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DEFINING RELATIONS FOR CLASSICAL LIE SUPERALGEBRAS WITHOUT CARTAN MATRICES

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(*communicated by Clas Löfwall*)

Abstract

The analogs of Chevalley generators are offered for simple (and close to them) $\mathbb{Z}$-graded complex Lie algebras and Lie superalgebras of polynomial growth without Cartan matrix. We show how to derive the defining relations between these generators and explicitly write them for a "most natural" ("distinguished" in terms of Penkov and Serganova) system of simple roots. The results are given mainly for Lie superalgebras whose component of degree zero is a Lie algebra (other cases being left to the reader). Observe presentations presentations of exceptional Lie superalgebras and Lie superalgebras of hamiltonian vector fields.

To Jan–Erik Roos on his sixty–fifth birthday

§1. Preliminaries

After Berezin formulated to one of us (DL) the problem which in modern terms would sound "define supermanifolds and differential supergeometry" it was natural to look after examples of Lie superalgebras that naturally appear in mathematics. Homotopy rings with respect to Whitehead's product are examples of such Lie rings, but nobody, so far, described these natural rings in any case. By description we mean identification of the semisimple part and radical. A reason for such careless attention to these rings becomes clear soon after one tackles the problem: they are nilpotent, hence, not so interesting in a sense (simple Lie (super)algebras have a richer structure and interesting representation theory).

A paper by C. Löfvall and J.-E. Roos [**LR**] is a break-through: in a similar problem they made a very interesting observation: they not only found traces left by simple Lie superalgebras where nothing indicated them, they also identified these superalgebras as a "positive part" of certain twisted loops algebras with values in simple Lie superalgebras. The paper [**LR**] is, clearly, the first in a series to appear,

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where, among other things, Löfvall and Roos will need presentations (in other words, generators and defining relations) of the *positive* part $\mathfrak{g}_> = \sum\limits_{i>0} \mathfrak{g}_i$ of certain (twisted) loop superalgebra $\mathfrak{g} = \sum\limits_{i>0} \mathfrak{g}_i$, associated with the simple (or a close to the simple) finite dimensional Lie superalgebras.

The results obtained here (an extension of [**GLP**]), and those of [**GL1**], [**GL2**], [**Y**], [**LSe**], as well as those which, though listed as open problems, are supplied with an instruction how to solve them, describe how to shorthand the presentation needed, both generators and relations, compare with appalling presentation of [**T**] or implicit presentations of the positive parts of vectorial algebras in [**FF**].

We hope that our results contribute to the fest on the occasion of Jan-Erik's birthday and make his calculations, if not life, easier.

Namely, we observe that even in the absence of $\mathfrak{g}_0$ there is a concise way to encode the presentation. Indeed, in the majority of cases $\mathfrak{g}_>$ is the direct sum of irreducible $\mathfrak{g}_0$-modules $\mathfrak{g}_i$, and, as Lie superalgebra, $\mathfrak{g}_>$ is generated by $\mathfrak{g}_1$. If $\mathfrak{g}_1$ is irreducible, as in cases of Löfvall and Roos, then, instead of dim $\mathfrak{g}_1$ generators, it suffices to take just one, any one vector, say the lowest weight vector.

The space of relations (same as the space of generators in the general case) must not split into the direct sum of *irreducible* $\mathfrak{g}_0$-modules, but, nevertheless, one can list only the vacuum vectors, i.e., the lowest AND highest weight vectors (since some modules can be glued in indecomposable conglomerates, we need both). *Mathematica*-based package SuperLie ([**G**]) helps to find these vacuum vectors.

To reduce volume of the paper, we did not reproduce the standard homological interpretation of relations and spectral sequence leading to the answer; for Lie algebras it is expressed in [**LP**] and its superization is straightforward via the Sign Rule. Observe only that since relations represent homology class, they can be "pure" or "dirty" if defined modulo boundaries.

History and an overview. The traditional way to determine classical simple finite dimensional Lie algebra (over $\mathbb{C}$) is via Chevalley generators, though other generators are possible. For discussion of other possibilities with examples see [**GL1**]. Recently a presentation of simple Lie superalgebras of the four Cartan series of vector fields was given [**LP**] and, together with Serre relations for affine Kac–Moody algebras [**K1**], this completed description of presentations of simple $\mathbb{Z}$-graded Lie algebras of polynomial growth (presentations with respect to other choices of generators are certainly possible).

Here we consider simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth (and close to them "classical" Lie superalgebras, such as central extensions of the simple ones, their algebras of differentiations, etc.). Their list is conjecturally ([**LS1**]) completed and consists of

- the finite dimensional ones (classification results by Kaplansky and Nahm-Rittenberg-Scheunert [**FK**], [**NRS**] were skillfully rounded up by Kac [**K2**]),
- the vectorial algebras, i.e., algebras of vector fields, (classification announced [**LS1**] and partly proved [**LS2**] by Leites and Schepochkina; the proof was again quickly rounded up by Kac and Cheng [**K3**], bar some gaps, see [**Sh5**]),
- the twisted loop algebras (with symmetrizable Cartan matrix [**vdL**]), or ob-

tained as twisted loops [**FLS**], and

• the stringy (i.e., vectorial algebras pertaining to string theories) algebras (for their intrinsic definition and list see [**GLS1**]).

In terms of presentations, another subdivision is more natural:

(a) the algebras of the form $\mathfrak{g}(A)$ with Cartan matrix A (subdivided into subclasses (a_s) with symmetrizable Cartan matrix and (a_n) with non-symmetrizable Cartan matrix,

(q) the series $\mathfrak{psq}$ and its relatives (central extensions, exterior differentiations, etc.) and

(v) the vectorial algebras and their relatives, the members of the subclass are easily recognized by lack of the property "if α is a root, then so is $-\alpha$".

For $\mathfrak{g}(A)$ (both subcases) a very redundant presentation is given in [**LSS**] and, the minimal one, in [**GL2**]; their q-quantization (for symmetrizable A) is described in [**Y**]. The redundant presentation [**LSS**], though very long, has an advantage: it only involves Serre relations. Regrettably, it is so redundant that practically it is useless.

For $\mathfrak{q}(n)$ and $\mathfrak{q}(n)^{(1)}$ see [**LSe**]; presentation of twisted loops is an open problem.

For presentations of vectorial Lie algebras see [**LP**]; the series with $\mathfrak{g}_0$ a Lie algebra see [**GLP**]; here we also consider several cases left in [**GLP**] as open problems and the Lie superalgebras of Hamiltonian vector fields and its central extension: Poisson superalgebra. The last two cases are of interest in relation with spinor-oscillator representations and its q-quantization, see [**Kl**], [**LSh**].

Though several results describing presentations of simple and close to them vectorial superalgebras were obtained a while ago ([**Ko**], [**T**]) they are given in the form too bulky to grasp or implicit ([**U**]; [**FF**]). A simplification of presentations is desirable: for $\mathfrak{g}_0 = \mathfrak{sh}(0|n)$ the dimension of the space of relations computed in [**T**] for $n = 5$ is equal to 420 and grows with n, whereas the total number of vacum vectors in the space of relations is < 10 and does not grow with n, cf. Tables 2.1.2.

Problem formulation. Consider $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \underset{i\in\mathbb{Z}}{\oplus} \mathfrak{g}_i$ of the following two types:

(1) vectorial Lie superalgebras, i.e., of finite *depth* d (in the above sum $i \geqslant d$), cf. [**LP**];

(2) of infinite depth, but not of the form $\mathfrak{g}(A)$ (the algebras $\mathfrak{g}(A)$ being already considered in complementing each other papers [**GL2**] and [**Y**]) or of type $\mathfrak{q}$ (for whose presentation see [**LSe**]).

Among these algebras we will first consider the ones for which $\mathfrak{g}_0$ is a Lie algebra. The general case is an open problem, which can be solved any time for any given $\mathfrak{g}$ via the lines indicated here and with the help of Grozman's SuperLie package.

Observe that in [**GL2**] and [**Y**] all bases (systems of simple roots or, rather, corresponding generators) are considered. For the vectorial algebras and superalgebras and for loop algebras with values in vectorial superalgebras we have considered below just one of the possible bases. It can well happen that presentations corresponding to some other base is nicer in some sense: e.g., for a rank n simple Lie algebra, Serre relations corresponding to $3n$ Chevalley generators though numerous ($\sim n^2$) are very simple and easy to compute, unlike a handful of independent on n

but more intricate relations between the pair of "Jacobson generators" considered in [**GL1**]. Nevertheless, both presentations are needed. For a method of passage from base to base (an analog of the Weyl group) see [**PS**].

Let $\mathfrak{g}_+ = \underset{i>0}{\oplus} \mathfrak{g}_i$ and $\mathfrak{g}_- = \underset{i<0}{\oplus} \mathfrak{g}_i$; let $\mathfrak{n}_\pm$ be a maximal nilpotent subalgebra of $\mathfrak{g}_0$ described in textbooks (e.g., [**OV**]) if $\mathfrak{g}_0$ is a Lie algebra or in [**PS**] (see also refs. therein) if $\mathfrak{g}_0$ is a Lie superalgebra. We decompose $\mathfrak{g}$ into the sum $\mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$, where $\mathfrak{N}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{g}_\pm$, and for the cases when $\mathfrak{g}_0$ is a Lie algebra (purely even) describe the defining relations. The relations obtained for vectorial Lie superalgebras are not very simple-looking, cf. [**LP**].

Notice that, unlike the case of finite dimensional simple Lie algebras, the bases, i.e., systems of simple roots, correspond not to maximal solvable Lie superalgebras (described in [**Shc**]) but to what is called *Borel subalgebras* in [**PS**].

Open problems are listed in §4.

§1. Generators in some vectorial Lie superalgebras and associated loops

1.1. Generators of $\mathfrak{vect}(0|n)$

Set $\partial_i = \frac{\partial}{\partial x_i}$. Some of the generators of $\mathfrak{vect}(0|n)$ generate its subalgebras as indicated (i.e., the X_i^+ and Y generate $\mathfrak{N}_+$; the $X_i^\pm$ generate $\mathfrak{sl}(1|n)$):

	$\mathfrak{sl}(1\vert n)$	
$\mathfrak{N}_+$	$x_1\partial_2, \quad \dots, x_{n-1}\partial_n, \quad \mathbf{x_n \sum x_i \partial_i}$	$x_n x_{n-1}\partial_1$
$\mathfrak{N}_-$	$x_2\partial_1, \quad \dots, \quad x_n\partial_{n-1}, \quad \partial_n$	
notations	$X_1^\pm, \quad \dots, \quad X_{n-1}^\pm, \quad X_n^\pm$	Y

The generators of $\mathfrak{svect}(0|n)$ are the same as of $\mathfrak{vect}(0|n)$ but without the boldfaced element $X_n^+ = x_n \sum x_i \partial_i$. The loop algebras have two more generators: for $\mathfrak{vect}(0|n)$ set

$$X_0^- = x_1 \dots x_n \partial_n \cdot t^{-1} \text{ and } X_0^+ = \partial_1 \cdot t.$$

For $\mathfrak{svect}(0|n)$ set

$$X_0^- = x_1 \dots x_{n-1} \partial_n \cdot t^{-1} \text{ and } X_0^+ = \partial_1 \cdot t.$$

It is not clear that this choice of generators (the highest weight vector of $\mathfrak{g}_{-1}$ and the lowest weight vector of $\mathfrak{g}_1$) which gives nice-looking relations for Lie algebras (and even Lie superalgebra with Cartan matrix) is the best when Lie superalgebras are very non-symmetric.

1.2. Generators of $\mathfrak{k}(1|n)$

In what follows we will by abuse of language write just f instead of either H_f, the Hamiltonian vector field generated by f or K_f, the contact vector field generated by f; in so doing we must remember that in either case (H_f or K_f) the degree of the vector field generated by a monomial f of degree k is equal to $k-2$.

Some of the generators of $\mathfrak{k}(1|n)$ generate the following subalgebras (for $n = 2k > 6$ and $n = 2k + 1 > 5$, respectively):

	$\mathfrak{osp}(2k\|2)$	
$\mathfrak{N}_+$	$\mathbf{t\eta_1}$ $\xi_1\eta_2$ $\dots$ $\eta_n\xi_{n-1}$ $\xi_n\xi_{n-1}$	$\eta_1\eta_2\eta_3$
$\mathfrak{N}_-$	ξ_1 $\eta_1\xi_2$ $\dots$ $\eta_{n-1}\xi_n$ $\eta_n\eta_{n-1}$	
notations	$X_0^{\pm}$ $X_1^{\pm}$ $\dots$ $X_{n-1}^{\pm}$ $X_n^{\pm}$	Y
	$\mathfrak{osp}(2k+1\|2)$	
$\mathfrak{N}_+$	$\mathbf{t\eta_1}$ $\xi_1\eta_2$ $\dots$ $\eta_n\xi_{n-1}$ $\xi_n\theta$	$\eta_1\eta_2\eta_3$
$\mathfrak{N}_-$	ξ_1 $\eta_1\xi_2$ $\dots$ $\eta_{n-1}\xi_n$ $\eta_n\theta$	
notations	$X_0^{\pm}$ $X_1^{\pm}$ $\dots$ $X_{n-1}^{\pm}$ $X_n^{\pm}$	Y

The generators of $\mathfrak{h}(0|n)$ and $\mathfrak{po}(0|n)$ are those above without the boldfaced element $X_0^+ = t\eta_1$.

The loop algebras $\mathfrak{h}(0|n)^{(1)}$ and $\mathfrak{po}(0|n)^{(1)}$ have two more generators:

$$Y_0^- = \xi_1 \dots \xi_n\eta_n\eta_{n-1} \dots \eta_2 \cdot t^{-1} \quad \text{and} \quad Y_0^+ = \eta_1 \cdot t.$$

It is not clear that this choice of generators is the best and it is desirable to experiment with other choices.

In small dimensions ($n < 7$) relations look differently and are to be computed separately. Besides, the generators look different. Though presentations of some of these algebras were considered, it is advisabable to revise it (the results of A. Nilsson are unpublished and those of [**FNZ**], as well of [**T**], should be presented in a more user-friendly form).

§2. Relations

2.1. Relations for $\mathfrak{N}_-$ of $\mathfrak{k}(1|n)$, $\mathfrak{po}(0|n)$ and $\mathfrak{h}(0|n)$

Clearly, $\mathfrak{N}_-$ for $\mathfrak{vect}(0|n)$ and $\mathfrak{svect}(0|n)$ coincides with $\mathfrak{n}_-$ for $\mathfrak{sl}(1|n)$ while $\mathfrak{N}_-$ for $\mathfrak{k}(1|n)$ and $\mathfrak{po}(0|n)$ coincides with $\mathfrak{n}_-$ for $\mathfrak{osp}(n|2)$, which are known [**LSe**], [**GL2**].

2.1.1. Relations for $\mathfrak{N}_-$ of $\mathfrak{h}(0|n)$

The Lie algebra $\mathfrak{h}(0|n)$ is generated by the same elements as $\mathfrak{k}(1|n)$ and $\mathfrak{po}(0|n)$ but the relations are different: for $\mathfrak{h}(0|n)$ there is an additional relation of weight $(0, \dots, 0)$ with respect to $\mathfrak{o}(n)$ because (for $n > 1$)

$$H_2(\mathfrak{g}_{-1}) = S^2(\mathfrak{g}_{-1}) = R(2\pi) \oplus R(0).$$

The corresponding cycle of weight 0 is

$$\{\xi_1, \eta_1\} + \dots + \{\xi_n, \eta_n\} \quad (+\{\theta, \theta\} \text{ if } n \text{ is odd }). \tag{$*$}$$

The relation expressed in terms of generators looks awful. It can be beautified as follows. In the space of relations corresponding to the other irreducible component the subspace of relations of weight 0 is of dimension $n-1$. Therefore, $n-1$ summands in $(*)$ vanish; for role of survivor select the simplest one of them, say, the following one: $\{\xi_1, \eta_1\} = 0$.

2.1.2. Relations for $\mathfrak{N}_+$ of $\mathfrak{h}(0|n)$, $n > 4$

$H_2(\mathfrak{g}_+)$ is the direct sum of irreducible $\mathfrak{g}_0$-modules with the following lowest weights with respect to $\mathfrak{o}(n)$, for notations see Tables in [**OV**]:

For $n = 2l$, $l \geqslant 5$:

N	the lowest weight	the corresponding cycle
1	$-2(\varepsilon_1+\varepsilon_2+\varepsilon_3)$	$\eta_1\eta_2\eta_3 \wedge \eta_1\eta_2\eta_3$
2	$-2(\varepsilon_1+\varepsilon_2)$	$\sum_i \eta_1\eta_2\eta_i \wedge \eta_1\eta_2\xi_i$
3	$-2\varepsilon_1$	$\sum_{i,j} \eta_1\eta_i\xi_j \wedge \eta_1\eta_i\xi_j - 2\sum_{i<j} \eta_1\eta_i\eta_j \wedge \eta_1\xi_i\xi_j$
4	0	$\sum_{i,j}\sum_{k\leqslant l}(\eta_i\eta_j\xi_k \wedge \xi_i\xi_j\eta_k + \eta_i\eta_j\eta_k \wedge \xi_i\xi_j\xi_k)$
5	$-(2\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)$	$\eta_1\eta_2\eta_3 \wedge \eta_4\eta_5\eta_1 - \eta_1\eta_2\eta_4 \wedge \eta_3\eta_5\eta_1 +$ $+\eta_1\eta_2\eta_5 \wedge \eta_3\eta_4\eta_1$
6	$-\varepsilon_1$	$\eta_1 \sum_i \eta_i\eta_{i+1}\eta_{i+2} \wedge \xi_i\xi_{i+1}\xi_{i+2}$

For small l the relations look differently; the form of relations 1) – 3) is the same as in the general case, the new in form relations are (here $\sum_{\rm cycl}$ means the cyclic permutation of $\eta_1\eta_2\eta_3$):

l	N	the lowest weight	the corresponding cycle
3	5	$-2\varepsilon_1$	$\eta_1\eta_2\xi_2 \wedge \eta_1\eta_3\xi_3 - \eta_1\eta_2\xi_3 \wedge \eta_1\eta_3\xi_2 - \eta_1\eta_2\eta_3 \wedge \eta_1\xi_2\xi_3$
3	6	$-2(\varepsilon_3-\varepsilon_1-\varepsilon_2)$	$\eta_1\eta_2\xi_3 \wedge \eta_1\eta_2\xi_3$
4	5	$\varepsilon_4-\varepsilon_1+\varepsilon_2+\varepsilon_3$	$\sum_{\rm cycl}\eta_1\eta_2\xi_4 \wedge \eta_3\eta_4\xi_4 - \sum_i\sum_{\rm cycl}(\eta_1\eta_2\xi_4 \wedge \eta_3\eta_i\xi_i +$ $\eta_1\eta_2\eta_3 \wedge \sum_i \eta_i\xi_i\xi_4)$
4	6	$-(2\varepsilon_1+\varepsilon_2+\varepsilon_3)$	$\eta_1\eta_2\eta_3 \wedge \eta_1\eta_4\xi_4 - \eta_1\eta_2\eta_4 \wedge \eta_1\eta_3\xi_4 + \eta_1\eta_3\eta_4 \wedge \eta_1\eta_2\xi_4$
5	6	$-(2\varepsilon_1-\varepsilon_2-\varepsilon_3-$ $-\varepsilon_4+\varepsilon_5)$	$\eta_1\eta_2\eta_3 \wedge \eta_4\eta_5\eta_1 - \eta_1\eta_2\eta_4 \wedge \eta_3\eta_5\eta_1 + \eta_1\eta_2\eta_5 \wedge \eta_3\eta_4\eta_1$
6		$-\varepsilon_1$	$\eta_1 \sum_i(\eta_i\eta_{i+1}\eta_{i+2} \wedge \xi_i\xi_{i+1}\xi_{i+2})$

For $n = 2l+1$, $l \geqslant 5$:

N	the lowest weight	the corresponding cycle
1	$-2(\varepsilon_1+\varepsilon_2+\varepsilon_3)$	$\eta_1\eta_2\eta_3 \wedge \eta_1\eta_2\eta_3$
2	$-2(\varepsilon_1+\varepsilon_2)$	$\sum_i \eta_1\eta_2\eta_i \wedge \eta_1\eta_2\xi_i$
3	$-2\varepsilon_1$	$\sum_{i,j} \eta_1\eta_i\xi_j \wedge \eta_1\eta_i\xi_j - 2\sum_{i<j} \eta_1\eta_i\eta_j \wedge \eta_1\xi_i\xi_j$
4	0	$\sum_{i,j}\sum_{k\leqslant l}(\eta_i\eta_j\xi_k \wedge \xi_i\xi_j\eta_k + \eta_i\eta_j\eta_k \wedge \xi_i\xi_j\xi_k)$
5	$-(2\varepsilon_1+\varepsilon_2+\varepsilon_3$ $+\varepsilon_4+\varepsilon_5)$	$\eta_1\eta_2\eta_3 \wedge \eta_4\eta_5\eta_1 - \eta_1\eta_2\eta_4 \wedge \eta_3\eta_5\eta_1 + \eta_1\eta_2\eta_5 \wedge \eta_3\eta_4\eta_1$
6	$-\varepsilon_1$	$\eta_1 \sum_i \eta_i\eta_{i+1}\eta_{i+2} \wedge \xi_i\xi_{i+1}\xi_{i+2}$

The space $H_1(\mathfrak{n}_+; \mathfrak{g}_1)$ is responsible for the following relations (element indicated should vanish for $i > 3$):

$$\{\xi_1\eta_2, \eta_1\eta_2\eta_3\},\ \{\xi_2\eta_3, \eta_1\eta_2\eta_3\},\ (\text{ad } \xi_3\eta_4)^2(\eta_1\eta_2\eta_3),$$

$$\{\xi_i\eta_{i+1}, \eta_1\eta_2\eta_3\},\ \{\xi_{n-1}\xi_n, \eta_1\eta_2\eta_3\}.$$

2.1.3. Relations for $\mathfrak{N}_+$ of $\mathfrak{k}(1|n), n > 4$

For the X_i^+ the relations are the same as for $\mathfrak{n}_+$ of $\mathfrak{osp}(n|2)$, cf. [**GL2**].

The relations between the X_i^+, $1 \leqslant i \leqslant n$ and Y are the same as for $\mathfrak{h}(0|n)$. New relations involving X_0^+ and Y are:

N	the lowest weight	the corresponding cycle
1	$-4\varepsilon_1$	$\sum_i(-q_1^2q_i \wedge q_1^2p_i) + (n+2)tq_1 \wedge q_1^3$
2	$-3\varepsilon_1 - \varepsilon_2$	$q_1^3 \wedge tq_2 + q_1^2q_2 \wedge tq_1$

2.1.4. Relations between $\mathfrak{N}_+$ and $\mathfrak{N}_-$ for $\mathfrak{h}(0|n)$ and $\mathfrak{k}(1|n)$, $n > 4$

These relations are as for $\mathfrak{osp}(2|2n)$ unless they involve Y; and the new extra ones are:

$$[Y, X_0^-] = \eta_2\eta_3 \qquad [Y, X_i^-] = 0 \text{ for } \ i > 0.$$

2.2. Relations for $\mathfrak{vect}(0|n)$ and $\mathfrak{svect}(0|m)$, $m > 2$

2.2.1. Relations for $\mathfrak{N}_+$ of $\mathfrak{vect}(0|n)$, $n > 2$

The space $H_1(\mathfrak{n}_+; \mathfrak{g}_1)$ is spanned by $(\text{ad}p_1q_2)^3q_1^3 \wedge p_1q_2$, $\quad q_1^3 \wedge p_2q_3$, $\dots$, $q_1^3 \wedge p_{n-1}q_n$, and $q_1^3 \wedge p_n^2$.

$H_2(\mathfrak{g}_+)$ is the direct sum of irreducible $\mathfrak{g}_0$-modules with the following lowest weights:

N	the lowest weight	the corresponding cycle
1	$2(\varepsilon_n + \varepsilon_{n-1} - \varepsilon_1)$	$\xi_n\xi_{n-1}\partial_1 \wedge \xi_n\xi_{n-1}\partial_1$
2	$2\varepsilon_n + \varepsilon_{n-1} + \varepsilon_{n-2} - \varepsilon_1 - \varepsilon_2$	$\xi_n\xi_{n-1}\partial_2 \wedge \xi_n\xi_{n-2}\partial_1 - \xi_n\xi_{n-1}\partial_1 \wedge \xi_n\xi_{n-2}\partial_2$
3	$2\varepsilon_n + \varepsilon_{n-1} - \varepsilon_1$	$\sum_i \xi_n\xi_i\partial_1 \wedge \xi_n\xi_{n-1}\partial_i$
4	$\varepsilon_n + \varepsilon_{n-1} + \varepsilon_{n-2} +$ $+\varepsilon_{n-3} - 2\varepsilon_1$	$\xi_n\xi_{n-1}\partial_1 \wedge \xi_{n-2}\xi_{n-3}\partial_1 - \xi_n\xi_{n-2}\partial_1 \wedge \xi_{n-1}\xi_{n-3}\partial_1 +$ $\xi_n\xi_{n-3}\partial_1 \wedge \xi_{n-1}\xi_{n-2}\partial_1$
5	$2\varepsilon_n$	$\sum_{i,j} \xi_n\xi_i\partial_i \wedge \xi_n\xi_j\partial_j$
6	$\varepsilon_n + \varepsilon_{n-1} + \varepsilon_{n-2} - \varepsilon_1$	$\sum_i(\xi_n\xi_{n-1}\partial_1 \wedge \xi_{n-2}\xi_i\partial_i + \xi_{n-2}\xi_{n-3}\partial_1 \wedge \xi_n\xi_i\partial_i +$ $\xi_{n-2}\xi_n\partial_1 \wedge \xi_{n-1}\xi_i\partial_i)$
7	$2\varepsilon_n + \varepsilon_{n-1} - \varepsilon_1$	$\sum_i \xi_n\xi_i\partial_i \wedge \xi_n\xi_{n-1}\partial_1$
8	$2\varepsilon_n$	$(\sum \xi_n\xi_i\partial_i) \wedge (\sum \xi_n\xi_j\partial_j)$

The corresponding relations for $\mathfrak{svect}$ are the relations 1) – 4).

The relations for $\mathfrak{vect}(0|4)$ are

N	the lowest weight	the corresponding cycle
1	$2(\varepsilon_4+\varepsilon_3-\varepsilon_1)$	$\xi_4\xi_3\partial_1\wedge\xi_4\xi_3\partial_1$
2	$2\varepsilon_4+\varepsilon_3-\varepsilon_1$	$\sum\xi_4\xi_i\partial_1\wedge\xi_4\xi_3\partial_i$
3	$2\varepsilon_4$	$\sum_{i,j}\xi_4\xi_i\partial_j\wedge\xi_4\xi_j\partial_i$
4	$\varepsilon_4+\varepsilon_3+\varepsilon_2-\varepsilon_1$	$\sum(\xi_4\xi_3\partial_1\wedge\xi_2\xi_i\partial_i+\xi_4\xi_2\partial_1\wedge\xi_3\xi_i\partial_i+\xi_3\xi_2\partial_1\wedge\xi_4\xi_i\partial_i)$
5	$2\varepsilon_4$	$(\sum\xi_4\xi_i\partial_i)\wedge(\sum\xi_4\xi_i\partial_i)$
6	$2\varepsilon_4+\varepsilon_3-\varepsilon_1$	$(\sum\xi_4\xi_i\partial_i)\wedge\xi_4\xi_3\partial_1$

The corresponding relations for $\mathfrak{svect}(0|4)$ are the relations 1) – 3). The relations for $\mathfrak{vect}(0|3)$ are

N	the lowest weight	the corresponding cycle
1	$2(\varepsilon_3+\varepsilon_2-\varepsilon_1)$	$\xi_3\xi_2\partial_1\wedge\xi_3\xi_2\partial_1$
2	$2\varepsilon_3$	$\xi_3\xi_2\partial_1\wedge\xi_3\xi_1\partial_2+\xi_3\xi_1\partial_2\wedge\xi_3\xi_2\partial_1$
3	$2\varepsilon_3$	$(\sum\xi_3\xi_i\partial_i)\wedge(\sum\xi_3\xi_i\partial_i)$
4	$2\varepsilon_3+\varepsilon_2-\varepsilon_1$	$(\sum\xi_3\xi_i\partial_i)\wedge\xi_3\xi_2\partial_1$

The corresponding relations for $\mathfrak{svect}(0|3)\simeq\mathfrak{spe}(3)$ are the relations 1) – 2).

2.2.2. Relations between $\mathfrak{N}_+$ and $\mathfrak{N}_-$ for $\mathfrak{vect}(0|n)$, $n>3$, and $\mathfrak{svect}(0|m)$, $m>2$

These relations are as for $\mathfrak{sl}(1|n)$ unless they involve Y; the extra relations are:

$$[Y,X_n^-]=(x_{n-1}\partial_1)=[X_{n-1}^-,[\dots,[X_2^-,X_1^-]]\dots];\qquad [Y,X_i^-]=0 \text{ for } i>0.$$

2.2.3. Relations for $\mathfrak{N}_\pm$ of $\mathfrak{vect}(0|n)^{(1)}$, $n>3$

The new relations that involve X_0^+ are (we only indicate the terms to be equated to zero):

For $n=3$, $\mathfrak{N}_+$: $[X_0^+,X_0^+]$, $[X_0^+,X_2^+]$, $[X_0^+,Y]$, $[X_1^+,[X_0^+,X_1^+]]$, $[Y,[X_0^+,X_1^+]]-[X_0^+,X_3^+]$, $(\mathrm{ad}X_1^+)^2[X_1^+,Y]$, $[Y,(\mathrm{ad}X_1^+)^2Y]$, $[[X_0^+,X_1^+],[X_0^+,X_3^+]]$, $[[X_0^+,X_3^+],[X_2^+,X_3^+]]$, $[[X_3^+,[X_0^+,X_1^+]]$, $[Y,[X_1^+,X_2^+]]]+\frac{1}{2}[(\mathrm{ad}X_1^+)^2Y,[X_2^+,[X_0^+,X_3^+]]]$.

For $n=3$, $\mathfrak{N}_-$: $[X_0^-,X_1^-]$, $(\mathrm{ad}X_0^-)^2X_2^-$, $(\mathrm{ad}X_0^-)^2X_3^-$, $(\mathrm{ad}X_2^-)^2X_0^-$, $[[X_0^-,X_2^-],[X_0^-,X_3^-]]$, $[[X_2^-,[X_0^-,X_3^-]]$, $[X_3^-,[X_0^-,X_2^-]]]$, $[[[X_0^-,X_3^-],[X_1^-,X_2^-]]$, $[[X_0^-,X_3^-],[X_2^-,X_3^-]]]+2[[X_3^-,[X_1^-,[X_0^-,X_2^-]]]$, $[[X_0^-,X_3^-],[X_2^-,X_3^-]]]$

For $n=4$, $\mathfrak{N}_+$: $[X_0^+,X_0^+]$, $[X_0^+,X_2^+]$, $[X_0^+,X_3^+]$, $[X_0^+,Y]$, $(\mathrm{ad}X_1^+)^2X_0^+$, $[Y,[X_0^+,X_1^+]]$, $[[X_0^+,X_1^+],[X_0^+,X_4^+]]$, $[[X_0^+,X_1^+],[X_2^+,Y]]-[X_0^+,X_4^+]$, $[[X_0^+,X_4^+],[X_3^+,X_4^+]]$, $[[X_2^+,Y],[X_3^+,[X_0^+,X_4^+]]]$, $[[X_4^+,[X_0^+,X_1^+]]$, $[Y,[X_2^+,X_3^+]]]-\frac{1}{2}[[X_3^+,[X_0^+,X_4^+]]$, $[Y,[X_1^+,X_2^+]]]$, $[[[X_0^+,X_4^+],[X_1^+,X_2^+]]$, $[[X_1^+,Y],[X_2^+,X_3^+]]]+[[[X_0^+,X_1^+],[X_3^+,X_4^+]]$, $[[X_1^+,X_2^+],[X_2^+,Y]]]$

<u>For $n=4$, $\mathfrak{N}_-$</u>: $[X_0^-, X_0^-]$, $[X_0^-, X_1^-]$, $[X_0^-, X_2^-]$, $(\mathrm{ad}X_3^-)^2 X_0^-$, $[[X_0^-, X_3^-], [X_0^-, X_4^-]]$, $[[X_0^-, X_4^-], [X_3^-, [X_0^-, X_4^-]]]$,

$$[[X_2^-, [X_0^-, X_3^-]], [X_4^-, [X_0^-, X_3^-]]],$$
$$[[X_4^-, [X_0^-, X_3^-]], [[X_0^-, X_4^-], [X_2^-, X_3^-]]],$$
$$[[X_0^-, X_4^-], [X_3^-, X_4^-]], [[X_2^-, X_3^-], [X_4^-, [X_0^-, X_3^-]]]],$$

$$[[[X_1^-, X_2^-], [[X_0^-, X_4^-], [X_3^-, X_4^-]]], [[X_4^-, [X_0^-, X_3^-]], [X_4^-, [X_2^-, X_3^-]]]]+$$
$$\tfrac{1}{2}\,[[[X_1^-, X_2^-], [[X_0^-, X_4^-], [X_3^-, X_4^-]]], [[X_3^-, [X_0^-, X_4^-]], [X_4^-, [X_2^-, X_3^-]]]]$$

<u>For $n=5$, $\mathfrak{N}_+$</u>: $[X_0^+, X_0^+]$, $[X_0^+, X_2^+]$, $[X_0^+, X_3^+]$, $[X_0^+, X_4^+]$, $[X_0^+, Y]$, $(\mathrm{ad}X_1^+)^2 X_0^+$, $[Y, [X_0^+, X_1^+]]$, $[[X_0^+, X_1^+], [X_0^+, X_5^+]]$, $[[X_0^+, X_5^+], [X_4^+, X_5^+]]$, $[[X_3^+, Y], [X_2^+, [X_0^+, X_1^+]]]+[X_0^+, X_5^+], [[X_3^+, Y], [X_4^+, [X_0^+, X_5^+]]] = 0$, $[[X_5^+, [X_0^+, X_1^+]], [Y, [X_3^+, X_4^+]]]$, $[[Y, [X_3^+, X_4^+]], [[X_0^+, X_5^+], [X_1^+, X_2^+]]] + [[Y, [X_1^+, X_2^+]], [[X_0^+, X_5^+], [X_3^+, X_4^+]]]$

$$[[[X_3^+, Y], [X_3^+, [X_1^+, X_2^+]]], [[X_4^+, X_5^+], [X_2^+, [X_0^+, X_1^+]]]]-$$
$$[[[X_2^+, X_3^+], [X_5^+, [X_0^+, X_1^+]]], [[X_3^+, X_4^+], [Y, [X_1^+, X_2^+]]]],$$

$$[[[X_4^+, X_5^+], [[X_1^+, Y], [X_2^+, X_3^+]]], [[X_3^+, [X_1^+, X_2^+]], [Y, [X_3^+, X_4^+]]]] =$$
$$\tfrac{1}{2}\,[[[X_2^+, X_3^+], [X_4^+, X_5^+]], [[X_4^+, X_5^+], [X_3^+, [X_1^+, Y]]]]+$$
$$\tfrac{1}{2}\,[[[X_3^+, Y], [X_4^+, X_5^+]], [[X_4^+, X_5^+], [X_3^+, [X_1^+, X_2^+]]]]$$

<u>For $n=5$, $\mathfrak{N}_-$</u>: $[X_0^-, X_1^-]$, $[X_0^-, X_2^-]$, $[X_0^-, X_3^-]$, $(\mathrm{ad}X_0^-)^2 X_4^-$, $(\mathrm{ad}X_4^-)^2 X_0^-$, $[[X_0^-, X_4^-], [X_0^-, X_5^-]]$, $[[X_4^-, [X_0^-, X_5^-]], [X_5^-, [X_0^-, X_4^-]]]$, $[[X_5^-, [X_0^-, X_4^-]], [[X_0^-, X_5^-], [X_3^-, X_4^-]]]$, $[[[X_0^-, X_5^-], [X_3^-, X_4^-]], [[X_0^-, X_5^-], [X_4^-, X_5^-]]]$, $[[[X_2^-, X_3^-], [X_5^-, [X_0^-, X_4^-]]], [[X_4^-, X_5^-], [X_3^-, [X_0^-, X_4^-]]]]$, $[[[X_4^-, X_5^-], [X_3^-, [X_0^-, X_4^-]]], [[X_2^-, X_3^-], [[X_0^-, X_5^-], [X_4^-, X_5^-]]]]$

2.2.4. The periplectic series

Recall that the compatible (with parity) $\mathbb{Z}$-gradings of $\mathfrak{spe}(n)$ are of the form $\mathfrak{spe}(n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and there are two such cases both with $\mathfrak{g}_0 = \mathfrak{sl}(n)$: (here id is the standard $\mathfrak{sl}(n)$-module):

a) $\mathfrak{g}_1 = S^2(\mathrm{id})$, $\quad \mathfrak{g}_{-1} = E^2(\mathrm{id}^*)$;

b) $\mathfrak{g}_1 = E^2(\mathrm{id})$, $\quad \mathfrak{g}_{-1} = S^2(\mathrm{id}^*)$.

Let $\mathfrak{n}^\pm$ be the maximal nilpotent subalgebras of $\mathfrak{g}_0$. Set

$$\mathfrak{m}^+ = \mathfrak{n}^+ \oplus \mathfrak{g}_1; \quad \mathfrak{m}^- = \mathfrak{n}^- \oplus \mathfrak{g}_{-1}$$

Denote by X^+ (resp. X^-) a vector of lowest (highest) weight in the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$). The first term $\underset{p+q=2}{\oplus} E_1^{p,q}$ of the spectral sequence converging to $H_2(\mathfrak{m}^\pm)$ consists of

$$E_1^{2,0} = H_2(\mathfrak{n}^\pm), \quad E_1^{1,1} = H_1(\mathfrak{n}^\pm; \mathfrak{g}_{\pm 1}), \quad E_1^{0,2} = H_0(\mathfrak{n}^\pm; E^2(\mathfrak{g}_{\pm 1})).$$

Since we already know $H_2(\mathfrak{n}^\pm)$, we are only interested in the other two summands. In case (a), (resp. (b)), $H_1(\mathfrak{n}^\pm; \mathfrak{g}_{\pm 1})$ is the same as for $\mathfrak{m}^+$ of $\mathfrak{sp}(2n)$ and for $\mathfrak{m}^-$ of $\mathfrak{o}(2n)$ (resp. for $\mathfrak{m}^-$ of $\mathfrak{o}(2n)$ and $\mathfrak{sp}(2n)$ of $\mathfrak{m}^+$), we explicitly have:

$$(\mathrm{ad}X_n^+)^3(X^+) = 0, \quad (\mathrm{ad}X_n^-)^2(X^-) = 0 \qquad (a)$$

$$(\mathrm{ad}X_n^+)^2(X^+) = 0, \quad (\mathrm{ad}X_n^-)^3(X^-) = 0 \qquad (b)$$

with $[X_i^+, X^+] = [X_i^-, X^-] = 0$ for $i < n$ in both cases (a) and (b).

Let φ_i be the ith fundamental weight of $\mathfrak{g}_0$, $R(\chi)$ the (space of the) irreducible representation with highest weight χ. Now, for the $\mathfrak{sl}(n)$-modules $\mathfrak{g}_1 = R(2\varphi_1)$ in case (a) and $\mathfrak{g}_1 = R(\varphi_2)$ in case (b), we have:

$$S^2(R(2\varphi_1)) = R(4\varphi_1) \oplus R(2\varphi_2)$$
$$S^2(R(\varphi_2)) = \begin{cases} R(2\varphi_2) \oplus R(\varphi_4) & \text{if } n > 3 \\ R(2\varphi_2) & \text{if } n = 3. \end{cases}$$

Therefore, we have the relations

$$[X^\pm, X^\pm] = 0 \qquad \text{for both cases a) and b)}$$

and the relations

$$\begin{array}{ll} \text{(a)} & [X^+, [X_n^+, [X_n^+, X^+]]] = 0, \quad [X^-, [X_n^-, [X_{n-1}^-, X^-]]] = 0; \\ \text{(b)} & [X^+, [X_n^+, [X_n^+, X^+]]] = 0, \quad [X^-, [X_n^-, [X_{n-1}^-, X^-]]] = 0. \end{array} \qquad (PeR_\pm)$$

of which the first in case (a) and the second in case (b) are only defined if $n > 3$.

2.3.1 Exceptional loop algebras: $\mathfrak{d}(\varepsilon)^{(3)}$

Let ε be a primitive cubic root of 1 and $\mathfrak{d}(\varepsilon)$ the deform of the Lie superalgebra $\mathfrak{osp}(4|2)$ corresponding to the value of parameter equal to ε, i.e., $\mathfrak{d}(\varepsilon) = \mathfrak{g}(A)$ for any of the following Cartan matrices (cf. [**GL2**]):

$$\begin{pmatrix} 0 & -1 & \varepsilon^2 \\ 1 & 0 & \varepsilon \\ \varepsilon^2 & -\varepsilon & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 2 & -1 & 0 \\ \varepsilon & 0 & \varepsilon^2 \\ 0 & -1 & 2 \end{pmatrix}, \text{ or } \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & \varepsilon \\ 0 & -1 & 2 \end{pmatrix}, \text{ or } \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & \varepsilon^2 \\ 0 & -1 & 2 \end{pmatrix}.$$

The algebra $\mathfrak{d}(\varepsilon)$ has an outer automorphism of order 3; select the generators of the maximal nilpotent subalgebras of $\mathfrak{d}(\varepsilon)^{(3)}$ as follows. Let $X_1^\pm$, $X_2^\pm$, $X_3^\pm$ be the Chevalley generators of $\mathfrak{d}(\varepsilon)$. Set

$$\begin{array}{ll} Y_1^+ = \varepsilon X_1^+ + \varepsilon^2 X_2^+ + X_3^+, & Y_2^+ = [X_3^-, [X_1^-, X_2^-]]; \\ Y_1^- = \varepsilon X_1^- + \varepsilon^2 X_2^- + X_3^-, & Y_2^+ = [X_1^+, X_2^+] - [X_1^+, X_3^+] + [X_2^+, X_3^+]. \end{array}$$

The relations between these generators are:

$$
\begin{gathered}
[Y_2^+, Y_2^+] = 0,\\
[[Y_1^+, Y_2^+], [Y_2^+, [Y_1^+, Y_1^+]]] = 0,\\
(\mathrm{ad}[Y_1^+, Y_1^+])^3([Y_1^+, Y_2^+]) = 0,\\
[[Y_1^+, Y_1^+], [[Y_2^+, [Y_1^+, Y_1^+]], [[Y_2^+, [Y_1^+, Y_1^+]], [[Y_1^+, Y_1^+], [Y_2^+, [Y_1^+, Y_1^+]]]]]] = 64Y_1^+,\\
[[Y_1^+, Y_2^+], [[Y_2^+, [Y_1^+, Y_1^+]], [[Y_2^+, [Y_1^+, Y_1^+]], [[Y_1^+, Y_1^+], [Y_2^+, [Y_1^+, Y_1^+]]]]]] = -96Y_2^+.
\end{gathered}
$$

$$
\begin{gathered}
(\mathrm{ad}Y_2^-)^3 Y_1^- = 0,\\
[[Y_1^-, Y_2^-], [Y_1^-, Y_2^-]] - 2[Y_2^-, [Y_2^-, [Y_1^-, [Y_1^-]] = 0,\\
(\mathrm{ad}[Y_1^-, Y_1^-])^2[Y_1^-, Y_2^-] = 0,\\
[[Y_1^-, Y_1^-], [[Y_2^-, [Y_1^-, Y_1^-]], [[Y_1^-, Y_2^-], [Y_2^-, [Y_1^-, Y_1^-]]]]] = -64Y_1^-,\\
[[Y_1^-, Y_2^-], [[Y_2^-, [Y_1^-, Y_1^-]], [[Y_1^-, Y_2^-], [Y_2^-, [Y_1^-, Y_1^-]]]]] = -64Y_2^-.
\end{gathered}
$$

2.3.2. Stringy superalgebras

For the exceptional stringy superalgebra $\mathfrak{kas}^L$ introduced in [**GLS1**] (and in [**CK**]) the relations are computed in [**GLS1**] and, in another form, in [**CK**]. Observe that the relations of $\mathfrak{kas}$ are *not* simply the relations for $\mathfrak{kas}^L$ that do not involve the extra generators.

Among the stringy superalgebras $\mathfrak{k}^L(1|6)$ is one of the most interesting: it possesses a nondegenerate invariant symmetric bilinear form and, therefore, can be q-quantized, cf. [**LSp**].

The basis of $\mathfrak{g}_+$ for the standard $\mathbb{Z}$-grading of $\mathfrak{k}^L(1|6)$:

$$
\begin{array}{llll}
X_1^+ = \xi_1\eta_2, & X_2^+ = \xi_2\eta_3, & X_3^+ = \xi_2\xi_3, & X_0^+ = t\eta_1,\\
\tilde{X}_1^+ = \frac{1}{t}\xi_1\xi_3\eta_2\eta_3, & \tilde{X}_2^+ = \frac{1}{t}\xi_1\xi_2\eta_1\eta_3, & \tilde{X}_3^+ = \frac{1}{t}\xi_1\xi_2\xi_3\eta_1, & \tilde{X}_0^+ = \eta_1\eta_2\eta_3
\end{array}
$$

The generators $X_i^{\pm}$ for $i \neq 0$, clearly, generate $\mathfrak{o}(6)$ while all of them generate $\mathfrak{osp}(6|2)$. One expects the same relations between them, but the other generators interfere and the final result is as follows (we skip the superscript):

$$
[X_2, X_3] = 0,\ \ [X_2, X_0] = 0,\ \ [X_3, X_0] = 0,\ \ [X_0, X_0] = 0;
$$

$$
(\mathrm{ad}X_1)^2 X_2 = 0;\ (\mathrm{ad}X_1)^2 X_3 = 0,\ (\mathrm{ad}X_1)^2 X_0 = 0,\ (\mathrm{ad}X_2)^2 X_1 = 0,\ (\mathrm{ad}X_3)^2 X_1 = 0;
$$

$$
\begin{gathered}
[X_1, \tilde{X}_0] = 0,\ \ [X_1, \tilde{X}_1] = 0,\ \ [X_2, \tilde{X}_0] = 0,\ \ [X_2, \tilde{X}_2] = 0,\\
[X_2, \tilde{X}_3] = 0,\ \ [X_3, \tilde{X}_2] = 0,\ \ [X_3, \tilde{X}_3] = 0,\ \ [X_0, \tilde{X}_0] = 0\\
[\tilde{X}_0, \tilde{X}_0] = 0,\ \ [\tilde{X}_0, \tilde{X}_1] = 0,\ \ [\tilde{X}_0, \tilde{X}_2] = 0,\\
[\tilde{X}_1, \tilde{X}_2] = 0,\ \ [\tilde{X}_1, \tilde{X}_3] = 0,\ \ [\tilde{X}_2, \tilde{X}_3] = 0;
\end{gathered}
$$

$$
\begin{gathered}
[X_2, \tilde{X}_1] + [X_1, \tilde{X}_2] = 0,\ \ [X_3, \tilde{X}_1] + [X_1, \tilde{X}_3] = 0;\\
(\mathrm{ad}X_3)^2 \tilde{X}_0 - 2[X_0, \tilde{X}_3] = 0;
\end{gathered}
$$

$$[\tilde X_0,[X_0,\tilde X_3]]=0,\ \ [\tilde X_1,[X_1,X_2]]=0,\ \ [\tilde X_1,[X_1,X_3]]=0,\ \ [\tilde X_1,[X_1,X_0]]=0,$$
$$[\tilde X_2,[X_1,X_2]]=0,\ \ [\tilde X_3,[X_1,X_3]]=0,$$
$$[\tilde X_3,[X_3,\tilde X_0]]=0,\ \ [\tilde X_3,[X_0,\tilde X_2]]=0;$$

$$(\mathrm{ad}\tilde X_1)^2X_0=0,\ \ (\mathrm{ad}\tilde X_2)^2X_0=0,\ \ (\mathrm{ad}\tilde X_3)^2X_0=0,\ \ (\mathrm{ad}\tilde X_3)^2\tilde X_0=0;$$

$$[[X_1,X_2],[X_3,\tilde X_0]]+2[\tilde X_2,[X_0,\tilde X_1]]+[X_0,[X_1,\tilde X_2]]+[\tilde X_2,[X_1,X_0]]=0;$$

$$[[X_1,X_2],[X_0,\tilde X_2]]=0,\ \ [[X_1,X_3],[X_0,\tilde X_3]]=0,$$

$$[[X_1,\tilde X_2],[X_0,\tilde X_2]]=0,[[X_1,\tilde X_3],[X_0,\tilde X_3]]=0,$$

$$[[X_1,\tilde X_2],[X_3,\tilde X_0]]-[\tilde X_2,[X_0,\tilde X_1]]=0,\ \ [[X_0,\tilde X_3],[\tilde X_2,[X_0,\tilde X_1]]=0,$$

$$[[X_0,\tilde X_3],[\tilde X_2,[X_1,X_0]]+[[X_0,\tilde X_3],[\tilde X_0,[X_1,\tilde X_2]]=0,$$

$$2[[X_1,\tilde X_2],[X_0,\tilde X_3]]+2[\tilde X_3,[X_0,[X_1,X_2]]]+$$

$$[[X_1,X_3],[X_0,\tilde X_2]]-[[X_1,X_2],[X_0,\tilde X_3]]=0;$$

$$2[[X_1,\tilde X_3],[X_0,\tilde X_2]]-2[\tilde X_3,[X_0,[X_1,X_2]]]+$$

$$[[X_1,X_2],[X_0,\tilde X_3]]-[[X_1,X_3],[X_0,\tilde X_2]]=0.$$

§4. Presentation of vectorial Lie superalgebras: an overview of open problems

4.1. Vectorial Lie superalgebras with polynomial coefficients and loops with values in them

Lemma . ([**ShV**]) *There are embeddings of Lie superalgebras such that their negative parts as $\mathbb{Z}$-graded Lie superalgebras coincide:*

$$\mathfrak{sl}(n+1|m)\subset\mathfrak{vect}(n|m),\quad \mathfrak{osp}(m|2n+2)\subset\mathfrak{k}(2n+1|m),$$
$$\mathfrak{pe}(n+1)\subset\mathfrak{m}(n),\quad \mathfrak{spe}(n+1)\subset\mathfrak{sm}(n).$$

This is already something: it remains to establish presentation of $\mathfrak{N}^+$ only. The rest of the known information is gathered in the following statement, cf. [**HP**]. The cases when the Lie superalgebra is not simple are marked by a + sign; we disregard them in this paper.

Theorem . *The following table lists the restrictions for the relations of the subalgebra $\mathfrak{N}^+$ of a simple vectorial Lie superalgebra to be "pure"* (blank space) *or "dirty"* (marked by a dot).

$\mathfrak{vect}(n|m)$

m				
	+			
3	+			
2	+			
1	+	•		
0	+	•	•	
	0	1	2	3 . . . n

$\mathfrak{svect}(n|m)$

m				
	+	+		
	+	+		
3	+	+		
2	+	+		
1	+	+		
0	+	+	•	
	0	1	2	3 . . . n

$\mathfrak{h}(2n|m)$

m				
	+			
	+			
3	+			
2	+			
1	+	•		
0	+	•	•	
	0	1	2	3 . . . n

$\mathfrak{k}(2n+1|m)$

m				
3				
2	•			
1	•			
0	•	•		
	0	1	2	3 . . . n

$\mathfrak{svect}^{\circ}(n)$

+	+	•		
0	1	2	3	4 . . . n

$\mathfrak{sh}(n)$

+	+	+	+	
0	1	2	3	4 . . . n

$\mathfrak{m}(n)$

•				
0	1	2	3	4 . . . n

$\mathfrak{b}_{\lambda}(n),\ \mathfrak{le}(n),\ \mathfrak{sle}(n)$

+	•			
0	1	2	3	4 . . . n

$\mathfrak{vect}(0|n)^{(1)},\ \mathfrak{sh}(2n)^{(2)}$

+				
0	1	2	3	4 . . . n

$\mathfrak{svect}(0|n)^{(1)}$

+	+			
0	1	2	3	4 . . . n

$\mathfrak{sh}(n)^{(1)}$

+	+	+		
0	1	2	3	4 . . . n

4.2. Twisted loops

There remains to compute the relations in the following twisted loop superalgebras which have no Cartan matrix: see [**LSS**]: $\mathfrak{psl}(n|n)^{(2)}_{\Pi}$; $\mathfrak{psl}(n|n)^{(2)}_{\Pi\circ(-st)}$; $\mathfrak{psq}(n)^{(4)}_{\Pi}$.

4.3. Stringy and vectorial superalgebras: other bases

We have only computed the relations in several cases. There remains to compute the relations in the cases marked by a thick dot (and a cross) in Table 4.1 and for the exceptional algebras [**Sh5**] (for the consistent gradings thereof this is done in [**GLS2**]; for the other W-gradings it is desirable, at least, for completeness). It is desirable to compute them for all simple stringy superalgebras or, at least, for the distinguished algebras that admit nontrivial central extensions most often used in applications, cf. [**GLS1**], and for other bases of vectorial algebras and superalgebras.

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