is a regular epi.

Now, let $\overline{f}$ and i both factorize through an arrow $h: E \to A$ in $\mathcal{C}$, so that there are two arrows $l: P \to E$ and $m: B \to E$ such that $h \circ l = \overline{f}$ and $h \circ m = i$

$$\begin{tikzcd}
& & E \arrow[dl, "h"'] \\
P \arrow[r, "\overline{f}"] \arrow[urr, bend left, "l"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] & \\
C \arrow[r, "f"] & B \arrow[u, shift right, "i"'] \arrow[uur, bend right, "m"'] &
\end{tikzcd}$$

Then the fact that the pair of arrows $\overline{f}$ and i is jointly strongly epimorphic implies that h is a strong epi in $\mathcal{A}$ and then a regular epi, because $\mathcal{A}$ is regular. Since the arrow h is a regular epi in $\mathcal{A}$ between projective objects, it splits, proving that $\mathcal{C}$ is weak protomodular.

Conversely, let us assume that $\mathcal{C}$ is a weak protomodular category. We first remark that in any weak split pullback in $\mathcal{C}$

$$\begin{tikzcd}
P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] \\
C \arrow[r, "f"'] & B \arrow[u, shift right, "i"']
\end{tikzcd}$$

the pair of arrows $\overline{f}$ and i are jointly strongly epimorphic in $\mathcal{A}$. Indeed, if $h: E \to A$

is a monomorphism in $\mathcal{A}$ such that the actual pullbacks h_1 and h_2 of h along $\overline{f}$ and i are isomorphisms, then the arrows $\overline{f}$ and i both factorize through h. Consequently h is a split epi and then an iso, as desired.

In order to complete the proof, let us consider an actual split pullback in $\mathcal{A}$

$$\begin{tikzcd} C \underset{B}{\times} A \arrow[r, "f_1"] \arrow[d, "g_1"'] & A \arrow[d, shift right, "g"'] \\ C \arrow[r, "f"'] & B \arrow[u, shift right, "i"'] \end{tikzcd}$$

and we are going to prove that f_1 and i are jointly strongly epimorphic in $\mathcal{A}$. It is not difficult to show that the split epimorphism g in $\mathcal{A}$ can be "covered" by a split epimorphism $\tilde{g}$ in $\mathcal{C}$ so that in the diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "a"] \arrow[d, shift right, "\tilde{g}"'] & A \arrow[d, shift right, "g"'] \\ \tilde{B} \arrow[r, "b"'] \arrow[u, shift right, "\tilde{i}"'] & B \arrow[u, shift right, "i"'] \end{tikzcd}$$

$a \circ \tilde{i} = i \circ b$, $b \circ \tilde{g} = g \circ a$ and a, b are regular epimorphisms. Moreover, if $c: \tilde{C} \to C$ is a projective cover of C, then there is an arrow $\tilde{f}: \tilde{C} \to \tilde{B}$ with $b \circ \tilde{f} = f \circ c$. Let us then form a weak pullback $\tilde{P}$ of $\tilde{f}$ along $\tilde{g}$

$$\begin{tikzcd} \tilde{P} \arrow[r, "\tilde{f}_1"] \arrow[d, "\tilde{g}_1"'] & \tilde{A} \arrow[d, shift right, "\tilde{g}"'] \\ \tilde{C} \arrow[r, "\tilde{f}"'] & \tilde{B} \arrow[u, shift right, "\tilde{i}"'] \end{tikzcd}$$

and we know that the pair of arrows $\tilde{f}_1$ and $\tilde{i}$ in this weak pullback is jointly strongly epimorphic in $\mathcal{A}$. There is a factorization $d: \tilde{P} \to C \underset{B}{\times} A$

$$\begin{tikzcd}
\tilde{P} \arrow[rr, "\tilde{f}_1"] \arrow[dr, dotted, "d"] \arrow[dd, "\tilde{g}_1"'] & & \tilde{A} \arrow[dr, "a"] \arrow[dd, shift right, "\tilde{g}"'] & \\
& C \underset{B}{\times} A \arrow[rr, "f_1"] \arrow[dd, "g_1"] & & A \arrow[dd, shift right, "g"'] \\
\tilde{C} \arrow[rr, "\tilde{f}"] \arrow[dr, "c"'] & & \tilde{B} \arrow[uu, shift right, "\tilde{i}"'] \arrow[dr, "b"] & \\
& C \arrow[rr, "f"] & & B \arrow[uu, shift right, "i"']
\end{tikzcd}$$

with $f_1 \circ d = a \circ \tilde{f}_1$ and $g_1 \circ d = c \circ \tilde{g}_1$. Now, $\tilde{f}_1$ and $\tilde{i}$ are jointly strongly epimorphic, so that the pair of arrows $(a \circ \tilde{f}_1, a \circ \tilde{i}) = (f_1 \circ d, i \circ b)$ is jointly strongly epimorphic, and finally f_1 and i are jointly strongly epimorphic, proving that $\mathcal{A}$ is protomodular, as desired. $\square$

In the presence of binary coproducts, the weak protomodularity property is also equivalent to the following one:

2.4. Proposition. *Let $\mathcal{C}$ be a weakly lex category with binary coproducts. Then $\mathcal{C}$ is weak protomodular if and only if in any weak split pullback*

$$\begin{tikzcd}
P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, shift right, "g"'] \\
C \arrow[r, "f"'] & B \arrow[u, shift right, "i"']
\end{tikzcd}$$

the canonical arrow $(\overline{f}, i): P + B \to A$ is a split epimorphism.

Proof: Clearly any weak protomodular category with finite coproducts satisfies the property here above. On the other hand, assume that this latter property holds in $\mathcal{C}$, and let the square (1) be a weak split pullback such that there exists an arrow $h: E \to A$ with the property that the arrows $\overline{f}$ and i both factorize through h. If $l: P \to E$ and $m: B \to E$ are two arrows such that $h \circ l = \overline{f}$ and $h \circ m = i$, then the universal property of the coproduct $P + B$ gives a unique arrow $(l, m): P + B \to E$ with the property that $h \circ (l, m) = (\overline{f}, i)$. By assumption the arrow $(\overline{f}, i)$ is a split epi, and consequently the arrow h is a split epi as well. $\square$

When $\mathcal{C}$ has binary coproducts and a zero object the property of weak protomodularity can be expressed in terms of weak (split) kernels:

2.5. Proposition. *Let $\mathcal{C}$ be a weakly lex category with binary coproducts and a zero object. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is weak protomodular*
2. *in any weak split kernel in $\mathcal{C}$*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & A \arrow[d, shift right, "g"'] \\ 0 \arrow[r] & B \arrow[u, shift right, "i"'] \end{tikzcd}$$

the canonical arrow $(k, i): K + B \to A$ is a split epimorphism

Proof: Only the implication $2 \Rightarrow 1$ needs to be proved. Given any weak split pullback

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] \\ C \arrow[r, "f"'] & B \arrow[u, shift right, "i"'] \end{tikzcd}$$

a weak kernel of g is given by the outer rectangle

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & A \arrow[d, shift right, "g"'] \\ 0 \arrow[r] & C \arrow[r, "f"] & B, \arrow[u, shift right, "i"'] \end{tikzcd}$$

where the left hand square is a weak kernel of $\overline{g}$. Now by assumption the induced arrow $(\overline{f} \circ k, i): K + B \to A$ is a split epi. Since $(\overline{f}, i) \circ (k + 1_B) = (\overline{f} \circ k, i)$, this implies that $(\overline{f}, i)$ is a split epi.

□

3. Semi-abelian exact completions

The aim of this section is to study the exact categories with enough projectives which are semi-abelian.

Let us first introduce a new definition:

3.1. Definition. A category is *weak semi-abelian* if it is weak protomodular, it has finite coproducts and a zero object.

By Remark 2.2, if $\mathcal{C}$ is finitely complete and weak semi-abelian, then $\mathcal{C}$ is semi-abelian. It is also clear that a weakly lex category with finite coproducts and a zero object is weak semi-abelian exactly when the weak (split) kernels satisfy the condition 2 in Proposition 2.5.

3.2. Proposition. *Let $\mathcal{C}$ be a weakly lex category. The following conditions are equivalent:*

1. *the Cauchy completion $\mathcal{C}_{cc}$ of $\mathcal{C}$ is weak semi-abelian*
2. *$\mathcal{C}_{ex}$ is semi-abelian*

Proof: 1 $\Rightarrow$ 2 Since $(\mathcal{C}_{cc})_{ex} \simeq \mathcal{C}_{ex}$, we just need to prove that $(\mathcal{C}_{cc})_{ex}$ is semi-abelian. By Proposition 2.3 we know that $(\mathcal{C}_{cc})_{ex}$ is protomodular, so in particular $(\mathcal{C}_{cc})_{ex}$ is exact Maltsev, and this will imply that $(\mathcal{C}_{cc})_{ex}$ has finite coproducts. Indeed, if (R,X) and (S,Y) are two pseudo-equivalence relations, then their coproduct is given by the central part of the diagram

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & R+S \arrow[d, shift right=2, "d+\delta"'] \arrow[d, shift left=2, "c+\gamma"] & S \arrow[l, "i_S"'] \arrow[d, shift right=2, "\delta"'] \arrow[d, shift left=2, "\gamma"] \\ X \arrow[r, "i_X"'] \arrow[u] & X+Y \arrow[u] & Y, \arrow[l, "i_Y"] \arrow[u] \end{tikzcd}$$

which is a reflexive graph in an exact Maltsev category, ans so a pseudo-equivalence relation (the same argument holds for any finite coproduct). Moreover, the functor $\Gamma\colon \mathcal{C}_{cc} \to (\mathcal{C}_{cc})_{ex}$ preserves the zero object, and $(\mathcal{C}_{cc})_{ex}$ is semi-abelian.

2 $\Rightarrow$ 1 If $\mathcal{C}_{ex}$ is semi-abelian, then $\mathcal{C}_{cc}$ is weak protomodular, being a projective cover of $(\mathcal{C}_{cc})_{ex} \simeq \mathcal{C}_{ex}$. $\mathcal{C}_{cc}$ has finite coproducts, since a coproduct of two regular projective objects is regular projective. It is also clear that $\mathcal{C}_{cc}$ has a zero object, since $(\mathcal{C}_{cc})_{ex}$ has a zero object 0, and 0 is regular projective. $\square$

The following useful property of exact Maltsev categories will be needed:

3.3. Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be exact Maltsev categories with finite coproducts, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor that preserves finite coproducts. Then the categories $\mathcal{A}$ and $\mathcal{B}$ have finite colimits and the functor F preserves them.*

Proof: Given two arrows f and g

$$\begin{tikzcd} G \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \end{tikzcd}$$

in a category $\mathcal{A}$ with finite coproducts, it is easy to check that an arrow p is the coequalizer of f and g precisely when it is the coequalizer of the arrows $(f, 1_A, g)$ and $(g, 1_A, f)$

$$\begin{tikzcd} G+A+G \arrow[r, shift left=2, "{(f,1_A,g)}"] \arrow[r, shift right=2, "{(g,1_A,f)}"'] & A. \arrow[l] \end{tikzcd}$$

This fact will be useful to construct the coequalizer of f and g in any exact Maltsev category $\mathcal{A}$. Indeed, let I be the regular image factorization of this reflexive graph

$$\begin{tikzcd}
G+A+G \arrow[r, shift left=2, "{(f,1_A,g)}"] \arrow[r, shift right=2, "{(g,1_A,f)}"'] \arrow[dr, "q"'] & A \arrow[l] \arrow[r, "p"] \arrow[d] & B \\
& I. \arrow[u, shift left=2, "d"] \arrow[u, shift right=2, "c"']
\end{tikzcd}$$

I determines a reflexive relation on A in an exact Maltsev category, hence an equivalence relation on A. Accordingly, the quotient $p: A \to B$ of this (effective) equivalence relation exists, and p also is the coequalizer of $(f, 1_A, g)$ and $(g, 1_A, f)$, since q is an epimorphism. An exact functor preserving finite coproducts preserves each part of this construction and so preserves the coequalizer p. $\square$

Since the definition of semi-abelian category is given only in terms of finite limits and finite colimits, it is reasonable to give the following

3.4. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be two semi-abelian categories. A functor $F: \mathcal{A} \to \mathcal{B}$ is called *semi-abelian* if it preserves finite limits and finite colimits.

Let us denote by $SA[\mathcal{A}, \mathcal{B}]$ the category of semi-abelian functors from $\mathcal{A}$ to $\mathcal{B}$, and by $FCLCo[\mathcal{C}, \mathcal{D}]$ the category of finite coproduct preserving left covering functors from a weakly lex category $\mathcal{C}$ with coproducts to a semi-abelian category $\mathcal{D}$.

In the context of semi-abelian categories Theorem 1.1 gives the following result (see also [**15**] and [**24**]), which can be interpreted as the fact that $\mathcal{C}_{ex}$ is the "semi-abelian completion of $\mathcal{C}$":

3.5. Corollary. *Let $\mathcal{C}$ be a weak semi-abelian category. Then, for any semi-abelian category $\mathcal{B}$, the functor $(\Gamma \circ -): SA[\mathcal{C}_{ex}, \mathcal{B}] \to FCLCo[\mathcal{C}, \mathcal{B}]$ gives an equivalence of categories.*

Proof: As shown in the proof of Proposition 3.2, if $\mathcal{C}$ is weak semi-abelian, then $\mathcal{C}_{ex}$ is semi-abelian. By Lemma 33 in [**14**], the functor $\Gamma: \mathcal{C} \to \mathcal{C}_{ex}$ preserves finite coproducts. By using the description of coproducts in the semi-abelian category $\mathcal{C}_{ex}$ given in Proposition 3.2, one can then show that a left covering functor $F: \mathcal{C} \to \mathcal{B}$ preserves finite coproducts precisely when its exact extension $\overline{F}: \mathcal{C}_{ex} \to \mathcal{B}$ preserves finite coproducts. By Lemma 3.3 and by Theorem 1.1 the proof is complete. $\square$

4. Semi-abelian varieties

Any finitary variety of universal algebras is the exact completion of its (Kleisli) subcategory of free algebras [**14**]. In this last section we consider the special situation when $\mathcal{C}_{ex}$ is a semi-abelian variety of universal algebras and $\mathcal{C}$ is its subcategory of free algebras.

Varieties of universal algebras which are semi-abelian were characterized in [**10**]. A slightly different proof of this fact can be deduced from the general categorical results of the previous section.

4.1. Corollary. *Let $\mathcal{A}$ be a variety of universal algebras and $\mathcal{C}$ its full subcategory of free algebras. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ is semi-abelian*
2. *$\mathcal{C}_{cc}$ is weak semi-abelian*
3. *$\mathcal{C}$ is weak semi-abelian*
4. *$\mathcal{A}$ has a unique constant 0, n binary terms $\alpha_1(x,y), \ldots, \alpha_n(x,y)$ and a $(n+1)$-ary term β such that*

$$\beta(\alpha_1(x,y), \ldots, \alpha_n(x,y), y) = x, \quad \text{and} \quad \alpha_i(x,x) = 0 \quad \text{for } i = 1, \ldots, n$$

Proof: Conditions 1 and 2 are equivalent by Proposition 3.2, while 3 is also equivalent since $\mathcal{C}$ and $\mathcal{A}$ both have coproducts.

We are now going to prove that 3 and 4 are equivalent. It is clear that $\mathcal{C}$ has a zero object exactly when the theory has a unique constant, that we shall denote by 0.

Let us first assume that $\mathcal{C}$ is a weak semi-abelian category. Let $F(x,y)$ and $F(y)$ be the free algebras on two and one generators, respectively, and let us consider the actual (split) kernel in $\mathcal{A}$

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & F(x,y) \arrow[d, shift right, "g"'] \\ 0 \arrow[r] & F(y) \arrow[u, shift right, "i"'] \end{tikzcd}$$

where $g: F(x,y) \to F(y)$ is determined by $g(x) = y = g(y)$ and $i: F(y) \to F(x,y)$ is determined by $i(y) = y$. Let $j: K \vee F(y) \to F(x,y)$ be the union of $k: K \to F(x,y)$ and $i: F(y) \to F(x,y)$ as subobjects of $F(x,y)$:

$$\begin{tikzcd} K \arrow[r, dotted, "l"] \arrow[dr, "k"'] & K \vee F(y) \arrow[d, dotted, "j"] & F(y) \arrow[l, dotted, "m"'] \arrow[dl, "i"] \\ & F(x,y) & \end{tikzcd}$$

Let $p: \overline{K} \to K$ be a regular epi, with $\overline{K}$ in $\mathcal{C}$, so that the square

$$\begin{tikzcd} \overline{K} \arrow[r, "k \circ p"] \arrow[d] & F(x,y) \arrow[d, shift right, "g"'] \\ 0 \arrow[r] & F(y) \arrow[u, shift right, "i"'] \end{tikzcd}$$

is a weak kernel. Then the fact that $\mathcal{C}$ is weak protomodular and that $k \circ p$ and i both factorize through j implies that j is a split epi, and then an iso. The element x therefore belongs to $K \vee F(y)$, and this proves the existence of $k_1, k_2, \ldots, k_n$ in K and an $(n+1)$-ary term β such that $\beta(k_1, \ldots, k_n, y) = x$. But K is a subalgebra of $F(x, y)$ and then there exist n binary terms $\alpha_i(x, y) = k_i$ for $i = 1, \ldots, n$, so that $\beta(\alpha_1(x, y), \ldots, \alpha_n(x, y), y) = x$. By definition of K, $\alpha_i(x, x) = g(\alpha_i(x, y)) = 0$ for $i = 1, \ldots, n$.

Conversely, let us assume that the terms satisfying the conditions in 4 exist in the variety $\mathcal{A}$. Let the exterior of the diagram

$$\begin{tikzcd}
\overline{K} \arrow[dr, dotted, "p"] \arrow[drr, bend left, "k\circ p"] \arrow[ddr, bend right, "\overline{g}"'] & & \\
& K \arrow[r, "k"'] \arrow[d] & A \arrow[d, shift right, "g"'] \\
& 0 \arrow[r, "f"] & B. \arrow[u, shift right, "i"']
\end{tikzcd}$$

be a weak split kernel of the split epi g obtained by "covering" the actual kernel K of g with a regular epi $p: \overline{K} \to K$, with $\overline{K}$ in $\mathcal{C}$. It is clear that the induced arrow $p + 1_B: \overline{K} + B \to K + B$ is a regular epi in $\mathcal{A}$. Accordingly, it suffices to prove that $(k, i): K + B \to A$ is a regular epi in $\mathcal{A}$, so that the arrow $(k \circ p, i): \overline{K} + B \to A$ will be a regular epi in $\mathcal{A}$, and then it will be split, because it lies in $\mathcal{C}$.

For any a in A we have

$$g(\alpha_i(a, i \circ g(a)) = \alpha_i(g(a), g \circ i \circ g(a)) = \alpha_i(g(a), g(a)) = 0,$$

so that $\alpha_i(a, i \circ g(a)) = k(x_i)$, for some x_i in K. Now

$$\begin{aligned}
&(k, i) \circ \beta(i_K(x_1), \ldots, i_K(x_n), i_B \circ g(a)) \\
&\qquad = \beta(k(x_1), \ldots, k(x_n), i \circ g(a)) \\
&\qquad = \beta(\alpha_1(a, i \circ g(a)), \ldots, \alpha_n(a, i \circ g(a)), i \circ g(a)) \\
&\qquad = a,
\end{aligned}$$

and the arrow (k, i) is then surjective.

$\square$

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Marino Gran `gran@lmpa.univ-littoral.fr`

Université du Littoral
Laboratoire de Mathématiques Pures et Appliquées
50 Rue F. Buisson BP 699
62228 Calais
France