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NONSPLIT EXTENSIONS OF MODULAR LIE ALGEBRAS OF RANK 2

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(*communicated by Larry Lambe*)

Abstract

Second cohomology groups of irreducible representations of classical Lie algebras A_2, B_2 and G_2 over an algebraically closed field of characteristic $p > h$ are calculated. Here h is the Coxeter number.

To Jan–Erik Roos on his sixty–fifth birthday

1. Formulation of the main result

Levi-Mal'cev theorem has cohomological origin. It states that any finite-dimensional extension of a finite-dimensional simple Lie algebra over a field of characteristic 0 is split. In case of characteristic $p > 0$ any Lie algebra has at least one nonsplit extension and the number of irreducible modules with such a property is finite [**8**]. For example, the 3-dimensional simple Lie algebra $A_1 = sl_2$ has exactly one irreducible module, namely the $(p-1)$-dimensional module V_{p-2}, with $H^2(A_1, V_{p-2}) \neq 0$ [**7**].

The aim of our paper is to calculate second cohomology groups with coefficients in an irreducible module for simple Lie algebras of rank 2: $\mathfrak{g} = A_2, B_2$ and G_2. The field $\mathcal{K}$ is algebraically closed and has characteristic $p > h$, where h is the Coxeter number. An irreducible $\mathfrak{g}$-module V is called *2-peculiar*, if $H^2(\mathfrak{g}, V) \neq 0$. Let $\kappa_2(\mathfrak{g})$ be the number of peculiar modules. From our results it follows that $\kappa_2(\mathfrak{g}) = 2, 3, 3,$ for $\mathfrak{g} = A_2, B_2, G_2$ respectively. Let $L(\lambda)$ be an irreducible module with highest weight λ.

The main result of this paper is the following

Theorem 1.1. *Let $\mathfrak{g} = A_2, B_2, G_2$, $p > h$ and V be an irreducible $\mathfrak{g}$-module. Then $H^2(\mathfrak{g}, V)$ is trivial except in the following cases:*

(a) $\mathfrak{g} = A_2$, $H^2(\mathfrak{g}, L((p-3)\lambda_i)) \cong L(\lambda_i)^{(1)}, \;\; i = 1, 2;$

(b) $\mathfrak{g} = B_2$, $H^2(\mathfrak{g}, L((p-3)\lambda_1 + 2\lambda_2)) \cong L(\lambda_1)^{(1)},$

$$H^2(\mathfrak{g}, L(\lambda_1 + (p-4)\lambda_2)) \cong L(\lambda_2)^{(1)},$$

$$H^2(\mathfrak{g}, L((p-2)(\lambda_1 + \lambda_2))) \cong L(\lambda_2)^{(1)};$$

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(c) $\mathfrak{g} = G_2$, $H^2(\mathfrak{g}, L((p-5)\lambda_1 + 2\lambda_2)) \cong L(\lambda_1)^{(1)}$,

$$H^2(\mathfrak{g}, L(4\lambda_1 + (p-3)\lambda_2)) \cong L(\lambda_2)^{(1)},$$

$$H^2(\mathfrak{g}, L(3\lambda_1 + (p-2)\lambda_2))) \cong L(0)^{(1)}.$$

Here we use notations from [**11**]. Let $\mathfrak{g}$ be a classical Lie algebra over the field $\mathcal{K}$, G an algebraic group of the Lie algebra $\mathfrak{g}$. Recall that the Frobenius map is defined as the morphism $G \to G$ of the $\mathcal{K}$-group functor G induced by the map $x \mapsto x^p$ on the function algebra $\mathcal{K}[G]$ [**11**]. The normal subgroup G_1 is the scheme-theoretic kernel of this map. Let T be the maximal torus on G, $X(T)$ be the character group of T, $R \subset X(T)$ be the root system and R^+ be the set of positive roots on R. The simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ corresponds to the Bourbaki table [**3**]. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the fundamental weights, $X(T)_+$ be the set of dominant weights, $X_1(T)$ be the set of restricted dominant weights, i.e., $X_1(T) = \{\lambda = \sum_{i=1}^n r_i\lambda_i \in X(T) : r_i \in \mathbb{Z}, 0 \leqslant r_i < p, \text{ for all } i\}$. Endow $X(T)$ by the usual order : $\lambda \leqslant \mu$ if and only if, there exist integers $r_i \geqslant 0$ such that $\mu - \lambda = \sum_{i=1}^n r_i\alpha_i$. For any T-module V and any $\mu \in X(T)$ denote by V_μ its weight subspace in V.

There exists an algebra $\mathfrak{g}_\mathbb{Z}$ over $\mathbb{Z}$ such that $\mathfrak{g}_\mathbb{Z} \otimes \mathcal{K} \cong \mathfrak{g}$. In $\mathfrak{g}_\mathbb{Z}$ one can choose a Chevalley basis, that coincides with a basis of the semi-simple complex Lie algebra. To any root α there corresponds a basic vector e_α of the Lie algebra $\mathfrak{g}_\mathbb{Z}$. If α, $\beta \in R$, then $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$ for some integer $N_{\alpha,\beta}$. Identify e_α with $e_\alpha \otimes 1$. Note that the p-map $e \mapsto e^{[p]}$, defined on $\mathfrak{g}$, has the property $e_\alpha^{[p]} = 0$ for any $\alpha \in R$.

Recall the definition of a Weyl module. Let $\mathfrak{g}_\mathbb{C}$ be a Lie algebra over the field of complex numbers $\mathbb{C}$. Consider an irreducible $\mathfrak{g}_\mathbb{C}$-module $V(\lambda)_\mathbb{C}$ with highest weight λ. It is known that there exists a $\mathbb{Z}$-submodule $V(\lambda)_\mathbb{Z}$ of the $\mathfrak{g}_\mathbb{C}$-module $V(\lambda)_\mathbb{C}$. Then $V(\lambda) = V(\lambda)_\mathbb{Z} \otimes \mathcal{K}$ is a $\mathfrak{g}$-module. The obtained module is called a *Weyl module.*

Let B be the Borel subgroup of G corresponding to the negative roots, U be the unipotent radical of B and $\mathfrak{u}$ be the Lie algebra of U. The Lie algebra $\mathfrak{u}$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ and it spans basic vectors $e_{-\alpha}$, $\alpha \in R^+$. The Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a Lie algebra of the maximal torus T of G. For any $\lambda \in X(T)$ one can define a one-dimensional module $\mathcal{K}_\lambda$ over B using the isomorphism $B/U \cong T$. The induced G-module $H^0(\lambda) = Ind_B^G\mathcal{K}_\lambda$ is non-zero if and only if $\lambda \in X(T)_+$. If so, the socle $L(\lambda)$ of the induced module $H^0(\lambda)$ is a simple G-module with highest weight λ. It can also be constructed as the unique irreducible factor of the Weyl module $V(\lambda)$.

If $\lambda \in X_1(T)$, then $L(\lambda)$ remains simple as a G_1-module. Any simple G_1-module is defined uniquely by the highest weight $\lambda \in X_1(T)$ and it is isomorphic to $L(\lambda)$. The theory of restricted representations of the restricted Lie algebra $\mathfrak{g}$ is equivalent to the theory of representations of the group G_1.

A composition of a representation of G in a vector space V with the Frobenius map gives us a new representation with trivial action of G_1. Denote the obtained module by $V^{(1)}$. Thus this module as a module over the Lie algebra $\mathfrak{g}$ is a module with a trivial action. To any weight $\mu \in X(T)$ of the space V there corresponds the weight $p\mu$ of the space $V^{(1)}$. On the other hand, if V_1 is a G-module with trivial action of G_1 (or $\mathfrak{g}$) then there exists a unique G-module V, such that $V_1 = V^{(1)}$.

Denote this G-module V by $V^{(-1)}$. For example, if L is a G-module, then any cohomology group $H^i(G_1, L)$ is a G-module with trivial action of G_1 (or $\mathfrak{g}$). Therefore the module $H^i(G_1, L)^{(-1)}$ is a G-module with the above mentioned property.

Second cohomology groups of the adjoint representation of the Lie algebra B_2 in characteristic 3 was studied in [**13**], [**6**]. In [**16**], [**11**] first cohomology groups of modular Lie algebras with coefficients in irreducible modules are calculated. In [**16**] the non-triviality of first cohomology groups with coefficients in irreducible restricted modules with highest weights $p\lambda_i - \alpha_i,\ i = 1, 2, \ldots, n$, are proved. Here $\alpha_i, \lambda_i,\ \ i = 1, 2, \ldots, n$ are the simple roots and fundamental weights. In [**17**] a connection between first cohomology groups of irreducible modules and second cohomology groups of restricted Weyl modules are studied.

2. Connection between ordinary and restricted second cohomology groups

Consider the algebra $\mathfrak{g}$ as a restricted Lie algebra with the p-map $e \mapsto e^{[p]},\ e \in \mathfrak{g}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $U(\mathfrak{g})^+$ be a two sided ideal in $U(\mathfrak{g})$ such that $U(\mathfrak{g})$ is a direct sum of $\mathcal{K}$ and $U(\mathfrak{g})^+$. Let $P(\mathfrak{g})$ be the ideal generated by the elements $e^p - e^{[p]},\ e \in \mathfrak{g}$. The factor-algebra $U_0(\mathfrak{g}) = U(\mathfrak{g})/P(\mathfrak{g})$ is called the restricted universal enveloping algebra of $\mathfrak{g}$.

Restricted cohomology groups of restricted Lie algebras were introduced by G.Hochschild in ([**9**]). The cohomology groups $H^i(G_1, V)$ for a G_1-module V are equivalent to the restricted cohomology of the corresponding $\mathfrak{g}$-module ([**11**], I.9, p.145). Let $H^i_*(\mathfrak{g}, V)$ denote the i-th restricted cohomology group of the restricted Lie algebra $\mathfrak{g}$ with coefficients in a restricted $\mathfrak{g}$-module V. By definition $H^i_*(\mathfrak{g}, V) = Ext^i_{U_0(\mathfrak{g})}(\mathcal{K}, V)$.

The projection $U(\mathfrak{g}) \to \mathcal{K}$ induces the projection $U_0(\mathfrak{g}) \to \mathcal{K}$. Denote its kernel by $U_0(\mathfrak{g})^+$. Then $U_0(\mathfrak{g})^+$ is the image of $U(\mathfrak{g})^+$ in $U_0(\mathfrak{g})$ of the canonical map $U(\mathfrak{g}) \to U_0(\mathfrak{g})$. A map of the corresponding cochain complexes is induced by the homomorphism $\psi \mapsto \psi^0$, where $\psi^0(s_1, s_2, \ldots, s_i) = \psi(s_1', s_2', \ldots, s_i'),\ s_j \in U(\mathfrak{g})^+$ and s_j' are the canonical images in $U_0(\mathfrak{g})^+$.

Let now $C(V)$ be the cochain complex for the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module V.

Let $C^0(V)$ stand for the subcomplex consisting of the cochains of the form ψ^0, where ψ is a cochain for $U_0(\mathfrak{g})^+$ with coefficients in V. Then we have an exact sequence of cochain complexes

$$0 \to C^0(V) \to C(V) \to C(V)/C^0(V) \to 0.$$

Since the map $\psi \mapsto \psi^0$ is an isomorphism, we may identify $H^i(C^0(V))$ with $H^i_*(\mathfrak{g}, V)$. This gives us the following exact sequence:

$$\cdots \to H^i_*(\mathfrak{g}, V) \to H^i(\mathfrak{g}, V) \to H^i(C(V)/C^0(V)) \to H^{i+1}_*(\mathfrak{g}, V) \to \cdots$$

In [**9**] Hochschild shows that for $i = 1, 2$

$$H^i(C(V)/C^0(V)) \cong S(\mathfrak{g}, H^{i-1}(\mathfrak{g}, V)),$$

where $S(\mathfrak{g}, H^{i-1}(\mathfrak{g}, V))$ is the space of p-semilinear maps $\mathfrak{g} \to H^{i-1}(\mathfrak{g}, V)$. In [**11**] it was proved that for $i = 1, 2$ there is an isomorphism of G-modules

$$S(\mathfrak{g}, H^{i-1}(\mathfrak{g}, V)) \cong H^{i-1}(\mathfrak{g}, V) \otimes \mathfrak{g}^*$$

(proposition 9.20, p. 160). It is evident that $H^0(C(V)/C^0(V)) = 0$. The identification of $H^i_*(\mathfrak{g}, V)$ with $H^i(G_1, V)$ gives us the following exact sequence of G-modules:

$$0 \to H^1(G_1, V) \to H^1(\mathfrak{g}, V) \to H^0(\mathfrak{g}, V) \otimes \mathfrak{g}^* \to H^2(G_1, V) \to$$
$$\to H^2(\mathfrak{g}, V) \to H^1(\mathfrak{g}, V) \otimes \mathfrak{g}^* \to H^3(G_1, V). \tag{1}$$

Lemma 2.1. *Let V be a nontrivial irreducible $\mathfrak{g}$-module and $H^1(\mathfrak{g}, V) = 0$. Then $H^2(\mathfrak{g}, V) \cong H^2(G_1, V)$ as G-modules.*

Proof. Since V is a nontrivial irreducible $\mathfrak{g}$-module, $H^0(\mathfrak{g}, V) = 0$. The isomorphism follows from the exact sequence (1). □

3. Peculiar irreducible modules

Call an irreducible $\mathfrak{g}$-module V *peculiar*, if $H^*(\mathfrak{g}, V) \neq 0$. Let $\mathfrak{g}$ be a simple classical Lie algebra, $p > 0$, $U_0(\mathfrak{g})$ its restricted universal enveloping algebra, $Z_0(\mathfrak{g})$ be the center of $U_0(\mathfrak{g})$. The central character $c_V : Z(\mathfrak{g}) \to \mathcal{K}$, maps each element $C \in Z_0(\mathfrak{g})$ to its unique eigenvalue $c_V(C)$ on V.

Let λ, $\mu \in X(T)$. We will say, that λ *and* μ *are connected*, if $\lambda = w(\mu + \rho) - \rho$ for some $w \in W$. If λ and μ are connected, then according to the linkage principal, $L(\mu)$ is a composition factor of Weyl module $V(\lambda)$ ([**1**], Corollary 3 of theorem 1). This means that the maximal submodule of Weyl module $V(\lambda)$ is generated by highest vectors with weights connected with λ. If $M(\lambda)$ is a maximal submodule of the Weyl module $V(\lambda)$, then the following sequence is exact

$$0 \to M(\lambda) \to V(\lambda) \to V(\lambda)/M(\lambda) \to 0.$$

The corresponding long exact cohomological sequence shows that the highest weights of peculiar modules are connected.

Lemma 3.1. *Let $L(\lambda)$ be a peculiar module. Then $\lambda \in X_1(T)$ and $\lambda = w(\rho) - \rho + p\nu$ where $\nu \in X(T)$, $w \in W$.*

Proof. According to ([**10**], theorem 2.1) two modules with connected highest weights have just the same central characters. The trivial module is peculiar. It is evident that the central character of the trivial module is equal to zero. According to the linkage principle highest weights of peculiar modules are connected with the highest weight 0. It is known that cohomologies of non-restricted modules are trivial ([**7**]). Thus, the highest weight of a peculiar module has the form $\lambda = w(\rho) - \rho + p\nu \in X_1(T)$, where $\nu \in X(T)$ and w runs through elements of Weyl group W. □

Corollary 3.2. *The lists of possible highest weights of any peculiar module of a simple classical Lie algebra* $\mathfrak{g}$ *of rank two are given below*

$$\mathfrak{g} = A_2$$

$$0, (p-2)\lambda_1 + \lambda_2, \lambda_1 + (p-2)\lambda_2, (p-3)\lambda_1, (p-3)\lambda_2, (p-2)(\lambda_1 + \lambda_2);$$

$$\mathfrak{g} = B_2$$

$$0, (p-2)\lambda_1 + 2\lambda_2, \lambda_1 + (p-2)\lambda_2, (p-3)\lambda_1 + 2\lambda_2,$$

$$\lambda_1 + (p-4)\lambda_2, (p-3)\lambda_1, (p-4)\lambda_2, (p-2)(\lambda_1 + \lambda_2);$$

$$\mathfrak{g} = G_2$$

$$0, (p-2)\lambda_1 + \lambda_2, 3\lambda_1 + (p-2)\lambda_2, (p-5)\lambda_1 + 2\lambda_2, 4\lambda_1 + (p-3)\lambda_2,$$

$$(p-6)\lambda_1 + 2\lambda_2, 4\lambda_1 + (p-4)\lambda_2, (p-6)\lambda_1 + \lambda_2,$$

$$3\lambda_1 + (p-4)\lambda_2, (p-5)\lambda_1, (p-3)\lambda_2, (p-2)(\lambda_1 + \lambda_2).$$

Proof. We show detailed calculations only in the case of A_2. For other algebras the calculations are similar. So, let $\mathfrak{g} = A_2$. The Weyl group has 6 elements $1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1$. Here s_i corresponds to $s_i(\mu) = \mu - \frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. The half-sum of positive roots is equal to $\rho = \alpha_1 + \alpha_2$. It is evident that, to the neutral element 1 corresponds a peculiar highest weight $\lambda = 0$. For s_1 we have

$$\lambda = s_1(\rho) - \rho + p\nu =$$

$$s_1(\alpha_1 + \alpha_2) - \alpha_1 - \alpha_2 + p\nu = -\alpha_1 + p\nu = -2\lambda_1 + \lambda_2 + p\nu.$$

Since λ is restricted and dominant, $\nu = \lambda_1$. So $\lambda = s_1(\rho) - \rho + p\nu = (p-2)\lambda_1 + \lambda_2$ may be a peculiar weight corresponding to s_1. Similarly,

$$\lambda = s_2(\rho) - \rho + p\nu = -\alpha_2 + p\nu = \lambda_1 - 2\lambda_2 + p\nu = \lambda_1 + (p-2)\lambda_2,$$

$$\lambda = s_1s_2(\rho) - \rho + p\nu = -2\alpha_1 - \alpha_2 + p\nu = -3\lambda_1 + p\nu = (p-3)\lambda_1,$$

$$\lambda = s_2s_1(\rho) - \rho + p\nu = -\alpha_1 - 2\alpha_2 + p\nu = -3\lambda_2 + p\nu = (p-3)\lambda_2,$$

$$\lambda = s_1s_2s_1(\rho) - \rho + p\nu = -2\alpha_1 - 2\alpha_2 + p\nu = -2\lambda_1 - 2\lambda_2 + p\nu = (p-2)(\lambda_1 + \lambda_2).$$

□

Lemma 3.3. *Let* $\mathfrak{g} = A_2$. *Then as* G*-modules,*

$$H^0(0) = L(0),\ H^0((p-3)\lambda_1) = L((p-3)\lambda_1),\ H^0((p-3)\lambda_2) = L((p-3)\lambda_2);$$

$$H^0((p-2)\lambda_1+\lambda_2)/L((p-2)\lambda_1+\lambda_2)\cong L((p-3)\lambda_1);$$

$$H^0(\lambda_1+(p-2)\lambda_2)/L(\lambda_1+(p-2)\lambda_2)\cong L((p-3)\lambda_2);$$

$$H^0((p-2)(\lambda_1+\lambda_2))/L((p-2)(\lambda_1+\lambda_2))\cong L(0).$$

Proof. See [**4**], [**14**], [**12**]. □

Lemma 3.4. *Let $\mathfrak{g}=B_2$ and $p>3$. Then*

$$H^0(0)=L(0),\ H^0((p-3)\lambda_1)=L((p-3)\lambda_1),\ H^0((p-4)\lambda_2)= L((p-4)\lambda_2);$$

$$H^0((p-2)\lambda_1+2\lambda_2)/L((p-2)\lambda_1+2\lambda_2)\cong L((p-3)\lambda_1+2\lambda_2);$$

$$H^0(\lambda_1+(p-2)\lambda_2)/L(\lambda_1+(p-2)\lambda_2)\cong L(\lambda_1+(p-4)\lambda_2);$$

$$H^0((p-3)\lambda_1+2\lambda_2)/L((p-3)\lambda_1+2\lambda_2)\cong L((p-3)\lambda_1);$$

$$H^0(\lambda_1+(p-4)\lambda_2)/L(\lambda_1+(p-4)\lambda_2)\cong L((p-4)\lambda_2);$$

$$H^0((p-2)(\lambda_1+\lambda_2))/L((p-2)(\lambda_1+\lambda_2))\cong L(\lambda_1+(p-2)\lambda_1).$$

Proof. Recall that the element of maximal length of the Weyl group is $w_0=-1$ for $\mathfrak{g}=B_2$. Since in this case, $V(\lambda)=H^0(-w_0(\lambda))^*=H^0(\lambda)^*$, the maximal submodule of the Weyl module is isomorphic to the factor-module $H^0(\lambda)/L(\lambda)$. Therefore it is enough to prove that for any of the considered modules $H^0(\lambda)$, the maximal submodule of the corresponding Weyl module $V(\lambda)$ coincides with an irreducible module mentioned in the lemma.

Let $\{e_1,e_2,e_3,e_4,h_1,h_2,f_1,f_2,f_3,f_4\}$ be the Chevalley basis of the Lie algebra $\mathfrak{g}$. Vectors in the module $V(\lambda)$ can be presented as linear combinations of monomials like

$$v_{i,j,k,s}:=\frac{f_4^sf_1^kf_3^jf_2^i}{s!k!j!i!}\otimes v_\lambda,$$

where v_λ is the highest vector and $\{f_1,f_2,f_3,f_4\}$ is the basis of $\mathfrak{u}$. The actions of the elements e_1,e_2 on the monomials $v_{i,j,k,s}$ are defined by

$$e_1v_{i,j,k,s}=(s+1)v_{i,j-2,k,s+1}-(i+1)v_{i+1,j-1,k,s}+(m_1+1+i-j-k)v_{i,j,k-1,s},$$

$$e_2v_{i,j,k,s}=2(k+1)v_{i,j-1,k+1,s}-(j+1)v_{i,j+1,k,s-1}+(m_2+1-i)v_{i-1,j,k,s},$$

where $\lambda=m_1\lambda_1+m_2\lambda_2$.

Let

$$v_{i,k}^{m_1,m_2}=\sum_{0\leqslant j+s\leqslant k,\ 0\leqslant j+2s\leqslant i}a_{j,s}v_{i-j-2s,j,k-j-s,s}$$

be the vector in the space $V(\lambda)$ with weight $\lambda-k\alpha_1-i\alpha_2$. Call it *normal*, if $a_{0,0}\neq 0$. It is known that highest vectors of proper submodules of $V(\lambda)$ are normal ([**15**]).

It is evident that normal vectors of the modules $V((p-3)\lambda_1)$, $V((p-4)\lambda_2)$ cannot serve as highest vectors. So, the modules $V((p-3)\lambda_1)$, $V((p-4)\lambda_2)$ have no proper submodules. Therefore, they are irreducible and are equal to the corresponding induced modules.

Suppose now that $\lambda = (p-3)\lambda_1 + 2\lambda_2$. Then highest vectors can be found among the normal vectors $v_{i,k}^{p-3,2}$, $i \leqslant 2$, $k \leqslant p-3$.

We now show that $v_{1,1}^{p-3,2}$, $v_{2,2}^{p-3,2}$ cannot serve as a highest vector. Suppose that

$$v_{1,1}^{p-3,2} = a_1 v_{1,0,1,0} + b_1 v_{0,1,0,0},$$

$$v_{2,2}^{p-3,2} = a_2 v_{2,0,2,0} + b_2 v_{1,1,1,0} + c_2 v_{2,0,2,0} + d_2 v_{1,1,1,0}$$

are highest vectors. Then

$$e_1 v_{1,1}^{p-3,2} = e_1(a_1 v_{1,0,1,0} + b_1 v_{0,1,0,0}) = 0 \Rightarrow -2a_1 - b_1 = 0,$$

$$e_2 v_{1,1}^{p-3,2} = e_2(a_1 v_{1,0,1,0} + b_1 v_{0,1,0,0}) = 0 \Rightarrow 2a_1 + 2b_1 = 0;$$

$$e_1 v_{2,2}^{p-3,2} = e_1(a_2 v_{2,0,2,0} + b_2 v_{1,1,1,0} + c_2 v_{2,0,2,0} + d_2 v_{1,1,1,0}) = 0 \Rightarrow \\ c_2 = b_2 = a_2 = 0;$$

$$e_2 v_{2,2}^{p-3,2} = e_2(a_2 v_{2,0,2,0} + b_2 v_{1,1,1,0} + c_2 v_{2,0,2,0} + d_2 v_{1,1,1,0}) = 0 \Rightarrow$$

$$a_2 + 4b_2 = 0,\ 2b_2 + 2c_2 - d_2 = 0.$$

Therefore, $a_1 = b_1 = 0$, $a_2 = b_2 = c_2 = d_2 = 0$.

From the condition $e_1 v_{2,k}^{p-3,2} = 0$ it follows that $k \leqslant 2$. Since the normal vector $v_{2,2}^{p-3,2}$ cannot be a highest vector, we have that $k = 1$. Since

$$e_1 v_{2,1}^{p-3,2} = 2(p-1)v_{2,0,0,0} + 2v_{0,0,0,0} = 0,$$

$$e_2 v_{2,1}^{p-3,2} = 2v_{1,0,1,0} - 2v_{0,1,0,0} - 2v_{0,1,0,0} + 2v_{0,1,0,0} = 0,$$

we obtain the unique (up to scalar) highest vector $v_{2,1}^{p-3,2} = 2v_{2,0,1,2} - v_{1,1,0,0} - 2v_{0,0,0,1}$.

Since the module $V((p-3)\lambda_1 + 2\lambda_2)$ has no other highest vectors except $v_{2,1}^{p-3,2}$, the submodule generated by this vector is irreducible. The weight of the highest vector $v_{2,1}^{p-3,2}$ is $(p-3)\lambda_1 + 2\lambda_2 - \alpha_1 - 2\alpha_2 = (p-3)\lambda_1$. Therefore, the maximal submodule of $V((p-3)\lambda_1 + 2\lambda_2)$ is a module isomorphic to $L((p-3)\lambda_1)$. Analogous calculations show that the vectors

$$v_{1,1}^{p-2,2} = v_{1,0,1,0} - v_{0,1,0,0},$$

$$v_{2,1}^{1,p-2} = 2v_{2,0,1,0} + 3v_{1,1,0,0} - 6v_{0,0,0,1},$$

$$v_{1,1}^{1,p-4} = v_{1,0,1,0} + 2v_{0,1,0,0},$$

$$v_{p-3,p-3}^{p-2,p-2} = v_{p-3,0,p-3,0} +$$

$$\sum_{1 \leqslant j+2s \leqslant p-2} (-1)^s (p-3-j-s)!(p-1)\cdots(p-j-s) v_{p-3-j-2s,j,p-3-j-s,s}$$

are unique (up to scalar) highest vectors of the modules $V((p-2)\lambda_1+2\lambda_2)$, $V(\lambda_1+(p-2)\lambda_2)$, $V(\lambda_1+(p-4)\lambda_2)$, $V((p-2)(\lambda_1+\lambda_2))$ correspondingly.

Therefore, the submodules generated by one of these vectors are irreducible. Their highest weights are respectively $(p-3)\lambda_1+2\lambda_2$, $\lambda_1+(p-4)\lambda_2$, $(p-4)\lambda_2$, $\lambda_1+(p-2)\lambda_2$. Thus maximal submodules of the following modules $V((p-2)\lambda_1+2\lambda_2)$, $V(\lambda_1+(p-2)\lambda_2)$, $V(\lambda_1+(p-4)\lambda_2)$, $V((p-2)(\lambda_1+\lambda_2))$ are the irreducible modules $L((p-3)\lambda_1+2\lambda_2)$, $L(\lambda_1+(p-4)\lambda_2)$, $L((p-4)\lambda_2)$, $L(\lambda_1+(p-2)\lambda_2)$. The lemma is proved completely. □

By analogous methods the following lemma can be proved.

Lemma 3.5. *Let $\mathfrak{g}=G_2$ and $p>5$. Then*

$$H^0(0)=L(0),\ H^0((p-5)\lambda_1)=L((p-5)\lambda_1),\ H^0((p-3)\lambda_2)= L((p-3)\lambda_2);$$

$$H^0((p-2)\lambda_1+\lambda_2)/L((p-2)\lambda_1+\lambda_2)\cong L((p-5)\lambda_1+2\lambda_2);$$

$$H^0(3\lambda_1+(p-2)\lambda_2)/L(3\lambda_1+(p-2)\lambda_2)\cong L(4\lambda_1+(p-3)\lambda_2);$$

$$H^0((p-5)\lambda_1+2\lambda_2)/L((p-5)\lambda_1+2\lambda_2)\cong L((p-6)\lambda_1+2\lambda_2);$$

$$H^0(4\lambda_1+(p-3)\lambda_2)/L(4\lambda_1+(p-3)\lambda_2)\cong L(4\lambda_1+(p-4)\lambda_2);$$

$$H^0((p-6)\lambda_1+\lambda_2)/L((p-6)\lambda_1+\lambda_2)\cong L((p-5)\lambda_1);$$

$$H^0(4\lambda_1+(p-4)\lambda_2)/L(4\lambda_1+(p-4)\lambda_2)\cong L((p-3)\lambda_2);$$

$$H^0((p-2)(\lambda_1+\lambda_2))/L((p-2)(\lambda_1+\lambda_2))\cong L((2p-6)\lambda_1+2\lambda_1).$$

4. G_1-cohomology

Let $S(\mathfrak{u}^*)$ be the symmetric algebra of the Lie algebra $\mathfrak{u}^*$, w an element of Weyl group W, $l(w)$ length of the element w, ρ the half sum of positive roots and $w(\rho)-\rho+p\nu\in X_1(T)$. Below we use the following known facts about first cohomology groups of G_1 ([**12**], proposition 4.9(b) and 4.3) and Andersen-Jantzen general formula ([**2**], corollary 3.7(a),(b)):

$$H^1(G_1,L(p\lambda_i-\alpha_i))^{(-1)}\cong H^0(\lambda_i),\ i=1,2,\ldots,n, \tag{2}$$

$$H^1(G_1,L(\lambda))^{(-1)}\cong (H^0(\lambda)/L(\lambda))^{G_1},$$

$$\text{where } \lambda\neq p\lambda_i-\alpha_i\,, i=1,2,\ldots,n, \tag{3}$$

$$H^i(G_1,\mathcal{K})^{(-1)}\cong H^0(S^{i/2}(\mathfrak{u}^*)). \tag{4}$$

$$H^i(G_1,H^0(\mathcal{K}_{w(\rho)-\rho+p\nu}))^{(-1)}\cong H^0(S^{(i-l(w))/2}(\mathfrak{u}^*)\otimes\mathcal{K}_\nu). \tag{5}$$

Proposition 4.1. *Let $\mathfrak{g} = A_2, B_2, G_2$ and $p > h$. Then $H^2(G_1, H^0(\lambda)) = 0$, except in the following cases*

(a) $\mathfrak{g} = A_2$

$$H^2(G_1, H^0(0))^{(-1)} \cong \mathfrak{g}^* \cong H^0(\lambda_1 + \lambda_2) = L(\lambda_1 + \lambda_2),$$

$$H^2(G_1, H^0((p-3)\lambda_1))^{(-1)} \cong H^0(\lambda_1),$$

$$H^2(G_1, H^0((p-3)\lambda_2))^{(-1)} \cong H^0(\lambda_2);$$

(b) $\mathfrak{g} = B_2$

$$H^2(G_1, H^0(0))^{(-1)} \cong H^0(\lambda_1) \oplus H^0(2\lambda_2),$$

$$H^2(G_1, H^0((p-3)\lambda_1 + 2\lambda_2))^{(-1)} \cong H^0(\lambda_1),$$

$$H^2(G_1, H^0(\lambda_1 + (p-4)\lambda_2))^{(-1)} \cong H^0(\lambda_2);$$

(c) $\mathfrak{g} = G_2$

$$H^2(G_1, H^0(0))^{(-1)} \cong H^0(\lambda_1) \oplus H^0(\lambda_2),$$

$$H^2(G_1, H^0((p-5)\lambda_1 + 2\lambda_2))^{(-1)} \cong H^0(\lambda_1),$$

$$H^2(G_1, H^0(4\lambda_1 + (p-3)\lambda_2))^{(-1)} \cong H^0(\lambda_2).$$

Proof. follows from (4) and (5) . □

For any $\lambda \in X_1(T) \setminus \{0\}$ the following exact sequence holds

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0. \tag{6}$$

Consider the corresponding long exact sequence of G_1-cohomology groups

$$\cdots \to H^i(G_1, L(\lambda)) \to H^i(G_1, H^0(\lambda)) \to H^i(G_1, H^0(\lambda)/L(\lambda)) \to$$

$$H^{i+1}(G_1, L(\lambda)) \to H^{i+1}(G_1, H^0(\lambda)) \to H^{i+1}(G_1, H^0(\lambda)/L(\lambda)) \to \cdots$$

The triviality of $H^0(G_1, L(\lambda))$ is evident. The module $H^0(G_1, H^0(\lambda))$ is an invariant space for G_1 and a submodule of the G-module $H^0(\lambda)$. If it is non-zero, it contains the simple socle $L(\lambda)$ of the G-module $H^0(\lambda)$. Furthermore, G_1 acts on $L(\lambda)$ in a trivial way if and only if, $\lambda \in pX(T)$. For the restricted weight $\lambda \in X_1(T)$ this is possible only in the case $\lambda = 0$. Therefore,

$$H^0(\lambda)^{G_1} = 0 \tag{7}$$

for any $\lambda \in X_1(T) \setminus \{0\}$. Then the exact sequence of G_1-cohomology groups looks like

$$\begin{aligned} 0 \to H^0(G_1, H^0(\lambda)/L(\lambda)) \to H^1(G_1, L(\lambda)) \to H^1(G_1, H^0(\lambda)) \to \\ H^1(G_1, H^0(\lambda)/L(\lambda)) \to H^2(G_1, L(\lambda)) \to H^2(G_1, H^0(\lambda)) \to \\ H^2(G_1, H^0(\lambda)/L(\lambda)) \to H^3(G_1, L(\lambda)) \to H^3(G_1, H^0(\lambda)) \to \cdots \end{aligned} \tag{8}$$

Proposition 4.2. *Let* $\mathfrak{g} = A_2, B_2, G_2$ *and* $p > h$. *Then* $H^1(G_1, L(\lambda)) = 0$, *except in the following cases*

(a) $\mathfrak{g} = A_2$

$$H^1(G_1, L((p-2)(\lambda_1 + \lambda_2)))^{(-1)} \cong L(0),$$

$$H^1(G_1, L((p-2)\lambda_1 + \lambda_2))^{(-1)} \cong L(\lambda_1),$$

$$H^1(G_1, L(\lambda_1 + (p-2)\lambda_2))^{(-1)} \cong L(\lambda_2);$$

(b) $\mathfrak{g} = B_2$

$$H^1(G_1, L((p-2)\lambda_1 + 2\lambda_2))^{(-1)} \cong L(\lambda_1),$$

$$H^1(G_1, L(\lambda_1 + (p-2)\lambda_2))^{(-1)} \cong L(\lambda_2);$$

(c) $\mathfrak{g} = G_2$

$$H^1(G_1, L((p-2)\lambda_1 + \lambda_2))^{(-1)} \cong L(\lambda_1),$$

$$H^1(G_1, L(3\lambda_1 + (p-2)\lambda_2))^{(-1)} \cong L(\lambda_2).$$

Proof. As we mentioned above, the first ordinary cohomology groups and the corresponding cohomology groups for G_1 coincide. Statement (a) was proved in ([**5**],(3.6), p.112) and ([**12**], 6.10, p.314).

We now prove (b) and (c). We will use (2) and (3). Let us consider the induced modules $H^0(\lambda)$ corresponding to the weights from the list of corollary 3.2.

By lemmas 3.4 and 3.5 the factor-modules $H^0(\lambda)/L(\lambda)$ for the Lie algebras $\mathfrak{g} = B_2, G_2$ are simple and the highest weight of $H^0(\lambda)/L(\lambda)$ is not an element of $pX(T)$, therefore $(H^0(\lambda)/L(\lambda))^{G_1} = 0$ for peculiar modules. So, nontrivial first cohomology groups can appear only for modules of the form $L(p\lambda_i - \alpha_i)$ and they are given by (2).

Let $\mathfrak{g} = B_2$. We have $p\lambda_1 - \alpha_1 = p\lambda_1 - 2\lambda_1 + \lambda_2 = (p-2)\lambda_1 + \lambda_2$, $p\lambda_2 - \alpha_2 = p\lambda_2 + \lambda_1 - 2\lambda_2 = \lambda_1 + (p-2)\lambda_2$. So, according to (2) we obtain (b).

If $\mathfrak{g} = G_2$, then $p\lambda_1 - \alpha_1 = p\lambda_1 - 2\lambda_1 + \lambda_2 = (p-2)\lambda_1 + \lambda_2$, $p\lambda_2 - \alpha_2 = p\lambda_2 + 3\lambda_1 - 2\lambda_2 = 3\lambda_1 + (p-2)\lambda_2$. So, by (2) we obtain (c). □

Proposition 4.3. *Let* $\mathfrak{g} = A_2, B_2, G_2$ *and* $p > h$. *Then* $H^2(G_1, L(\lambda)) = 0$, *except in the following cases*

(a) $\mathfrak{g} = A_2$

$$H^2(G_1, L(0))^{(-1)} \cong \mathfrak{g}^* \cong H^0(\lambda_1 + \lambda_2) = L(\lambda_1 + \lambda_2),$$

$$H^2(G_1, L((p-3)\lambda_1))^{(-1)} \cong L(\lambda_1),$$

$$H^2(G_1, L((p-3)\lambda_2))^{(-1)} \cong L(\lambda_2);$$

(b) $\mathfrak{g} = B_2$

$$H^2(G_1, L(0))^{(-1)} \cong L(\lambda_1) \oplus L(2\lambda_2),$$

$$H^2(G_1, L((p-3)\lambda_1 + 2\lambda_2))^{(-1)} \cong L(\lambda_1),$$

$$H^2(G_1, L(\lambda_1 + (p-4)\lambda_2))^{(-1)} \cong L(\lambda_2);$$

$$H^2(G_1, L((p-2)(\lambda_1+\lambda_2))^{(-1)} \cong L(\lambda_2);$$

(c) $\mathfrak{g} = G_2$

$$H^2(G_1, L(0))^{(-1)} \cong L(\lambda_1) \oplus L(\lambda_2),$$

$$H^2(G_1, L((p-5)\lambda_1 + 2\lambda_2))^{(-1)} \cong L(\lambda_1),$$

$$H^2(G_1, L(4\lambda_1 + (p-3)\lambda_2))^{(-1)} \cong L(\lambda_2).$$

Proof. (a) follows from the exact sequence (8), lemma 3.3 and propositions 4.1, 4.2, part (a).

(b) follows from the exact sequence (8), lemma 3.4 and propositions 4.1 and 4.2, part (b).

(c) follows from the exact sequence (8), lemma 3.5 and propositions 4.1 and 4.2, part (c). □

5. $\mathfrak{g}$-cohomology

To prove theorem 1.1 we need some lemmas.

Lemma 5.1. *Let $\mathfrak{g}$ be a Lie algebra, V be a restricted $\mathfrak{g}$-module. For an associative 2-cocycle ψ, let ψ' be the function defined by $\psi'_x(y) = \psi(x^p - x^{[p]}, y) - \psi(y, x^p - x^{[p]})$. Then the map $\psi \to \psi'$ induces a $\mathcal{K}$-linear map of $H^2(\mathfrak{g}, V)$ into $S(\mathfrak{g}, H^1(\mathfrak{g}, V))$.*

Proof. [**9**], Theorem 3.1. □

Lemma 5.2. *Let $\mathfrak{g} = A_2,\ B_2, G_2$. Suppose that V is a restricted irreducible $\mathfrak{g}-$module and $H^1(\mathfrak{g}, V) \neq 0$. Then the lists of possible weights of the G-module $H^2(\mathfrak{g}, V)$ and the lists of possible dominant weights are the following*

$\mathfrak{g}$	V	*weights of* $H^2(\mathfrak{g}, V)$	*dominants*
A_2	$L((p-2)\lambda_1 + \lambda_2)$	$p\lambda_1$, $p(-\lambda_1+\lambda_2)$, $p(-\lambda_2)$	$p\lambda_1$
A_2	$L(\lambda_1 + (p-2)\lambda_2)$	$p\lambda_2$, $p(\lambda_1-\lambda_2)$, $-p\lambda_1$	$p\lambda_2$
A_2	$L((p-2)(\lambda_1+\lambda_2))$	0, $p(2\lambda_1-\lambda_2)$, $p(-\lambda_1+2\lambda_2)$, $p(\lambda_1+\lambda_2)$, $p(-2\lambda_1+\lambda_2)$, $p(\lambda_1-2\lambda_2)$, $p(-\lambda_1-\lambda_2)$	$p(\lambda_1+\lambda_2)$
B_2	$L((p-2)\lambda_1 + 2\lambda_2)$	0, $p\lambda_1$, $-p\lambda_1$, $p(-\lambda_1+2\lambda_2)$, $p(\lambda_1-2\lambda_2)$	0, $p\lambda_1$
B_2	$L(\lambda_1 + (p-2)\lambda_2)$	$p\lambda_2$, $-p\lambda_2$, $p(\lambda_1-\lambda_2)$ $p(-\lambda_1+\lambda_2)$	0, $p\lambda_2$
G_2	$L((p-2)\lambda_1 + \lambda_2)$	0, $p\lambda_1$, $-p\lambda_1$, $p(-\lambda_1+\lambda_2)$, $p(2\lambda_1-\lambda_2)$, $p(\lambda_1-\lambda_2)$, $p(-2\lambda_1+\lambda_2)$	0, $p\lambda_1$
G_2	$L(3\lambda_1 + (p-2)\lambda_2)$	0, $p\lambda_2$, $-p\lambda_2, p(2\lambda_1-\lambda_2)$, $p(-3\lambda_1+2\lambda_2)$, $p(-\lambda_1+\lambda_2)$, $p(3\lambda_1-\lambda_2)$, $p\lambda_1$, $p(-2\lambda_1+\lambda_2)$, $p(3\lambda_1-2\lambda_2)$, $p(\lambda_1-\lambda_2)$, $p(-3\lambda_1+\lambda_2)$, $-p\lambda_1$	0, $p\lambda_1$, $p\lambda_2$

Proof. It follows from lemma 5.1 and proposition 4.2. □

Lemma 5.3. *Let $\mathfrak{g} = B_2$ and $p \geqslant 5$. Then*

$$H^3(G_1, L((p-3)\lambda_1 + 2\lambda_2)) = H^3(G_1, L(\lambda_1 + (p-4)\lambda_2)) = 0.$$

Proof. By lemma 3.4 the following sequence is exact

$$0 \to L((p-3)\lambda_1 + 2\lambda_2) \to H^0((p-3)\lambda_1 + 2\lambda_2) \to L((p-3)\lambda_1) \to 0.$$

The corresponding exact sequence of G_1-cohomology groups gives us that the following sequence is exact

$$\begin{aligned} H^2(G_1, L((p-3)\lambda_1)) \to H^3(G_1, L((p-3)\lambda_1 + 2\lambda_2)) \to \\ H^3(G_1, H^0((p-3)\lambda_1 + 2\lambda_2)) \end{aligned} \tag{9}$$

By (5) $H^3(G_1, H^0((p-3)\lambda_1 + 2\lambda_2)) = 0$, since $i = 3$, $l(w) = 2$. By proposition 4.3 we have $H^2(G_1, L((p-3)\lambda_1)) = 0$. Therefore, from the exact sequence (9) we obtain $H^3(G_1, L((p-3)\lambda_1 + 2\lambda_2)) = 0$.

The second statement $H^3(G_1, L(\lambda_1 + (p-4)\lambda_2)) = 0$ can be proved in an analogous way.

Lemma 5.4. *Let $\mathfrak{g} = G_2$ and $p > 5$. Then*

$$H^3(G_1, L((p-5)\lambda_1 + 2\lambda_2)) = 0.$$

Proof. By lemma 3.5 the following sequence is exact

$$0 \to L((p-5)\lambda_1 + 2\lambda_2) \to H^0((p-5)\lambda_1 + 2\lambda_2) \to L((2p-6)\lambda_1 + 2\lambda_2) \to 0.$$

Therefore, the following sequence of G_1-cohomology groups is exact

$$\begin{aligned} H^2(G_1, L((2p-6)\lambda_1 + 2\lambda_2)) \to H^3(G_1, L((p-5)\lambda_1 + 2\lambda_2)) \to \\ H^3(G_1, H^0((p-5)\lambda_1 + 2\lambda_2)) \end{aligned} \tag{10}$$

By (5) $H^3(G_1, H^0((p-5)\lambda_1 + 2\lambda_2)) = 0$, since $i = 3$, $l(w) = 2$. By proposition 4.3 $H^2(G_1, L((2p-6)\lambda_1 + 2\lambda_2)) = 0$. Then from the exact sequence (10) it follows that $H^3(G_1, L((p-5)\lambda_1 + 2\lambda_2)) = 0$. □

Proof of theorem 1.1. The proof is divided into two parts. In the first part we prove all isomorphisms mentioned in theorem 1.1. In the second part we establish that for all other weights given in corollary 3.2 the second cohomology groups are trivial.

Part 1. By lemma 2.1 all isomorphisms, except the case $\mathfrak{g} = G_2$ and $V = L(3\lambda_1 + (p-2)\lambda_2)$, follow from proposition 4.3.

Let us prove the last isomorphism of (c). If $H^2(G_1, L(\lambda)) = 0$, then from the exact sequence (1) it follows that $H^2(\mathfrak{g}, L(\lambda))$ is isomorphic to the kernel of the map

$$f : H^1(\mathfrak{g}, L(\lambda)) \otimes \mathfrak{g}^* \to H^3(G_1, L(\lambda)). \tag{11}$$

By proposition 4.3 $H^2(G_1, L(3\lambda_1 + (p-2)\lambda_2)) = 0$. Hence $H^2(\mathfrak{g}, L(3\lambda_1 + (p-2)\lambda_2))$ is isomorphic to $\ker f$. By (5)

$$H^3(G_1, H^0(3\lambda_1 + (p-2)\lambda_2))^{(-1)} \cong$$

$$H^0(2\lambda_1) \oplus H^0(2\lambda_2) \oplus H^0(3\lambda_1) \oplus H^0(\lambda_1+\lambda_2). \tag{12}$$

By lemma 3.5 $H^0(3\lambda_1+(p-2)\lambda_2)/L(3\lambda_1+(p-2)\lambda_2) \cong L(4\lambda_1+(p-3)\lambda_2)$ and by proposition 4.3 $H^2(G_1, L(4\lambda_1+(p-3)\lambda_2))^{(-1)} \cong L(\lambda_2)$. Therefore, by the exact sequence (8), $H^3(G_1, L(3\lambda_1+(p-2)\lambda_2))$ as a G-module has (possible) composition factors $H^0(2\lambda_1)$, $H^0(2\lambda_2)$, $H^0(3\lambda_2)$, $H^0(\lambda_1+\lambda_2)$ and $H^0(\lambda_2)$.

By proposition 4.2 $H^1(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2)) \cong L(\lambda_2)$. Therefore,

$$H^1(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2)) \otimes \mathfrak{g}^* \cong L(\lambda_2) \otimes \mathfrak{g} \cong$$

$$H^0(2\lambda_1) \oplus H^0(2\lambda_2) \oplus H^0(3\lambda_1) \oplus H^0(\lambda_2) \oplus H^0(0). \tag{13}$$

From the decompositions of $H^1(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2)) \otimes \mathfrak{g}^*$ and $H^3(G_1, L(3\lambda_1 + (p-2)\lambda_2))$ we obtain that $H^0(0) = L(0) \subseteq \ker f$.

If $\ker f$ contains some of the G-modules $H^0(2\lambda_1)$, $H^0(2\lambda_2)$, $H^0(3\lambda_1)$ and $H^0(\lambda_2)$, then the G-module $H^2(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2))$ has nontrivial elements with weights $2p\lambda_1$, $2p\lambda_2$, $3p\lambda_1$, or $p\lambda_2$. We will prove that this is impossible.

For $\mathfrak{g} = G_2$ the list of dominant weights of adjoint G-module is $\{0, p\lambda_1, p\lambda_2\}$ (see lemma 5.2). Therefore, the only non-zero dominant weights of the G-module $H^2(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2))$ are $p\lambda_1$ or $p\lambda_2$. Thus cocycles in $Z^2(\mathfrak{g}, L(3\lambda_1+(p-2)\lambda_2))$ with weights $2p\lambda_1$, $2p\lambda_2$, $3p\lambda_1$ are coboundaries.

Now we prove that the classes of cocycles with weights $p\lambda_2$ are also trivial. To do it we use the realization of the $\mathfrak{g}$-module $L(3\lambda_1+(p-2)\lambda_2)$ as a factor-module of the Weyl module.

Let $f_i, h_j, e_i \,:\, i = 1,\ldots,6,\ j = 1,2$ be a Chevalley basis of $\mathfrak{g}$, where $f_i = e_{-\alpha_i}$, $e_i = e_{\alpha_i}$ for $i = 1,2$ i $f_3 = e_{-\alpha_1-\alpha_2}$, $f_4 = e_{-2\alpha_1-\alpha_2}$, $f_5 = e_{-3\alpha_1-\alpha_2}$, $f_6 = e_{-3\alpha_1-2\alpha_2}$, $e_3 = e_{\alpha_1+\alpha_2}$, $e_4 = e_{2\alpha_1+\alpha_2}$, $e_5 = e_{3\alpha_1+\alpha_2}$, $e_6 = e_{3\alpha_1+2\alpha_2}$. The Weyl module $V(m_1\lambda_1+m_2\lambda_2)$ can be defined on the vector space

$$v_{i,j,k,l,s,t} := \frac{f_6^t f_5^s f_4^l f_3^k f_2^j f_1^i}{t!s!l!k!j!i!} \otimes v_{m_1\lambda_1+m_2\lambda_2},$$

where $v_{m_1\lambda_1+m_2\lambda_2}$ is the highest weight, by

$$e_1 v_{i,j,k,l,s,t} = (l+1)v_{i,j,k+1,l,s-1,t} - 3(t+1)v_{i,j,k,l-2,s,t+1} -$$
$$2(k+1)v_{i,j,k+1,l-1,s,t} - 3(j+1)v_{i,j+1,k-1,l,s,t} + (m_1+1-i)v_{i-1,j,k,l,s,t},$$

$$\begin{aligned} e_2 v_{i,j,k,l,s,t} = {} & (s+1)v_{i,j,k,l,s+1,t-1} - (l+1)v_{i,j,k-2,l+1,s,t} + \\ & (i+1)v_{i+1,j,k-1,l,s,t} - 2(t+1)v_{i,j,k-3,l,s,t+1} + \\ & (m_2+1+i-j-k)v_{i,j-1,k,l,s,t}, \end{aligned}$$

$$f_1 v_{i,j,k,l,s,t} = -3(s+1)v_{i,j,k,l-1,s+1,t} - 3(t+1)v_{i,j,k-2,l,s,t+1}$$
$$-2(l+1)v_{i,j,k-1,l+1,s,t} - (k+1)v_{i,j-1,k+1,l,s,t} + (i+1)v_{i+1,j,k,l,s,t},$$

$$f_2 v_{i,j,k,l,s,t} = -(t+1)v_{i,j,k,l,s-1,t+1} + (j+1)v_{i,j+1,k,l,s,t}.$$

If some cocycle ψ of the module $L(3\lambda_1 + (p-2)\lambda_2)$ has weight $p\lambda_2$, then

$$\psi(e_1, f_3) = x_1 v_{0,0,0,0,0,0}, \;\; \psi(e_5, f_6) = x_2 v_{0,0,0,0,0,0},$$

$$\psi(f_1, f_2) = x_3 v_{1,0,0,0,0,0}, \;\; \psi(e_1, f_4) = x_4 v_{1,0,0,0,0,0},$$

$$\psi(e_4, f_6) = x_5 v_{1,0,0,0,0,0}, \;\; \psi(f_1, f_3) = x_6 v_{2,0,0,0,0,0},$$

$$\psi(e_1, f_5) = x_7 v_{2,0,0,0,0,0}, \;\; \psi(e_3, f_6) = x_8 v_{2,0,0,0,0,0},$$

$$\psi(f_1, f_4) = x_9 v_{3,0,0,0,0,0}, \;\; \psi(e_2, f_6) = x_{10} v_{3,0,0,0,0,0},$$

$$\psi(f_2, f_3) = x_{11} v_{1,1,0,0,0,0} + x_{12} v_{0,0,1,0,0,0},$$

$$\psi(f_1, f_6) = x_{13} v_{3,0,1,0,0,0} + x_{14} v_{2,0,0,1,0,0} + x_{15} v_{1,0,0,0,1,0},$$

$$\psi(e_1, f_6) = x_{16} v_{2,1,0,0,0,0} + x_{17} v_{1,0,1,0,0,0} + x_{18} v_{0,0,0,1,0,0},$$

$$\psi(f_2, f_4) = x_{19} v_{2,1,0,0,0,0} + x_{20} v_{1,0,1,0,0,0} + x_{21} v_{0,0,0,1,0,0},$$

$$\psi(f_2, f_5) = x_{22} v_{3,1,0,0,0,0} + x_{23} v_{2,0,1,0,0,0} + x_{24} v_{1,0,0,1,0,0} + x_{25} v_{0,0,0,0,1,0},$$

$$\psi(f_3, f_4) = x_{26} v_{3,1,0,0,0,0} + x_{27} v_{2,0,1,0,0,0} + x_{28} v_{1,0,0,1,0,0} + x_{29} v_{0,0,0,0,1,0},$$

$$\psi(f_2, f_6) = x_{30} v_{3,2,0,0,0,0} + x_{31} v_{2,1,1,0,0,0} + x_{32} v_{1,0,2,0,0,0} + x_{33} v_{1,1,0,1,0,0} +$$

$$x_{34} v_{0,0,1,1,0,0} + x_{35} v_{0,1,0,0,1,0} + x_{36} v_{0,0,0,0,0,1},$$

$$\psi(f_3, f_5) = x_{37} v_{3,0,1,0,0,0} + x_{38} v_{2,0,0,1,0,0} + x_{39} v_{1,0,0,0,1,0},$$

$$\psi(f_3, f_6) = x_{40} v_{3,1,1,0,0,0} + x_{41} v_{2,0,2,0,0,0} + x_{42} v_{2,1,0,1,0,0} + x_{43} v_{0,0,0,2,0,0} +$$

$$x_{44} v_{1,0,1,1,0,0} + x_{45} v_{1,1,0,0,1,0} + x_{46} v_{0,0,1,0,1,0} + x_{47} v_{1,0,0,0,0,1},$$

$$\psi(f_4, f_5) = x_{48} v_{3,0,0,1,0,0} + x_{49} v_{2,0,0,0,1,0},$$

$$\psi(f_4, f_6) = x_{50} v_{3,0,2,0,0,0} + x_{51} v_{3,1,0,1,0,0} + x_{52} v_{1,0,0,2,0,0} + x_{53} v_{2,0,1,1,0,0} +$$

$$x_{54} v_{2,1,0,0,1,0} + x_{55} v_{1,0,1,0,1,0} + x_{56} v_{0,0,0,1,1,0} + x_{57} v_{2,0,0,0,0,1},$$

$$\psi(f_5, f_6) = x_{58} v_{3,0,1,1,0,0} + x_{59} v_{2,0,0,2,0,0} + x_{60} v_{3,1,0,0,1,0} + x_{61} v_{0,0,0,0,2,0} +$$

$$x_{62} v_{2,0,1,0,1,0} + x_{63} v_{1,0,0,1,1,0} + x_{64} v_{3,0,0,0,0,1},$$

for some $x_i \in \mathcal{K}$ $(i = 1, \ldots, 64)$. Non-written components of $\psi(X, Y)$ are equal to 0 (sums of weights of X and Y can not be a weight of the module).

The conditions

$$d\psi(h_i, f_2, f_6) = 0, \; d\psi(h_i, f_2, f_5) = 0, \; d\psi(h_i, e_1, f_3) = 0, \; i = 1, 2,$$

gives us a system of linear equations. Solving this system (we omit the standard but long calculations) gives us that

$$\psi(h_i, f_2) = \psi(h_i, f_3) = \psi(h_i, f_4) = \psi(h_i, f_5) = \psi(h_i, f_6) = 0,\ i = 1, 2.$$

Furthermore, from the following system of linear equations with 16 conditions

$$\begin{gathered} d\psi(f_4, f_5, f_6) = 0,\ d\psi(f_3, f_5, f_6) = 0,\ d\psi(f_3, f_4, f_5) = 0, \\ d\psi(f_2, f_4, f_5) = 0, d\psi(f_2, f_5, f_6) = 0,\ d\psi(f_1, f_4, f_6) = 0, \\ d\psi(e_1, f_5, f_6) = 0,\ d\psi(e_2, f_5, f_6) = 0, d\psi(e_3, f_5, f_6) = 0, \\ d\psi(e_4, f_5, f_6) = 0,\ d\psi(e_5, f_4, f_5) = 0,\ d\psi(e_6, f_5, f_6) = 0, \\ d\psi(e_2, f_4, f_6) = 0,\ d\psi(e_3, f_3, f_4) = 0,\ d\psi(e_1, f_3, f_4) = 0, \\ d\psi(e_1, f_4, f_6) = 0 \end{gathered}$$

we obtain that ψ should have the following form

$$\psi(f_1, f_2) = x_3 v_{1,0,0,0,0,0},\ \ \psi(f_1, f_3) = x_6 v_{2,0,0,0,0,0},$$

$$\psi(f_2, f_3) = x_{11} v_{1,1,0,0,0,0} + x_{12} v_{0,0,1,0,0,0}.$$

Then from the equation $d\psi(e_1, f_1, f_2) = 0$ we obtain that $x_3 = 0$. Since $\psi(f_1, f_2) = 0$, then the condition $d\psi(e_1, f_1, f_3) = 0$ gives us that $x_6 = 0$. Finally, from the condition $d\psi(f_2, f_3, f_6) = 0$, we have that $\psi(f_2, f_3) = 0$. Therefore, any cocycle with weight $p\lambda_1$ is trivial.

So, in the case $L(\lambda) = L(3\lambda_1 + (p-2)\lambda_2)$, the kernel of f is isomorphic to the G-module $H^0(0) = L(0)$. Thus,

$$H^2(G_2, L(3\lambda_1 + (p-2)\lambda_2)) \cong L(0)^{(1)}.$$

Part 2. We now prove that the second cohomology groups of irreducible peculiar modules which are not mentioned in the formulation of theorem 1.1 are trivial.

The lists of highest weights of peculiar modules are given in corollary 3.2. By proposition 4.3 the second cohomology groups of G_1 for these modules are trivial. Therefore, it is enough to show that $\ker f = 0$ for any of these modules. We will do it by considering each case of algebras separately.

The case (a). By proposition 4.2 only the following 3 modules have nontrivial first cohomology groups $L((p-2)(\lambda_1 + \lambda_2))$, $L((p-2)\lambda_1 + \lambda_2)$, $L(\lambda_1 + (p-2)\lambda_2)$. By (4) and (5) we have

$$H^3(G_1, H^0((p-2)(\lambda_1 + \lambda_2)))^{(-1)} \cong H^0(\lambda_1 + \lambda_2), \tag{14}$$

$$H^3(G_1, H^0((p-2)\lambda_1 + \lambda_2))^{(-1)} \cong H^0(2\lambda_2) \oplus H^0(2\lambda_1 + \lambda_2), \tag{15}$$

$$H^3(G_1, H^0(\lambda_1 + (p-2)\lambda_2))^{(-1)} \cong H^0(2\lambda_1) \oplus H^0(\lambda_1 + 2\lambda_2). \tag{16}$$

$$H^3(G_1, H^0(\lambda))^{(-1)} = 0 \text{ if } \lambda = 0,\ (p-3)\lambda_1,\ (p-3)\lambda_2. \tag{17}$$

According to (14)–(17), lemma 3.3 and proposition 4.3 the exact sequence (8) gives us the following exact sequences

$$0 \to H^0(\lambda_1+\lambda_2) \to H^3(G_1, L((p-2)(\lambda_1+\lambda_2))) \to H^0(\lambda_1+\lambda_2) \to 0, \tag{18}$$

$$\begin{gathered} 0 \to L(\lambda_1) \to H^3(G_1, L((p-2)\lambda_1+\lambda_2)) \to \\ H^0(2\lambda_2) \oplus H^0(2\lambda_1+\lambda_2) \to 0, \end{gathered} \tag{19}$$

$$\begin{gathered} 0 \to L(\lambda_2) \to H^3(G_1, L(\lambda_1+(p-2)\lambda_2)) \to \\ H^0(2\lambda_1) \oplus H^0(\lambda_1+2\lambda_2) \to 0. \end{gathered} \tag{20}$$

On the other hand, according to proposition 4.2, we have

$$H^1(\mathfrak{g}, L((p-2)(\lambda_1+\lambda_2)) \otimes \mathfrak{g}^* \cong L(0) \otimes \mathfrak{g} \cong \mathfrak{g} \cong H^0(\lambda_1+\lambda_2), \tag{21}$$

$$\begin{gathered} H^1(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2)) \otimes \mathfrak{g}^* \cong L(\lambda_1) \otimes \mathfrak{g} \cong \\ H^0(2\lambda_2) \oplus H^0(2\lambda_1+\lambda_2) \oplus H^0(\lambda_1), \end{gathered} \tag{22}$$

$$\begin{gathered} H^1(g, L(\lambda_1+(p-2)\lambda_2)) \otimes \mathfrak{g}^* \cong \\ L(\lambda_2) \otimes \mathfrak{g} \cong H^0(2\lambda_1) \oplus H^0(\lambda_1+2\lambda_2) \oplus H^0(\lambda_2). \end{gathered} \tag{23}$$

Consider each case of these modules.

Let $V = L((p-2)(\lambda_1+\lambda_2))$. From (14) and (21) we see that $H^3(G_1, L((p-2)(\lambda_1+\lambda_2))$ contains a composition factor $H^0(\lambda_1+\lambda_2)$ isomorphic to $H^1(\mathfrak{g}, L((p-2)(\lambda_1+\lambda_2))) \otimes \mathfrak{g}^*$.

If $\ker f$ contains $H^0(\lambda_1+\lambda_2)$, then the second cohomology group $H^2(\mathfrak{g}, L((p-2)(\lambda_1+\lambda_2)))$ as a G-module contains classes of cocycles of weights $p(\lambda_1+\lambda_2)$. The highest weight $p(\lambda_1+\lambda_2) - 2\alpha_1 - 2\alpha_2$ can not be presented as a sum of $p(\lambda_1+\lambda_2)$ and two roots. Therefore, such a case is impossible. Therefore, $H^0(\lambda_1+\lambda_2)$ can not be in $\ker f$. Then the exact sequence (1) for $L((p-2)(\lambda_1+\lambda_2))$ gives us that $\ker f = 0$.

Let $f_i, h_j, e_i \,:\, i = 1,2,3,\, j = 1,2$ be the Chevalley basis of $\mathfrak{g}$, where $f_i = e_{-\alpha_i}$, $e_i = e_{\alpha_i}$ for $i = 1,2$ and $f_3 = e_{-\alpha_1-\alpha_2}$, $e_3 = e_{\alpha_1+\alpha_2}$. The Weyl module $V(m_1\lambda_1+m_2\lambda_2)$ can be defined on the vector space with basis

$$v_{i,j,k} := \frac{f_3^k f_2^j f_1^i}{k!j!i!} \otimes v_{m_1\lambda_1+m_2\lambda_2},$$

where $v_{m_1\lambda_1+m_2\lambda_2}$ is the highest weight.

The action of $\mathfrak{g}$ is given by

$$e_1 v_{i,j,k} = -(j+1)v_{i,j+1,k-1} + (m_1+1-i)v_{i-1,j,k},$$

$$e_2v_{i,j,k} = (i+1)v_{i+1,j,k-1} + (m_2+1+i-j-k)v_{i,j-1,k},$$

$$f_1v_{i,j,k} = -(k+1)v_{i,j-1,k+1},$$

$$f_2v_{i,j,k} = (j+1)v_{i,j+1,k}.$$

Let now $V = L((p-2)\lambda_1 + \lambda_2)$. From (19) and (22) we see that $H^1(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2)) \otimes \mathfrak{g}^*$ and $H^3(G_1, L((p-2)\lambda_1+\lambda_2))$ have isomorphic composition factors $H^0(2\lambda_1+\lambda_2)$, $H^0(2\lambda_2)$, $H^0(\lambda_1)$.

By lemma 5.2 any nontrivial cocycle of $Z^2(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))$ must have a dominant weight of the form $p\lambda_1$. Therefore, cocycles of $Z^2(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))$ with weights $p(2\lambda_1+\lambda_2)$, $2p\lambda_2$ are 0. Thus $H^0(2\lambda_1+\lambda_2)$, $H^0(2\lambda_2)$ can not be in $\ker f$. We now prove that $H^0(\lambda_1)$ also can not be in $\ker f$. To do this, we prove that cocycles with weight $p\lambda_1$ are coboundaries.

Let ψ be some cocycle of weight $p\lambda_1$. Then

$$\psi(h_1,f_1) = x_1v_{0,0,0},\ \psi(h_2,f_1) = x_2v_{0,0,0},\ \psi(e_2,f_3) = x_3v_{0,0,0},$$

$$\psi(h_1,f_3) = x_4v_{0,1,0},\ \psi(h_2,f_3) = x_5v_{0,1,0,},$$

$$\psi(f_1,f_2) = x_6v_{0,1,0},\ \psi(f_1,f_3) = x_7v_{0,0,1},$$

for some $x_i \in \mathcal{K}$ $(i = 1,\dots,7)$. Non-written components are zero.

From the cocyclicity conditions we obtain one cocycle of weight $p\lambda_1$ with the following non-zero components

$$\psi(e_2,f_3) = v_{0,0,0},\ \psi(f_1,f_2) = v_{0,1,0},\ \psi(f_1,f_3) = v_{0,0,1}.$$

It is easy to see that, ψ is a coboundary $d\omega$, where $\omega(f_1) = v_{0,0,0}$. So, any cocycle of the weight $p\lambda_1$ is trivial. Thus $L(\lambda_1) \not\subset \ker f$. The case of the dual module $V = L(\lambda_1 + (p-2)\lambda_2)$ may be treated in a similar way.

The case (b). According to proposition 4.2 the first cohomology groups are non-zero only for the following two modules $L((p-2)\lambda_1+2\lambda_2)$, $L(\lambda_1+(p-2)\lambda_2)$. By (5) we have

$$H^3(G_1, H^0((p-2)\lambda_1+2\lambda_2))^{(-1)} \cong H^0(2\lambda_1) \oplus H^0(2\lambda_2) \oplus H^0(\lambda_1+2\lambda_2), \tag{24}$$

$$H^3(G_1, H^0(\lambda_1+(p-2)\lambda_2))^{(-1)} \cong H^0(3\lambda_2) \oplus H^0(\lambda_1+\lambda_2). \tag{25}$$

According to (24), (25), lemma 3.4, 5.3 and proposition 4.3 the exact sequence (8) can be rewritten as follows

$$\begin{aligned} 0 \to L(\lambda_1) \to H^3(G_1, L((p-2)\lambda_1+2\lambda_2)) \to \\ H^0(2\lambda_1) \oplus H^0(2\lambda_2) \oplus H^0(\lambda_1+2\lambda_2) \to 0, \end{aligned} \tag{26}$$

$$\begin{aligned} 0 \to L(\lambda_2) \to H^3(G_1, L(\lambda_1+(p-2)\lambda_2)) \to \\ H^0(3\lambda_2) \oplus H^0(\lambda_1+\lambda_2) \to 0. \end{aligned} \tag{27}$$

By proposition 4.2,

$$H^1(\mathfrak{g}, L((p-2)\lambda_1 + 2\lambda_2)) \otimes \mathfrak{g}^* \cong L(\lambda_1) \otimes \mathfrak{g} \cong$$
$$H^0(2\lambda_2) \oplus H^0(\lambda_1 + 2\lambda_2) \oplus H^0(\lambda_1), \tag{28}$$

$$H^1(\mathfrak{g}, L(\lambda_1 + (p-2)\lambda_2)) \otimes \mathfrak{g}^* \cong L(\lambda_2) \otimes \mathfrak{g} \cong$$
$$H^0(3\lambda_2) \oplus H^0(\lambda_1 + \lambda_2) \oplus H^0(\lambda_2). \tag{29}$$

Let $V = L((p-2)\lambda_1 + 2\lambda_2)$. From (26) and (28) we see that all composition factors $H^0(\lambda_1 + 2\lambda_2)$, $H^0(2\lambda_2)$, $H^0(\lambda_1)$ of the G-module $H^1(\mathfrak{g}, L((p-2)\lambda_1 + 2\lambda_2)) \otimes \mathfrak{g}^*$ are in the module $H^3(G_1, L((p-2)\lambda_1 + 2\lambda_2))$. By lemma 5.2 cocycles have dominant weights equal to $p\lambda_1$. Therefore, $Z^2(\mathfrak{g}, L((p-2)\lambda_1 + 2\lambda_2))$ has no cocycles with weights $p(\lambda_1 + 2\lambda_2)$, $2p\lambda_2$. Therefore, $\ker f$ has no submodules isomorphic to $H^0(\lambda_1 + 2\lambda_2)$ or $H^0(2\lambda_2)$. We now prove that $H^0(\lambda_1) \not\subseteq \ker f$. We prove that any cocycle ψ with weight $p\lambda_1$ is a coboundary.

For any cocycle ψ with weight $p\lambda_1$

$$\psi(h_1, f_1) = x_1 v_{0,0,0,0},\ \psi(h_2, f_1) = x_2 v_{0,0,0,0},\ \psi(e_2, f_3) = x_3 v_{0,0,0,0},$$

$$\psi(h_1, f_3) = x_4 v_{1,0,0,0},\ \psi(h_2, f_3) = x_5 v_{1,0,0,0},\ \psi(f_1, f_2) = x_6 v_{1,0,0,0},$$

$$\psi(e_2, f_4) = x_7 v_{1,0,0,0},\ \psi(h_1, f_4) = x_8 v_{2,0,0,0},\ \psi(h_2, f_2) = x_9 v_{2,0,0,0},$$

$$\psi(f_2, f_3) = x_{10} v_{2,0,0,0},\ \psi(f_1, f_3) = x_{11} v_{0,1,0,0},$$

$$\psi(f_1, f_4) = x_{12} v_{1,1,0,0} + x_{13} v_{0,0,0,1},\ \psi(f_3, f_4) = x_{14} v_{1,0,0,1},$$

for some $x_i \in \mathcal{K}$ $(i = 1, \ldots, 14)$. Non-written components are 0.

From the cocyclicity conditions we obtain that such cocycles are linear combinations of the following two cocycles

$$\psi_1(e_2, f_3) = 2v_{0,0,0,0},\ \psi_1(f_1, f_2) = v_{1,0,0,0},$$

$$\psi_1(f_1, f_3) = v_{0,1,0,0},\ \psi_1(f_1, f_4) = v_{0,0,0,1};$$

$$\psi_2(e_2, f_4) = -v_{1,0,0,0},\ \psi_2(f_2, f_3) = -2v_{2,0,0,0},$$

$$\psi_2(f_1, f_4) = v_{0,0,0,1},\ \psi_2(f_3, f_4) = v_{1,0,0,1}.$$

Both cocycles are coboundaries: $\psi_1 = d\omega_1$ and $\psi_2 = d\omega_2$, where non-zero components of ω_1 and ω_2 are given by $\omega_1(f_1) = -v_{0,0,0,0}$ and $\omega_2(f_4) = -v_{2,0,0,0}$. So, any cocycle of weight $p\lambda_1$ is a coboundary. Therefore, $L(\lambda_1) \not\subseteq \ker f$.

Consider now the case $V = L(\lambda_1 + (p-2)\lambda_2)$.

From (27) and (29) we see that $H^1(\mathfrak{g}, L(\lambda_1 + (p-2)\lambda_2)) \otimes \mathfrak{g}^*$ and $H^3(G_1, L(\lambda_1 + (p-2)\lambda_2))$ have equal composition factors $H^0(3\lambda_2)$, $H^0(\lambda_2 + \lambda_2)$, $H^0(\lambda_2) = L(\lambda_2)$. By lemma 5.2 a non-trivial 2-cocycle has dominant weight equal to $p\lambda_2$. Therefore, all cocycles of $Z^2(\mathfrak{g}, L(\lambda_1 + (p-2)\lambda_2))$, with G-weights $3p\lambda_2$, $p(\lambda_1 + \lambda_2)$ are trivial. Thus, the modules $H^0(3\lambda_2)$, $H^0(\lambda_1 + \lambda_2)$ are not in $\ker f$.

We now prove that $H^0(\lambda_2)$ is also not in $\ker f$. So, we need to prove that any 2-cocycle with weight $p\lambda_2$ is trivial.

Let ψ be some cocycle of weight $p\lambda_2$. Then all components $\psi(X,Y)$ are zero except the following components

$$\psi(h_1,f_2)=x_1v_{0,0,0,0},\ \psi(h_2,f_2)=x_2v_{0,0,0,0},\ \psi(e_1,f_3)=x_3v_{0,0,0,0},$$

$$\psi(e_3,f_4)=x_4v_{0,0,0,0},\ \psi(h_1,f_3)=x_5v_{0,0,1,0},\ \psi(h_2,f_3)=x_6v_{0,0,1,0},$$

$$\psi(f_1,f_2)=x_7v_{0,0,,0},\ \psi(e_2,f_4)=x_8v_{0,0,1,0},$$

$$\psi(h_1,f_4)=x_9v_{1,0,1,0}+x_{10}v_{0,1,0,0},$$

$$\psi(h_2,f_4)=x_{12}v_{1,0,1,0}+x_{12}v_{0,1,0,0},\ \psi(f_2,f_3)=x_{13}v_{1,0,1,0}+x_{14}v_{0,1,0,0},$$

$$\psi(f_2,f_4)=x_{15}v_{1,1,0,0}+x_{16}v_{0,0,0,1},\ \psi(f_1,f_4)=x_{17}v_{0,1,1,0},$$

$$\psi(f_3,f_4)=x_{18}v_{1,1,1,0}+x_{19}v_{0,0,1,1},\ \psi(e_1,f_4)=x_{20}v_{1,0,0,0},$$

for some $x_i\in\mathcal{K}\ \ (i=1,\dots,20)$.

We have

$$d\psi(e_1,f_1,f_4)=0\Rightarrow x_{10}=0, x_9-x_{17}-x_{20}=0.$$

Then the equations $d\psi(h_1,e_1,f_3)=0$, $d\psi(h_1,e_3,f_4)=0$, $d\psi(h_1,f_2,f_3)=0$ gives us $x_1=x_5=x_9=0$.

Furthermore, from

$$d\psi(f_2,f_3,f_4)=0,\ \ d\psi(e_1,f_3,f_4)=0,\ \ d\psi(f_1,f_3,f_4)=0,$$

$$d\psi(e_1,f_1,f_4)=0,\ \ d\psi(e_1,f_2,f_4)=0,\ \ d\psi(e_3,f_2,f_4)=0$$

we have $x_4=x_{16}=x_{14}=-x_3$ and $x_{18}=x_{13}=x_{15}=x_{20}=x_{17}=0$.

From $d\psi(e_2,f_2,f_4)=0$, $d\psi(e_2,f_3,f_4)=0$, $d\psi(e_3,f_3,f_4)=0$ we obtain $x_8=x_{11}=x_{12}=x_{16}=x_{14}=x_{13}=0$. Finally, from $d\psi(e_1,f_1,f_2)=0$, $d\psi(e_3,f_2,f_3)=0$ we obtain $x_7=x_2=0$. So, $\psi=0$, if it has weight $p\lambda_2$. Thus $H^0(\lambda_2)\not\subseteq\ker f$.

The case (c). By proposition 4.2 only the following two modules have non-zero first cohomology groups, $L((p-2)\lambda_1+\lambda_2)$ and $L(3\lambda_1+(p-2)\lambda_2)$. We have proved above that $H^2(\mathfrak{g},L(3\lambda_1+(p-2)\lambda_2))\cong L(0)^{(1)}$. Therefore, it is enough to check that $\ker f=0$ for the module $L((p-2)\lambda_1+\lambda_2)$. According to (5),

$$H^3(G_1,H^0((p-2)\lambda_1+\lambda_2))^{(-1)}\cong H^0(\lambda_2)\oplus H^0(2\lambda_1)\oplus H^0(\lambda_1+\lambda_2). \tag{30}$$

So, (30), lemma 3.5, 5.4 and proposition 4.3 and the exact sequence (8), gives us the following exact sequence

$$\begin{aligned}0\to L(\lambda_1)\to H^3(G_1,L((p-2)\lambda_1+\lambda_2))\to\\ H^0(\lambda_2)\oplus H^0(2\lambda_1)\oplus H^0(\lambda_1+\lambda_2)\to 0.\end{aligned}\tag{31}$$

By proposition 4.2, we have

$$H^1(\mathfrak{g},L((p-2)\lambda_1+\lambda_2))\otimes\mathfrak{g}^*\cong L(\lambda_1)\otimes\mathfrak{g}\cong$$

$$H^0(2\lambda_1) \oplus H^0(\lambda_1+\lambda_2) \oplus H^0(\lambda_1). \tag{32}$$

By (31), (32) all composition factors of the $G-$module $H^1(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))\otimes \mathfrak{g}^*$ appear as a composition factors of $H^3(G_1, L((p-2)\lambda_1+\lambda_2))$. If $\ker f$ has such a composition factor, then $H^2(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))$ has non-trivial cocycles with weights $2p\lambda_1$, $p(\lambda_1+\lambda_2)$, $p\lambda_1$. We will prove that this is impossible.

By lemma 5.2 the dominant weight of a non-trivial cocycle in $Z^2(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))$ is $p\lambda_1$. Therefore, cocycles in $Z^2(\mathfrak{g}, L((p-2)\lambda_1+\lambda_2))$ with weights $2p\lambda_1$, $p(\lambda_1+\lambda_2)$ are 0. we now prove that any cocycle ψ with weight $p\lambda_1$ is 0. All components $\psi(X,Y)$ of such a cocycle except the following components are 0:

$$\psi(e_2,f_3)=x_1v_{0,0,0,0,0,0,0},\ \psi(e_3,f_4)=x_2v_{0,0,0,0,0,0,0},$$

$$\psi(e_4,f_5)=x_3v_{0,0,0,0,0,0,0},\ \psi(f_1,f_2)=x_4v_{0,1,0,0,0,0,0},$$

$$\psi(e_1,f_4)=x_5v_{0,1,0,0,0,0,0},\ \psi(e_4,f_6)=x_6v_{0,1,0,0,0,0,0},$$

$$\psi(f_1,f_3)=x_7v_{1,1,0,0,0,0,0}+x_8v_{0,0,1,0,0,0,0},$$

$$\psi(e_1,f_5)=x_9v_{1,1,0,0,0,0,0}+x_{10}v_{0,0,1,0,0,0,0},$$

$$\psi(e_3,f_6)=x_{11}v_{1,1,0,0,0,0,0}+x_{12}v_{0,0,1,0,0,0,0},\ \psi(e_2,f_4)=x_{13}v_{1,0,0,0,0,0,0},$$

$$\psi(e_3,f_5)=x_{14}v_{1,0,0,0,0,0,0},\ \psi(e_2,f_5)=x_{15}v_{2,0,0,0,0,0,0},$$

$$\psi(f_2,f_4)=x_{16}v_{0,1,1,0,0,0,0},\ \psi(f_2,f_6)=x_{17}v_{0,1,2,0,0,0,0},$$

$$\psi(e_1,f_6)=x_{33}v_{0,1,1,0,0,0,0},$$

$$\psi(f_1,f_4)=x_{18}v_{2,1,0,0,0,0,0}+x_{19}v_{1,0,1,0,0,0,0}+x_{20}v_{0,0,0,1,0,0,0},$$

$$\psi(e_2,f_6)=x_{21}v_{2,1,0,0,0,0,0}+x_{22}v_{1,0,1,0,0,0,0}+x_{23}v_{0,0,0,1,0,0,0},$$

$$\psi(f_2,f_5)=x_{24}v_{1,1,1,0,0,0,0}+x_{25}v_{0,0,2,0,0,0,0}+x_{26}v_{0,1,0,1,0,0,0},$$

$$\psi(f_3,f_4)=x_{27}v_{1,1,1,0,0,0,0}+x_{28}v_{0,0,2,0,0,0,0}+x_{29}v_{0,1,0,1,0,0,0},$$

$$\psi(f_1,f_5)=x_{30}v_{2,0,1,0,0,0,0}+x_{31}v_{1,0,0,1,0,0,0}+x_{32}v_{0,0,0,0,0,1,0},$$

$$\psi(f_1,f_6)=x_{34}v_{2,1,1,0,0,0,0}+x_{35}v_{1,0,2,0,0,0,0}+x_{36}v_{1,1,0,1,0,0,0}+$$

$$x_{37}v_{0,0,1,1,0,0,0}+x_{38}v_{0,1,0,0,0,1,0}+x_{39}v_{0,0,0,0,0,0,1},$$

$$\psi(f_3,f_5)=x_{40}v_{2,1,1,0,0,0,0}+x_{41}v_{1,0,2,0,0,0,0}+x_{42}v_{1,1,0,1,0,0,0}+$$

$$x_{43}v_{0,0,1,1,0,0,0}+x_{44}v_{0,1,0,0,0,1,0}+x_{45}v_{0,0,0,0,0,0,1},$$

$$\psi(f_3,f_6)=x_{46}v_{1,1,2,0,0,0,0}+x_{47}v_{0,0,3,0,0,0,0}+x_{48}v_{0,1,1,1,0,0,0}+x_{49}v_{0,1,0,0,0,0,1},$$

$$\psi(f_4,f_5)=x_{50}v_{3,1,1,0,0,0,0}+x_{51}v_{2,0,2,0,0,0,0}+x_{52}v_{2,1,0,1,0,0,0}+$$

$$x_{53}v_{0,0,0,2,0,0}+x_{54}v_{1,0,1,1,0,0}+x_{55}v_{0,0,1,0,1,0},$$

$$\psi(f_4,f_6)=x_{56}v_{2,1,2,0,0,0}+x_{57}v_{1,0,3,0,0,0}+x_{58}v_{0,1,0,2,0,0}+x_{59}v_{1,1,1,1,0,0}+$$

$$x_{60}v_{0,0,2,1,0,0}+x_{61}v_{1,1,0,0,0,1}+x_{62}v_{0,0,1,0,0,1}++x_{63}v_{0,1,1,0,1,0},$$

$$\psi(f_5,f_6)=x_{64}v_{3,1,2,0,0,0}+x_{65}v_{2,0,3,0,0,0}+x_{66}v_{1,1,0,2,0,0}+x_{67}v_{2,1,1,1,0,0}+$$

$$x_{68}v_{1,0,2,1,0,0}+x_{69}v_{0,0,1,2,0,0}+x_{70}v_{1,1,1,0,1,0}+x_{71}v_{0,0,2,0,1,0}+$$

$$x_{72}v_{0,1,0,1,1,0}+x_{73}v_{2,1,0,0,0,1}+x_{74}v_{1,0,1,0,0,1}+x_{75}v_{0,0,0,1,0,1},$$

where $x_i\in\mathcal{K}\ \ (i=1,\ldots,75)$.

From $d\psi(h_i,f_1,f_6)=0,\ d\psi(h_i,f_1,f_5)=0,\ d\psi(h_i,e_2,f_3)=0,\ i=1,2$, it follows that

$$\psi(h_i,f_1)=\psi(h_i,f_3)=\psi(h_i,f_4)=\psi(h_i,f_5)=\psi(h_i,f_6)=0,\ i=1,2.$$

From

$$d\psi(f_4,f_5,f_6)=0,\ d\psi(f_3,f_5,f_6)=0,\ d\psi(f_2,f_5,f_6)=0,$$

$$d\psi(f_3,f_4,f_6)=0,\ d\psi(f_3,f_4,f_5)=0,\ d\psi(f_1,f_2,f_6)=0,$$

$$d\psi(f,f_2,f_5)=0,\ d\psi(e_2,f,f_6)=0,\ d\psi(e_4,f_5,f_6)=0,$$

it follows that the following components $\psi(f_5,f_6)$, $\psi(f_3,f_6)$, $\psi(f_3,f_5)$, $\psi(f_2,f_6)$, $\psi(f_2,f_5)$, $\psi(f_1,f_6)$, $\psi(f_1,f_5)$, $\psi(f_1,f_2)$, $\psi(e_4,f_6)$, $\psi(e_4,f_5)$, $\psi(e_2,f_6)$ are zero and $\psi(f_4,f_6)=x_{62}v_{0,0,1,0,0,1}$, $\psi(f_4,f_5)=x_{62}v_{0,0,1,0,1,0}$, $\psi(f_3,f_4)=-2x_{62}v_{0,0,2,0,0,0}$.

From the conditions

$$d\psi(e_4,f_3,f_4)=0,\ d\psi(e_3,f_4,f_6)=0,\ d\psi(e_1,f_4,f_5)=0,$$
$$d\psi(e_1,f_4,f_6)=0$$

we see that the following components

$$\psi(e_1,f_4),\ \psi(f_4,f_6),\ \psi(f_1,f_6),\ \psi(f_3,f_4),\ \psi(e_1,f_5),\ \psi(f_2,f_4),\ \psi(e_3,f_6)$$

are zero. Finally, from the equations

$$d\psi(f_1,f_2,f_4)=0,\ d\psi(e_3,f_2,f_4)=0,\ d\psi(e_1,e_2,f_4)=0,$$

$$d\psi(e_1,e_3,f_5)=0,\ d\psi(e_6,f_3,f_5)=0,\ d\psi(e_3,f_3,f_4)=0$$

we have

$$\psi(f_1,f_4)=\psi(e_2,f_4)=\psi(e_3,f_5)=\psi(f_1,f_3)=\psi(e_3,f_4)=$$
$$\psi(e_2,f_3)=0.$$

Therefore, any cocycle in $Z^2(\mathfrak{g},L((p-2)\lambda_1+\lambda_2))$ with weight $p\lambda_1$ is zero. Thus $\ker f=0$.

Theorem 1.1 is proved completely. □

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