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TREES, FREE RIGHT-SYMMETRIC ALGEBRAS, FREE NOVIKOV ALGEBRAS AND IDENTITIES

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(*communicated by Larry Lambe*)

Abstract

An algebra with the identity $x \circ (y \circ z - z \circ y) = (x \circ y) \circ z - (x \circ z) \circ y$ is called right-symmetric. A right-symmetric algebra with the identity $x \circ (y \circ z) = y \circ (x \circ z)$ is called Novikov. We describe bases of free right-symmetric algebras and free Novikov algebras and give realizations of them in terms of trees. The free Lie algebra is obtained as a Lie subalgebra of the free right-symmetric algebra. We use our methods to study identities of Witt algebras.

To Jan–Erik Roos on his sixty–fifth birthday

1. Introduction

An algebra A over a commutative ring R is an R-module with a bilinear operation $\circ : A \times A \to A$. The R-algebra A is called right-symmetric if it satisfies the identity

$$a \circ (b \circ c - c \circ b) = (a \circ b) \circ c - (a \circ c) \circ b\,.$$

The reason for this name is the following. If $(a,b,c) = a \circ (b \circ c) - (a \circ b) \circ c$ is the associator, the identity can be rewritten as a right-symmetric condition of associators

$$(a,b,c) = (a,c,b).$$

A right-symmetric algebra is called Novikov if it also satisfies the left-commutativity identity

$$a \circ (b \circ c) = b \circ (a \circ c).$$

Right-symmetric algebras first appeared in Cayley's paper [**3**]. In this paper the identity $QPU = (Q \times P)U + (QP)U$ is mentioned. From this it is easy to obtain a right-symmetric identity for the right-symmetric Witt algebra. Cayley in fact also considered a realization of the right-symmetric Witt algebra as rooted trees. Right-symmetric algebras were re-examined in [**9**], [**10**], [**15**]. These algebras have many

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other names: Vinberg algebras, Koszul algebras, Gerstenhaber algebras and pre-Lie algebras. An algebra is called left-symmetric, if it satisfies the left-symmetric condition for associators

$$(a, b, c) = (b, a, c).$$

Any right (respectively left)-symmetric algebra is left (respectively right)-symmetric under the opposite multiplication $a \star b = b \circ a$. Novikov algebras firstly appeared in the study of hydrodynamical equations [**1**] (see also [**11**], [**12**], [**13**]).

Example 1.1. Let $U = k[x_1, \ldots, x_n]$ and $\partial_i = \partial/\partial_{x_i}$ be the partial derivative. Then $W_n^{rsym} = \{\sum u_i \partial_i : u_i \in U\}$ under multiplication $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$ is right-symmetric. This algebra is called the right-symmetric Witt algebra (in n variables). When $n = 1$ this algebra is Novikov. More generally, if A is a commutative algebra and ∂ is a derivation on A, then A is a Novikov algebra under multiplication $a * b = \partial(a)b$. In fact, we will prove in Section 7 that any Novikov algebra is a quotient of a subalgebra of an algebra of this kind.

In our paper we describe two bases of free right-symmetric algebras. The first uses the concept of rooted trees, and the basis elements are called t-elements. The other basis is obtained by considering a basis for the free (non-associative) algebra modulo the right-symmetric axiom. Here the basis elements are called r-elements. They have the same structure as the t-elements, but the multiplication rule is different. An r-element may be seen as a non-associative monomial and this basis is therefore suitable when the value of algebra homomorphisms are studied. This is indeed the case in section 10, where T-ideals in the free right-symmetric algebra appear in studying identities of the Witt algebra (see below).

On the other hand, the multiplication of t-elements has a nice explicit form, which sometimes makes it easier to use this basis.

The basis for the free right-symmetric algebra in terms of trees was established independently in [**4**]. Tree algebras also appears in quantum field theory [**5**] and in numerical analysis [**2**].

We also give a description of a basis in the free Novikov algebra. We prove that the number of basis elements in degree n of the free Novikov algebra with 1 generator is equal to the number of non-ordered partitions of $n - 1$.

For the free Novikov algebra on k generators, this number is the coefficient of $y^{-1}z^n$ in the series

$$\prod_{j=-1}^{\infty} \frac{1}{(1 - y^j z)^k}.$$

The identities studied in section 10 are the following, which we call the *left* and *right* standard polynomial.

$$s_{k+1}^l(X_0, \ldots, X_k) = \sum_{\sigma \in Sym_k} \operatorname{sign}(\sigma) X_{\sigma(1)} \circ (X_{\sigma(2)} \circ \cdots \circ (X_{\sigma(k)} \circ X_0) \cdots)$$

$$s_k^r(X_1, \ldots, X_k) = \sum_{\sigma \in Sym_k} \operatorname{sign}(\sigma)(\cdots (X_{\sigma(1)} \circ X_{\sigma(2)}) \circ \cdots \circ X_{\sigma(k-1)}) \circ X_{\sigma(k)}$$

They are obtained from the standard skew-symmetric associative identity

$$s_k(t_1,\ldots,t_k) = \sum_{\sigma\in Sym_k} \mathrm{sign}(\sigma) t_{\sigma(1)}\cdots t_{\sigma(k)}$$

in the following way. Let l_X and r_X be the left and right multiplication operators: $l_X(Y) = X\circ Y, r_X(Y) = Y\circ X$. Then s^l_{k+1} is obtained by substituting l_{X_i} for t_i in s_k and apply the result on X_0, while s^r_k is obtained by substituting r_{X_i} for t_i in s_k and apply the result on a unit e (which is added if necessary).

In [**8**] some identities of the right-symmetric Witt algebra was found. It was proved that W_n^{rsym} satisfies the left standard polynomial identity of degree $2n+1$

$$s^l_{2n+1} = 0.$$

It is not difficult to prove that W_n^{rsym} also satisfies the right standard polynomial identity

$$s^r_{n^2+2n} = 0.$$

A conjecture in [**8**] states that

$$s^r_{n^2+2n-1} = 0$$

is also a polynomial identity and that this identity is minimal for right polynomial identities. It is easy to see that in any right-symmetric algebra

$$s^l_3 = 0 \Rightarrow s^r_q = 0, \quad q\geqslant 3.$$

In our paper we prove that, given $k>3$, there is a right-symmetric algebra which satisfies the identity $s^l_k = 0$ but which does not satisfy the identity $s^r_q = 0$ for any q.

In particular, for $n>1$, the identity $s^r_{n^2+2n} = 0$ does not follow from $s^l_{2n+1} = 0$, so, W_n^{rsym} has at least two independent polynomial identities when $n>1$.

The key method here is to introduce in the free right-symmetric algebra

- a compatible order and
- a non-archimedian norm

The compatible order in the free right-symmetric algebra is very much like the orders studied in Gröbner theory for associative algebras and Lie algebras. Using this method, we prove that the Lie subalgebra generated by Ω of the free right-symmetric algebra on Ω under the Lie commutator $[X,Y] = X\circ Y - Y\circ X$ is free as a Lie algebra.

The non-archimedian norm allows us to make the free right-symmetric algebra to a topological algebra with a decreasing filtration of balls, such that these balls are closed under the action of algebra endomorphisms.

2. The set of rooted trees

Let Ω be any set. The set of rooted trees with nodes labelled from Ω, denoted $T(\Omega)$, is defined in the following way. Define recursively a set $\hat{T}(\Omega)$ by the rules given below, where t is just a formal symbol.

- $a \in \Omega \Rightarrow a \in \hat{T}(\Omega)$
- $a \in \Omega,\ x_1, \dots, x_n \in \hat{T}(\Omega) \Rightarrow t(a, x_1, \dots, x_n) \in \hat{T}(\Omega)$

We do not exclude the possibility that $n = 0$ in the second clause above, so $t(a) \in \hat{T}(\Omega)$ if $a \in \Omega$. The set $T(\Omega)$ is now defined as $\hat{T}(\Omega)/\sim$, where $\sim$ is the least equivalence relation that satisfies

- $a \sim t(a)$
- $t(a, x_1, \dots, x_n) \sim t(a, x_{i_1}, \dots, x_{i_n})$ if $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$.
- $t(a, x_1, \dots, x_n) \sim t(a, y_1, \dots, y_n)$ if $x_i \sim y_i$ for $i = 1, \dots, n$.

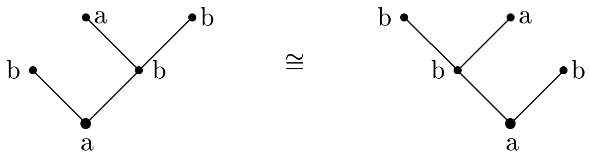
An equivalence class in $T(\Omega)$ is denoted by any of its representatives in $\hat{T}(\Omega)$.

Example 2.1. We have

$$t(a, b, t(b, a, b)) \in T(\{a, b\}) \text{ and } t(a, b, t(b, a, b)) = t(a, t(b, b, a), b)\,.$$

It is clear that the above definition of $T(\Omega)$ coincides with the graph theoretical notion of "rooted labelled trees", where two such trees are considered to be equal if there is an isomorphism of labelled graphs which sends the root to the root.

Example 2.2. The equality $t(a, b, t(b, a, b)) = t(a, t(b, b, a), b)$ may be illustrated by the following isomorphic rooted trees.



Definition 2.3. We make the following definitions for $y \in T(\Omega)$.

(i) The length of y, denoted $|y|$, is defined by

$$|t(a, x_1, \dots, x_n)| = 1 + |x_1| + \cdots + |x_n| \quad \text{for} \quad n \geqslant 0.$$

(ii) The number of branches in y, denoted $\mathrm{br}(y)$, is defined by

$$\mathrm{br}(t(a, x_1, \dots, x_n)) = n \quad \text{for} \quad n \geqslant 0.$$

(iii) The operation $\bullet$ on $T(\Omega)$ is defined as follows, where $a \in \Omega$ and $x_1, \dots, x_n, y \in T(\Omega)$,

$$t(a, x_1, \dots, x_n) \bullet y = t(a, x_1, \dots, x_n, y).$$

The operation $\bullet$ is non-associative and non-commutative. It satisfies however the right-commutative identity $(a \bullet b) \bullet c = (a \bullet c) \bullet b$ (in fact we will prove later that $T(\Omega)$ is the free right-commutative magma on Ω, cf. Proposition 6.3).

For a given set X, let $\mathrm{Mon}(X)$ denote the free commutative monoid on X; i.e., $\mathrm{Mon}(X)$ consists of words (including the empty word) in the alphabet X, where the letters may be written in any order. We have a bijection

$$T(\Omega) \to \Omega \times \mathrm{Mon}(T(\Omega)) \tag{1}$$

where the correspondence is $t(a, x_1, \ldots, x_n) \mapsto (a, x_1 \cdots x_n)$ and $a \mapsto (a, 1)$.

Let $T(\Omega)_n$ denote the subset of $T(\Omega)$ consisting of all elements of length n. Suppose Ω is finite, $|\Omega| = k$. Then $T(\Omega)_n$ is finite and we may form the generating series $f_k(z) = \sum_{n\geqslant 1} a(n,k) z^n$ where $a(n,k) = |T(\Omega)_n|$.

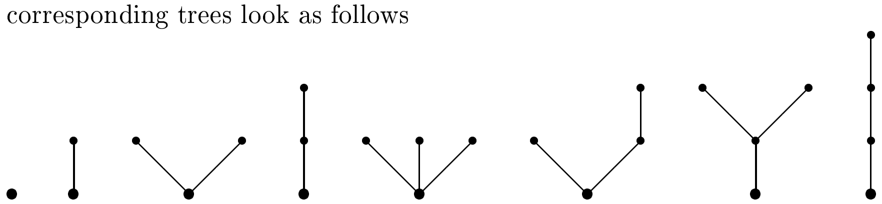
The length of a monomial $x_1 \cdots x_j$ is defined as $|x_1| + \cdots + |x_j|$. It is well-known that the generating series for $\mathrm{Mon}(T(\Omega))$ with respect to length is ($|\Omega| = k$)

$$\exp(f_k(z)) \stackrel{\text{def}}{=} \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^{a(n,k)}} .$$

Hence by (1), $f_k(z)$ satisfies the following functional equation, which may be found in [**3**].

$$f_k(z) = kz \exp(f_k(z)) \tag{2}$$

This equation makes it possible to compute the numbers $a(n,k)$ recursively. E.g., if $|\Omega| = 1$ then $a(1,1) = a(2,1) = 1$, $a(3,1) = 2$, $a(4,1) = 4$, $\ldots$ where the corresponding trees look as follows



It is easily seen that $a(n,k)$ is a polynomial in k of degree n for each n. The first polynomials are

$$a(1,k) = k, \quad a(2,k) = k^2$$
$$a(3,k) = \frac{3}{2}k^3 + \frac{1}{2}k^2, \quad a(4,k) = \frac{8}{3}k^4 + k^3 + \frac{1}{3}k^2 .$$

We may write $f_k(z)$ as a power series in k:

$$f_k(z) = \varphi_1(z)k + \varphi_2(z)k^2 + \varphi_3(z)k^3 + \cdots$$

We will now give recursive formulas for the functions $\varphi_1, \varphi_2, \ldots$. We have

$$\exp(f_k)(z) = \prod_{j=1}^{\infty} \exp(\varphi_j(z)k^j) = \prod_{j=1}^{\infty} (\exp(\varphi_j(z)))^{k^j} .$$

Hence by (2),

$$\frac{f_k(z)}{kz} = \prod_{j=1}^{\infty} (\exp(\varphi_j(z)))^{k^j} = e^{\sum_{j=1}^{\infty} \log(\exp(\varphi_j))k^j} .$$

Definition 2.4. For any series without constant term, $g(z) = \sum_{n\geqslant 1} b_n z^n$, let $\mathrm{logexp}(g)$ denote the series $\sum_{n\geqslant 1} -b_n \log(1-z^n)$. Also, let $y^{(n)} = y^n/n!$.

The multinomial theorem may be written

$$(y_1 + \cdots + y_r)^{(n)} = \sum_{\sum n_i = n} x_1^{(n_1)} \cdot \ldots \cdot x_r^{(n_r)}$$

and we get

$$\frac{f_k(z)}{kz} = \sum_{n=1}^{\infty} \sum_{\sum j n_j = n} (\psi_1^{(n_1)} \psi_2^{(n_2)} \cdot \ldots) k^n$$

where $\psi_j = \text{logexp}(\varphi_j)$.

Hence

$$\frac{1}{z}(\varphi_1(z) + \varphi_2(z)k + \varphi_3(z)k^2 + \cdots) =$$
$$1 + \psi_1(z)k + (\psi_1^{(2)}(z) + \psi_2(z))k^2 + (\psi_1^{(3)}(z) + \psi_1(z)\psi_2(z) + \psi_3(z))k^3 + \cdots$$

and we have proved the following theorem

Theorem 2.5. *Let Ω be a set with k elements and let $f_k(z) = \varphi_1(z)k + \varphi_2(z)k^2 + \varphi_3(z)k^3 + \cdots$ be the generating function for $T(\Omega)$. Then*

$$\begin{aligned}
\varphi_1(z) &= z \\
\varphi_2(z) &= z\text{logexp}(z) = -z\log(1-z) \\
\varphi_3(z) &= z(\frac{1}{2}(\log(1-z))^2 + \text{logexp}(\varphi_2)) \\
\varphi_4(z) &= z(-\frac{1}{6}(\log(1-z))^3 - \log(1-z)\text{logexp}(\varphi_2) + \text{logexp}(\varphi_3)) \\
&\ldots
\end{aligned}$$

and in general, given $\varphi_1, \ldots, \varphi_{n-1}$, we have

$$\varphi_n(z)/z = \sum_{\sum j n_j = n} (\psi_1^{(n_1)} \psi_2^{(n_2)} \cdot \ldots)$$

where $\psi_j = \text{logexp}(\varphi_j)$.

3. The tree algebra

Let R be a commutative ring. For any set Ω we define the "tree algebra", $\mathcal{T}(\Omega)$, as the free R-module on $T(\Omega)$. The operation $\bullet$ on $T(\Omega)$ is extended to $\mathcal{T}(\Omega)$ by linearity. A bilinear multiplication $\circ$ on $\mathcal{T}(\Omega)$ is defined recursively on basis elements as follows, where $a \in \Omega,\ x_1, \ldots, x_n, y \in T(\Omega)$.

$$\begin{aligned}
a \circ y &= a \bullet y = t(a, y) \\
t(a, x_1, \ldots, x_n) \circ y &= t(a, x_1, \ldots, x_n, y) + \\
&\quad \sum_{i=1}^{n} t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \circ y)
\end{aligned}$$

In other words, $y_1 \circ y_2$ is obtained as follows. Add a new branch to the root of y_2 and "plant" this graph to each node of y_1 and add the resulting trees. In this way $(\mathcal{T}(\Omega), \circ)$ becomes a positively graded (non-associative) R-algebra, $\mathcal{T}(\Omega) = \oplus_{n \geqslant 1} \mathcal{T}(\Omega)_n$, where $\mathcal{T}(\Omega)_n$ is the free R-module on $T(\Omega)_n$. We will use the notation $\mathcal{T}(\Omega)$ for this algebra (without mentioning the operation $\circ$).

Proposition 3.1. *The operations $\bullet$ and $\circ$ on $\mathcal{T}(\Omega)$ satisfy the following identities.*

(i)

$$(x \bullet y) \bullet z = (x \bullet z) \bullet y$$

i.e., $(\mathcal{T}(\Omega), \bullet)$ is right-commutative.

(ii)

$$(x \bullet y) \circ z = (x \circ z) \bullet y + x \bullet (y \circ z)$$

i.e., $(\mathcal{T}(\Omega), \circ)$ is a derivation algebra of $(\mathcal{T}(\Omega), \bullet)$

(iii)

$$(x \circ y) \circ z = (x \circ z) \circ y + x \circ (y \circ z) - x \circ (z \circ y)$$

i.e., $\mathcal{T}(\Omega)$ is right-symmetric.

Proof. The first two identities follow directly from the definitions. We use (i) and (ii) and induction over the length of x to prove (iii). Suppose first that $x = a \in \Omega$. Then $(a \circ y) \circ z - a \circ (y \circ z) = (a \bullet y) \circ z - a \bullet (y \circ z)$ and by (ii) this equals $(a \circ z) \bullet y = (a \bullet z) \bullet y$. In the same way $(a \circ z) \circ y - a \circ (z \circ y) = (a \bullet y) \bullet z$ and hence by (i) we get (iii). If $x \notin \Omega$ we may write $x = x_1 \bullet x_2$, where $|x_1| < |x|$ and $|x_2| < |x|$. By (ii)

$$\begin{aligned} ((x_1 \bullet x_2) \circ y) \circ z &= ((x_1 \circ y) \bullet x_2) \circ z + (x_1 \bullet (x_2 \circ y)) \circ z \\ &= ((x_1 \circ y) \circ z) \bullet x_2 + (x_1 \circ y) \bullet (x_2 \circ z) + \\ &\quad (x_1 \circ z) \bullet (x_2 \circ y) + x_1 \bullet ((x_2 \circ y) \circ z). \end{aligned}$$

Hence

$$\begin{aligned} (x \circ y) \circ z - (x \circ z) \circ y = {} & ((x_1 \circ y) \circ z - (x_1 \circ z) \circ y) \bullet x_2 + \\ & x_1 \bullet ((x_2 \circ y) \circ z - (x_2 \circ z) \circ y) \end{aligned}$$

and hence by induction and (ii)

$$\begin{aligned} (x \circ y) \circ z - (x \circ z) \circ y &= (x_1 \circ (y \circ z - z \circ y)) \bullet x_2 + \\ &\quad x_1 \bullet (x_2 \circ (y \circ z - z \circ y)) \\ &= (x_1 \bullet x_2) \circ (y \circ z - z \circ y) = x \circ (y \circ z - z \circ y). \end{aligned}$$

□

Now suppose Ω is well-ordered. The order may be extended to a well-order $<$ on $T(\Omega)$ in such a way that $y_1 < y_2$ if $|y_1| < |y_2|$ or if $|y_1| = |y_2|$ and $\mathrm{br}(y_1) < \mathrm{br}(y_2)$. Then the following properties are easily proven.

Proposition 3.2. *If a well-order $<$ on $T(\Omega)$ satisfies that $y_1 < y_2$ if $|y_1| = |y_2|$ and $\mathrm{br}(y_1) < \mathrm{br}(y_2)$, then the leading term of $t(a, x_1, \ldots, x_n) \circ y$ is $t(a, x_1, \ldots, x_n) \bullet y$ and the leading term of $(\ldots(a \circ x_1) \circ x_2) \circ \cdots \circ x_n)$ is $t(a, x_1, \ldots, x_n)$.*

Proof. It follows from the definition that $t(a, x_1, \ldots, x_n) \bullet y$ is the only term of $t(a, x_1, \ldots, x_n) \circ y$ which has $n+1$ branches and all terms have the same length (we ignore the coefficients, the leading term has however coefficient 1). In the same way, $t(a, x_1, \ldots, x_n)$ is the only term with n branches in $(\ldots(a \circ x_1) \circ x_2) \circ \cdots \circ x_n)$. □

Proposition 3.3. *As an R-algebra, $\mathcal{T}(\Omega)$ is generated by Ω.*

Proof. We prove by induction over a well-order on $T(\Omega)$ satisfying the assumptions in Proposition 3.2, that any element in $T(\Omega)$ is generated by Ω. Suppose this is true for all elements less than $t(a, x_1, \ldots, x_n)$. Since $x_i < t(a, x_1, \ldots, x_n)$ for all i, it follows by induction that $(\ldots(a \circ x_1) \circ x_2) \circ \cdots \circ x_n)$ is generated by Ω. By Proposition 3.2, this expression has $t(a, x_1, \ldots, x_n)$ as leading term. Since by induction all other terms are generated by Ω, it follows that $t(a, x_1, \ldots, x_n)$ is generated by Ω. □

4. Super-trees

We will shortly describe how signs may be introduced in the notions of tree and tree algebra. Suppose Ω is a super-set; i.e., the elements in Ω are divided into even and odd elements. We will use the notation $\epsilon(x, y)$ for the sign introduced when x and y are interchanged in a formula; i.e., $\epsilon(x, y) = -1$ if both x and y are odd and $+1$ otherwise. The function ϵ is supposed to be bi-additive (with values in the multiplicative group $\{-1, 1\}$).

The notion of odd and even elements is naturally extended to trees. A tree is a signed expression $\pm t(a, x_1, \ldots, x_n)$ or 0. The definition of equality for trees is changed to

$$t(a, x_1, \ldots, x_n) = \epsilon(x_i, x_{i+1}) t(a, x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

for any $i = 1, \ldots, n-1$. Also if y is odd, then for any $n \geqslant 0$

$$t(a, y, y, x_1, \ldots, x_n) = 0.$$

We let $\mathrm{Mon}(X)$, where X is a super-set, denote the monoid of signed monomials and 0, where an odd variable occurs at most once in a monomial. We have a bijection

$$T(\Omega) \to \Omega \times \mathrm{Mon}(T(\Omega))/(a, 0) = 0.$$

If Ω is finite, a functional equation for the generating series of all unsigned non-zero trees, $T(\Omega)^+$, may be given. The generating series for a graded super-set, which is finite in each degree, is a power series in z and y, where $y^2 = 1$. If $f(z, y) = \sum a_n z^n + y \sum b_n z^n$, then a_n is the number of even elements and b_n is the number of odd elements in degree n.

If $f(z, y) = \sum_{n \geqslant 1} a_n z^n + y(\sum_{n \geqslant 1} b_n z^n)$ is the generating function for a positively graded super-set, we make the following definition.

$$\exp(f(z, y)) \stackrel{\text{def}}{=} \prod_{n=1}^{\infty} \frac{(1 + y z^n)^{b_n}}{(1 - z^n)^{a_n}}$$

Suppose Ω consists of k_0 even elements and k_1 odd elements and let $f_{k_0,k_1}(z, y)$ be the generating series for $T(\Omega)^+$. Then we have

$$f_{k_0,k_1}(z, y) = (k_0 + k_1 y) z \exp(f_{k_0,k_1}(z, y)).$$

The definition of the operation $\bullet$ is the same as before. The tree algebra, $\mathcal{T}(\Omega)$, is defined as the free R-module on $T(\Omega)^+$. The operation $\bullet$ on $T(\Omega)$ is extended to $\mathcal{T}(\Omega)$ by linearity. The definition of $\circ$ on $\mathcal{T}(\Omega)$ is changed to

$$\begin{aligned} a \circ y &= a \bullet y = t(a, y) \\ t(a, x_1, \ldots, x_n) \circ y &= t(a, x_1, \ldots, x_n, y) + \\ &\quad \sum_{i=1}^{n} \epsilon(x_i, x_{i+1} + \ldots + x_n) t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \circ y). \end{aligned}$$

Proposition 4.1. *The operations $\bullet$ and $\circ$ on $\mathcal{T}(\Omega)$, where Ω is a super-set, satisfy the following identities.*

(i)

$$(x \bullet y) \bullet z = \epsilon(y, z)(x \bullet z) \bullet y$$

and

$$(x \bullet y) \bullet y = 0 \quad \textit{if} \;\; y \;\; \textit{is odd}$$

i.e., $(\mathcal{T}(\Omega), \bullet)$ is right-super-commutative.

(ii)

$$(x \bullet y) \circ z = \epsilon(y, z)(x \circ z) \bullet y + x \bullet (y \circ z)$$

$(\mathcal{T}(\Omega), \circ)$ is a derivation algebra of $(\mathcal{T}(\Omega), \bullet)$

(iii)

$$(x \circ y) \circ z = \epsilon(y, z)(x \circ z) \circ y + x \circ (y \circ z) - \epsilon(y, z) x \circ (z \circ y)$$

i.e., $\mathcal{T}(\Omega)$ is right-super-symmetric.

5. Compatible orders

In this section we assume that R is an integral domain. In the applications in later sections we will need that the order on $T(\Omega)$ is not just a well-order but also satisfies a compatibility condition which is made precise below.

Definition 5.1. Let $<$ be a well-order on $T(\Omega)$. We make the following definitions.

- The order $<$ is $\bullet$-compatible if
  $x < y \Rightarrow x \bullet z < y \bullet z$ and $z \bullet x < z \bullet y$ for $x, y, z \in T(\Omega)$.
- For $x \in \mathcal{T}(\Omega) \setminus \{0\}$, $lead(x) \in T(\Omega)$ is the maximal basis element which occurs in x with non-zero coefficient.
- The order $<$ is $\circ$-compatible if
  $x < y \Rightarrow lead(x \circ z) < lead(y \circ z)$ and $lead(z \circ x) < lead(z \circ y)$ for $x, y, z \in T(\Omega)$.
- The order $<$ is $\circ$-leading if $lead(x \circ y) = x \bullet y$ for $x, y \in T(\Omega)$.
- For $x, y \in \mathcal{T}(\Omega) \setminus \{0\}$, $x < y \Leftrightarrow lead(x) < lead(y)$.

Proposition 5.2. *Suppose $<$ is a well-order on $T(\Omega)$. Then the following holds.*

(i) If $<$ is $\bullet$-compatible and $\circ$-leading, then $<$ is $\circ$-compatible.

(ii) If $<$ is $\circ$-compatible, then
$lead(x \circ y) = lead(lead(x) \circ lead(y))$ *for* $x, y \in \mathcal{T}(\Omega) \setminus \{0\}$.

(iii) If $<$ is $\bullet$-compatible and $\circ$-leading, then $lead(x \circ y) = lead(x) \bullet lead(y)$ *for* $x, y \in \mathcal{T}(\Omega) \setminus \{0\}$.

(iv) If $<$ is $\circ$-compatible, then $x < y \Rightarrow x \circ z < y \circ z$ *and* $z \circ x < z \circ y$ *for* $x, y, z \in \mathcal{T}(\Omega) \setminus \{0\}$.

Proof. The first statement follows directly from the definitions. To prove (ii), suppose first that $x \in T(\Omega)$. Let $y = r_1 y_1 + r_2 y_2 + \ldots + r_n y_n$, where $r_i \in R$ and $y_i \in T(\Omega)$ for $i = 1, \ldots, n$, $r_1 \neq 0$ and $y_1 > y_2 > \ldots > y_n$. Since $<$ is $\circ$-compatible we have $x \circ y_1 > x \circ y_2 > \ldots > x \circ y_n$. Hence $lead(x \circ y) = lead(x \circ y_1) = lead(x \circ lead(y))$ and (ii) is proved when $x \in T(\Omega)$.

Next suppose $x = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$, where $s_i \in R$ and $x_i \in T(\Omega)$ for $i = 1, \ldots, n$, $s_1 \neq 0$ and $x_1 > x_2 > \ldots > x_n$. Then $x_1 \circ y > x_2 \circ y > \ldots > x_n \circ y$, since

$$lead(x_1 \circ y) = lead(x_1 \circ lead(y)) > lead(x_2 \circ lead(y)) = lead(x_2 \circ y),$$

where the equalities follow from above and the inequality follows from the assumption that $<$ is $\circ$-compatible. Hence $lead(x \circ y) = lead(x_1 \circ y)$ and again from above we get $lead(x \circ y) = lead(x_1 \circ lead(y)) = lead(lead(x) \circ lead(y))$.

(iii) follows from (i) and (ii).

To prove (iv), suppose $x, y \in \mathcal{T}(\Omega)$ and $x < y$. Then by (ii) and since $<$ is $\circ$-compatible we have

$$lead(x \circ z) = lead(lead(x) \circ lead(z)) < lead(lead(y) \circ lead(z)) = lead(y \circ z).$$

In the same way it is proven that $lead(z \circ x) < lead(z \circ y)$. □

To find a $\bullet$-compatible order on $T(\Omega)$ we will use a total order on $\mathrm{Mon}(X)$ defined for each totally ordered set X, which extends the order on X and is functorial in X. Such an order on $\mathrm{Mon}(X)$ is called "natural". It is called "compatible" if it is compatible with the monoid operation on $\mathrm{Mon}(X)$. A natural order on $\mathrm{Mon}(X)$ satisfies in particular the following,

$$a <_{X'} b \Leftrightarrow a <_X b \text{ for all } a, b \in \mathrm{Mon}(X')$$
$$\text{whenever } X' \subset X \text{ as totally ordered sets.}$$

Starting with a well-order on Ω and a compatible natural well-order on $\mathrm{Mon}(X)$ (such as the lexicographic order), we may use equation (1) in section 2 to define a total order on all elements in $T(\Omega)$ of length n, for $n = 1, 2, \ldots$. In this way we obtain a $\bullet$-compatible total order on $T(\Omega)$. The order might not be a well-order; e.g., if the number of branches (which corresponds to the degree of a monomial) is taken *before* the length (which corresponds to an extra weight that the variables have), we have the following infinite sequence in $T(\{a\})$.

$$t(a, a, a) > t(a, t(a, a, a)) > t(a, t(a, t(a, a, a))) > \ldots$$

On the other hand, if *length* is considered as the first criterion, then we obviously obtain a well-order on $T(\Omega)$. To obtain a $\circ$-compatible order, we also want the order to be $\circ$-leading. We give below the definition of two orders on $T(\Omega)$, which have all desired properties (proved in Proposition 5.4). The proof is easy for the first one, but the second one will be more useful for us in the applications.

Definition 5.3. Given a well-order on Ω, two orders $\prec_{deglex}$ and $\prec_{revlex}$ on $T(\Omega)$ are defined as follows.

(i) We have

$$x = t(a, x_1, \ldots, x_n) \prec_{deglex} t(b, y_1, \ldots, y_m) = y \quad \text{if}$$

- $|x| < |y|$
- $|x| = |y|$ and $n < m$
- $|x| = |y|$, $n = m$ and $(x_n, \ldots, x_1, a) \prec_{deglex} (y_n, \ldots, y_1, b)$ in lexicographic sense, where $x_1 \preceq_{deglex} \cdots \preceq_{deglex} x_n$ and $y_1 \preceq_{deglex} \cdots \preceq_{deglex} y_n$.

(ii) We have

$$x = t(a, x_1, \ldots, x_n) \prec_{revlex} t(b, y_1, \ldots, y_m) = y \quad \text{if}$$

- $|x| < |y|$
- $|x| = |y|$ and $t(a, x_1, \ldots, x_{n-1}) \prec_{revlex} t(b, y_1, \ldots, y_{m-1})$, where $x_i \preceq_{revlex} x_n$ for $i < n$ and $y_i \preceq_{revlex} y_m$ for $i < m$.
- $|x| = |y|$, $t(a, x_1, \ldots, x_{n-1}) = t(b, y_1, \ldots, y_{m-1})$ and $x_n \prec_{revlex} y_m$, where $x_i \preceq_{revlex} x_n$ for $i < n$ and $y_i \preceq_{revlex} y_m$ for $i < m$.

Proposition 5.4. *Given a well-order on Ω, the orders $\prec_{deglex}$ and $\prec_{revlex}$ defined above are $\bullet$-compatible, $\circ$-compatible and $\circ$-leading well-orders on $T(\Omega)$, and hence als*

Proof. The first order, $\prec_{deglex}$, is $\circ$-leading by Proposition 3.2 and the natural compatible well-order on $\mathrm{Mon}(X)$ used in the definition is "weight-degree-lexicographic" (so the order should rather be called "weightdeglex"). Hence, by Proposition 5.2, the order is $\circ$-compatible.

The order on $\mathrm{Mon}(X)$ used in the definition of $\prec_{revlex}$ may be described in the following way, which shows that it is a natural compatible well-order. A monomial m is written as a product of monomials $m_1 m_2 \cdots m_n$, where each monomial m_i is a product of variables of the same weight (i.e., *length*) and the weight of the variables in m_1 is greater than the weight of the variables in m_2 and so on. Now $m = m_1 m_2 \cdots m_n > m'_1 m'_2 \cdots m'_k = m'$ if $|m| > |m'|$, or $|m| = |m'|$ and $|m_i| = |m'_i|$ for $i = 1, \ldots j-1$ and $|m_j| < |m'_j|$ for some j, or $n = k$ and $|m_i| = |m'_i|$ for $i = 1, \ldots, n$ and $m > m'$ by the reverse lexicographic principle; i.e., the first variable which differs, counted from the right, is greater in m. This proves that the order is $\bullet$-compatible.

By Proposition 5.2 it is enough to prove that it is also $\circ$-leading; i.e., given $a \in \Omega$ and $x_1 \leqslant \ldots \leqslant x_n, y \in T(\Omega)$, we must prove that

$$t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \circ y) < t(a, x_1, \ldots, x_n, y)$$

for all i. We do this by induction over $|t(a, x_1, \ldots, x_n, y)|$. By the induction hypothesis we get that $lead(x_i \circ y) = x_i \bullet y$. Since $<$ is $\bullet$-compatible, we get

$$lead(t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \circ y)) = t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \bullet y)$$

and hence the claim to be proven is

$$t(a, x_1, \ldots, \hat{x}_i, \ldots, x_n) \bullet (x_i \bullet y) < t(a, x_1, \ldots, x_n, y).$$

Suppose $i < n$. By induction we get

$$t(a, x_1, \ldots, \hat{x}_i, \ldots, x_{n-1}) \bullet (x_i \bullet y) < t(a, x_1, \ldots, x_{n-1}, y)$$

and the claim follows by multiplying this to the right with x_n, using that $<$ is $\bullet$-compatible (and that $\bullet$ is right-commutative).

Suppose $i = n$. We have to prove that

$$t(a, x_1, \ldots, x_{n-1}, x_n \bullet y) < t(a, x_1, \ldots, x_{n-1}, x_n, y).$$

But this follows from the definition of the order, since $|x_n \bullet y| > |x_n|$ and $|x_n \bullet y| > |y|$. $\square$

6. Free algebras

The free magma on a set Ω, $M(\Omega)$, is recursively defined as a set of parenthesized strings in the alphabet Ω in the following way.

- $a \in \Omega \Rightarrow a \in M(\Omega)$
- $x, y \in M(\Omega) \Rightarrow (xy) \in M(\Omega)$

Any element in $M(\Omega) \setminus \Omega$ may be uniquely written as $((\ldots((ax_1)x_2)\ldots)x_n)$, where $a \in \Omega$ and $x_1, \ldots, x_n \in M(\Omega)$. This expression will be written as $r(a, x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are supposed to be of the same form (or belong to Ω) and we say in this case that $r(a, x_1, \ldots, x_n)$ is written in normal form. To be able to make substitutions, we also allow general expressions $r(x_1, \ldots, x_n)$ (which is equal to $((\ldots((x_1x_2)x_3)\ldots)x_n)$). The rule for normalizing a general expression is

$$r(r(x_1, \ldots, x_n), y_1, \ldots, y_m) = r(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

The length of an element is defined in the same way as for $T(\Omega)$. The multiplication $*$ on $M(\Omega)$ (defined by $x * y = (xy)$) satisfies the rule $r(a, x_1, \ldots, x_n) * y = r(a, x_1, \ldots, x_n, y)$ (and $a * y = r(a, y)$ if $a \in \Omega$). The free R-module on $M(\Omega)$ with multiplication defined as the linear extension of $*$ is called the free (non-associative) algebra on Ω and is denoted $\mathcal{M}(\Omega)$. Observe that $\mathcal{M}(\Omega)$ has no unit.

The free right-symmetric R-algebra on Ω, denoted $\mathcal{RS}(\Omega)$, is defined as the quotient of $\mathcal{M}(\Omega)$ with the twosided ideal generated by all elements of the form $(x * y) * z - (x * z) * y - x * (y * z) + x * (z * y)$.

The free right-commutative R-algebra on Ω, denoted $\mathcal{RC}(\Omega)$, is defined as the quotient of $\mathcal{M}(\Omega)$ by the twosided ideal generated by all elements of the form $(x * y) * z - (x * z) * y$. In the same way the free right-commutative magma on Ω, denoted $RC(\Omega)$, is defined as the quotient of $M(\Omega)$ by the congruence relation generated by $(x * y) * z \equiv (x * z) * y$.

Suppose Ω is well-ordered by $<$. The order may be extended to all of $M(\Omega)$ by taking length first and then for two elements of the same length, $(xy) < (x'y')$ if $x < x'$ or $x = x'$ and $y < y'$.

Definition 6.1. An element in $M(\Omega)$ is called right-symmetric normal if it is of the form $a \in \Omega$ or $r(a, x_1, \ldots, x_n)$, where $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ and $x_1, \ldots, x_n$ are right-symmetric normal.

Proposition 6.2. *The R-algebras $\mathcal{RS}(\Omega)$ and $\mathcal{RC}(\Omega)$ are generated as R-modules by elements represented by right-symmetric normal elements in $M(\Omega)$. Also any element in $RC(\Omega)$ may be written in right-symmetric normal form.*

Proof. We prove the statement by induction over the well-order $<$ defined above. Suppose the statement is true for all elements less than $r(a, x_1, \ldots, x_n)$. Then the statement is true for $r(a, x_1, \ldots, x_{n-1})$ and for x_n. We may therefore assume that $x_1, \ldots, x_n$ are of the right form and $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{n-1}$. Put $x = r(a, x_1, \ldots, x_{n-2})$, $y = x_{n-1}$ and $z = x_n$. If $y > z$, we may replace $r(a, x_1, \ldots, x_n) = (x*y)*z$ by $(x*z)*y + x*(y*z) - x*(z*y)$ in the case of $\mathcal{RS}(\Omega)$ and by $(x*z)*y$ in the case of $\mathcal{RC}(\Omega)$ or $RC(\Omega)$. The new elements are less than $(x*y)*z$ and we may use the induction hypothesis to get the result. □

Theorem 6.3. *There exist unique isomorphisms of free R-algebras over Ω extending the identity on Ω,*

$$\mathcal{RS}(\Omega) \cong (\mathcal{T}(\Omega), \circ)$$
$$\mathcal{RC}(\Omega) \cong (\mathcal{T}(\Omega), \bullet)$$

and a unique isomorphism of free magmas over Ω extending the identity on Ω,

$$RC(\Omega) \cong (T(\Omega), \bullet).$$

Proof. Since obviously $T(\Omega)$ is generated by Ω under $\bullet$, we get a surjective homomorphism $y \mapsto y^t$ from $M(\Omega)$ to $T(\Omega)$. The element y^t is obtained by replacing all the prefixes r by t in y. Since $(T(\Omega), \bullet)$ is right-commutative, the map factors through $RC(\Omega)$. To prove injectivity, assume we have two elements $x, y \in RC(\Omega)$ such that $x^t = y^t$ in $T(\Omega)$. By Proposition 6.2 we may assume that x and y are represented by elements where all branches are in linear order. But then x and y have to be identical, since $x^t = y^t$. This proves the last two statements.

By Proposition 3.3 we have a surjective homomorphism $\varphi : \mathcal{M}(\Omega) \to (\mathcal{T}(\Omega), \circ)$, and by Proposition 3.1 the map factors through $\mathcal{RS}(\Omega)$. To prove injectivity, suppose $\varphi(x) = 0$. Then there is a finite subset Ω_0 of Ω such that $x \in \mathcal{M}(\Omega_0)$. Hence we may assume that Ω is finite.

Let N be the number of normal right-symmetric elements in $M(\Omega)$ of length n. By Proposition 6.2 these elements generate the elements of length n in $\mathcal{M}(\Omega)$ modulo the right-symmetric axiom. Obviously we also have that N is the number of elements in $T(\Omega)$ of length n. Hence we get a surjective R-linear map $R^N \to R^N$ and we may apply the well-known Lemma 6.4 below. □

Lemma 6.4. *Let R be a commutative ring with unit and $f : R^N \to R^N$ an R-linear surjective map. Then f is an isomorphism.*

Proof. Let A be the matrix of f. Since f is surjective, there is a solution to the matrix equation $AX = I$. Taking determinants gives $\det(A) \cdot \det(X) = 1$. Hence $\det(A)$ is a unit in R and A is invertible (the standard formula for A^{-1} given in linear algebra is valid). □

Remark 6.5. The lemma is true even when R^N is replaced by a finitely generated R-module M (consider M as an $R[x]$-module).

It is possible to tell more about the isomorphisms in the theorem above.

Proposition 6.6. *Let $f : M(\Omega) \to (\mathcal{T}(\Omega), \circ)$ and $g : M(\Omega) \to (T(\Omega), \bullet)$ be the unique homomorphisms extending the identity on Ω. Given a $\circ$-compatible, $\circ$-leading well-order, $\prec$, on $T(\Omega)$ we have $lead(f(x)) = g(x)$ and the compatible order on $\mathcal{RS}(\Omega)$ which is defined through the isomorphism $\mathcal{RS}(\Omega) \cong (\mathcal{T}(\Omega), \circ)$ (Theorem 6.3) has the same definition on normal right-symmetric elements as $\prec$ has on $T(\Omega)$.*

Proof. Let $x = r(a, x_1, \ldots, x_n) \in M(\Omega)$. We have that $g(x) = x^t$ where x^t is obtained from x by replacing all the prefixes r by t. We prove that $lead(f(x)) = x^t$ by induction on the length of x. Indeed, by induction and by Proposition 5.2 (ii), we get that $lead(f(x)) = lead(t(a, x_1^t, \ldots, x_{n-1}^t) \circ x_n^t)$. Since the order is $\circ$-leading, this equals $t(a, x_1^t, \ldots, x_n^t) = x^t$. The second statement is now a direct consequence. $\square$

Remark 6.7. The injectivity of the map φ in the proof of Theorem 6.3 may also be deduced from Proposition 6.6.

7. Free Novikov algebras

A right-symmetric R-algebra is called Novikov if it is also left symmetric; i.e., it satisfies the axiom $x(yz) = y(xz)$. The free Novikov algebra on a set Ω is denoted $\mathcal{N}(\Omega)$ and is defined as the quotient of $\mathcal{M}(\Omega)$ by the two-sided ideal generated by all expressions

$$x(yz) - y(xz) \quad \text{and} \quad (xy)z - (xz)y - x(yz) + x(zy)\,.$$

We want to find a minimal generating set for $\mathcal{N}(\Omega)$ as an R-module. A step in this direction will be the "ordered nests".

A *subexpression* of $x = r(a, x_1 \ldots, x_n)$ is either x itself or a or a subexpression of some of $x_1, \ldots, x_n$.

Definition 7.1. An r-expression x in $M(\Omega)$ is called a nest if each subexpression has at most one branch which is not in Ω. This means that either $x \in \Omega$ or it has the form $x = r(a_0, a_1 \ldots, a_n, y)$, where $a_i \in \Omega$, $n \geqslant 0$ and y is a nest. Let in the first case $d_1(x) = 0$ and $\mathrm{root}(x)$ be undefined and in the second case $d_1(x) = n$ and $\mathrm{root}(x) = a_0$. A nest x is called ordered (given a total order on Ω) if either $x \in \Omega$ or, in the second case, $y \in \Omega$ or y is ordered and either $d_1(x) < d_1(y)$ or $d_1(x) = d_1(y)$ and $\mathrm{root}(x) \leqslant \mathrm{root}(y)$.

In any r-expression the atoms are divided into two groups, the roots and the leaves. A root of x is an element in Ω which occurs in x directly after a left parenthesis. The other atoms are called leaves. An element in Ω has by definition no root and one leaf, so any expression has at least one leaf and the expressions with exactly one leaf are the nests $r(a_1, r(a_2, \ldots, r(a_{n-1}, a_n)) \ldots)$, while the expressions with exactly one root are the (ordered) nests $r(a_0, a_1, \ldots, a_n)$. Given a total order on Ω, we now define the notion of a Novikov element.

Definition 7.2. An r-expression $x \in M(\Omega)$ is called a Novikov element if it is an ordered nest and the sequence of all leaves in x, read from the left, is ordered.

Example 7.3. If $a < b < c < d$, then $r(a, r(b, b, r(b, c, r(a, c, d, d))))$ is a Novikov element, while $r(a, r(b, b, r(b, c, r(a, b, d, d))))$ is not. The sequence of leaves is $bccdd$ in the first case and $bcbdd$ in the second case.

Proposition 7.4. *The residue classes in $\mathcal{N}(\Omega)$ of all Novikov elements constitute a generating set for $\mathcal{N}(\Omega)$ as an R-module.*

Let A be the submodule of $\mathcal{M}(\Omega)$ generated by the Novikov elements and B be the submodule generated by the ordered nests. Let I be the two-sided ideal in $\mathcal{M}(\Omega)$ generated by the left-commutative and right-symmetric axioms. Proposition 7.4 may be reformulated as $\mathcal{M}(\Omega) = A + I$. We begin the proof by proving the weaker statement that $\mathcal{M}(\Omega) = B + I$.

Lemma 7.5. *If $x \in \mathcal{M}(\Omega)$ has length N and L leaves, then $x \in B_{N,\leqslant L} + I$, where $B_{N,\leqslant L}$ is the submodule of $\mathcal{M}(\Omega)$ generated by ordered nests of length N and with at most L leaves.*

Proof. Suppose $x = r(a_0, x_1, \ldots, x_n)$ is of length N and has L leaves. We prove that $x \in B_{N,\leqslant L} + I$ by induction firstly over N and secondly over n, the number of branches in x. First we use the right-symmetric rule to re-order $x_1, \ldots, x_n$. The extra terms which are introduced in this process will have fewer branches and at most L leaves (and length N). Hence, by induction, we may assume that either $x_i \in \Omega$ for all i, and then x is an ordered nest, or $x_n \notin \Omega$. By induction $x_n \in B + I$ and we may assume that $x_n = r(b_0, b_1, \ldots, b_m, y)$ is an ordered nest, such that x has length N and has at most L leaves. Now we use the left-commutative axiom to get that x is equivalent modulo I to $r(b_0, b_1, \ldots, b_m, r(a_0, x_1, \ldots, x_{n-1}, y))$. By induction $r(a_0, x_1, \ldots, x_{n-1}, y) \in B_{N-m-1,\leqslant L-m} + I$ and hence we may assume that x is a nest of length N and with at most L leaves. Using the left-commutative axiom again, we easily get that $x \in B_{N,\leqslant L} + I$. ☐

Proof. Proof of Proposition 7.4 We will prove that any ordered nest x of length N, with L leaves and with $d_1(x) = d$ belongs to $A_{N,\leqslant(L,d)} + I$, where $A_{N,\leqslant(L,d)}$ is the submodule of $\mathcal{M}(\Omega)$ generated by all Novikov elements x of length N, with $< L$ leaves or with L leaves and with $d_1(x) \leqslant d$. We prove this by induction, firstly over N, secondly over L and thirdly over $P = d_1(x) + d_1(y)$, where $x = r(a_0, a_1, \ldots, a_n, y)$ and $a_i \in \Omega$ for $i = 0, \ldots, n$.

First suppose $x = r(a_0, a_1, \ldots, a_n)$ where $a_i \in \Omega$. We may re-order the elements $a_1, \ldots, a_n$ using the right-symmetric rule. The extra terms that appear will have $< L$ leaves. Hence by lemma 7.5 they belong to $B_{N,<L} + I$ and hence by induction they belong to $A_{N,<(L,d)} + I$.

Next suppose $x = r(a_0, \ldots, a_d, r(b_0, \ldots, b_m, y))$ is an ordered nest of length N, with L leaves and $d + m = P$. By induction $r(b_0, \ldots, b_m, y) \in A_{N',\leqslant(L',m)} + I$, where $N' + d + 1 = N$ and $L' + d = L$. By induction, we do not have to consider terms with $< L'$ leaves. Hence we may assume that $r(b_0, \ldots, b_m, y))$ is a Novikov element with L' leaves. Also, modulo terms with fewer leaves, we may assume that $a_1 \leqslant \ldots \leqslant a_d$.

Suppose $a_d > b_1$. The proof is finished if we can prove that a_d and b_1 may be interchanged. Indeed, the process may be repeated a finite number of steps, until the set of leaves $a_1, \ldots, a_d$ is minimal (of course there are only finitely many elements in Ω involved). This is in fact a fourth level of induction.

As before we may move b_1 to the right of b_m, but to pass y an extra term with L leaves will appear if $y \notin \Omega$, namely

$$r(a_0, \ldots, a_d, r(b_0, b_2 \ldots, b_m, r(y, b_1)))\,.$$

By induction $r(b_0, b_2 \ldots, b_m, r(y, b_1))$ may be replaced by a Novikov element with $d_1 \leqslant m - 1$ (modulo terms with fewer leaves). Hence we get an ordered nest with $d_1 \leqslant d$ and $d_1 + d_2 < P$ and hence by induction this term $\in A_{N,\leqslant(L,d)} + I$. We may now continue with the term

$$r(a_0, \ldots, a_d, r(b_0, b_2 \ldots, b_m, y, b_1))\,,$$

which modulo I is equal to $r(b_0, b_2 \ldots, b_m, y, r(a_0, \ldots, a_d, b_1))$. Now interchange a_d and b_1 (which is possible as before) and shift back to

$$r(a_0, \ldots, a_{d-1}, b_1, r(b_0, b_2 \ldots, b_m, y, a_d))\,.$$

Finally, by the same argument as above, a_d may be moved to the left of y. By induction $r(b_0, b_2 \ldots, b_m, a_d, y)$ may be replaced by a Novikov element with $d_1 \leqslant m$ and as was remarked above, we are done. $\square$

The Novikov elements may be coded in the following way. Associate to any root in a Novikov element a "weight", which is $n-1$, where n is the number of branches in the subexpression determined by the root. A Novikov element is given by an ordered sequence of roots with weights and an ordered sequence of leaves, such that the total weight of the roots is one less than the number of leaves. Giving the leaves the weight -1, we may hence consider a Novikov element as an ordered sequence of variables $a[i]$, where $a \in \Omega$ and $i = -1, 0, 1, 2, \ldots$, such that the total weight of the sequence is -1.

Example 7.6. The Novikov element $r(a, r(b, b, r(b, c, r(a, c, d, d))))$ is coded by the sequence $b[-1]c[-1]c[-1]d[-1]d[-1]a[0]b[1]b[1]a[2]$.

Hence, it is natural to consider the following R-algebra.

Definition 7.7. $\mathcal{NP}(\Omega)$= the (associative) commutative polynomial ring over R with the set of variables equal to $\{a[i];\ a \in \Omega,\ i \geqslant -1\}$. Let $\partial : \mathcal{NP}(\Omega) \to \mathcal{NP}(\Omega)$ be the R-derivation defined by $\partial(a[i]) = a[i+1]$ and let $\circ$ be the binary operation on $\mathcal{NP}(\Omega)$ defined by $p \circ q = \partial(p)q$.

The elements of degree -1 in $\mathcal{NP}(\Omega)$ is a free R-module with basis in one-to-one correspondence with the set of Novikov elements. Let $\mathcal{NP}_i(\Omega)$ denote the set of elements in $\mathcal{NP}(\Omega)$ of weight $i-1$. Then $p \circ q \in \mathcal{NP}_{i+j}(\Omega)$ if $p \in \mathcal{NP}_i(\Omega)$ and $q \in \mathcal{NP}_j(\Omega)$ and hence $(\mathcal{NP}(\Omega), \circ)$ is a graded R-algebra and $(\mathcal{NP}_0(\Omega), \circ)$ is a subalgebra. It is a general fact, mentioned in the introduction, that these algebras are Novikov.

Theorem 7.8. *The set of residue classes of Novikov elements is a basis for the free Novikov algebra $\mathcal{N}(\Omega)$ as an R-module. We have that $\mathcal{N}(\Omega)$ is isomorphic to $(\mathcal{NP}_0(\Omega), \circ)$, where $\mathcal{NP}_0(\Omega)$ consists of the elements of weight -1 in $\mathcal{NP}(\Omega)$. If Ω has k elements the generating function for the R-basis of $\mathcal{NP}(\Omega)$ consisting of monomials is*

$$y \prod_{j=-1}^{\infty} \frac{1}{(1-y^j z)^k},$$

where the coefficient of $y^m z^n$ is the number of elements in the basis of weight m and length n. The generating function for a basis of $\mathcal{N}(\Omega)$ is the formal power series in z obtained as the constant term of the series above considered as a series in y.

Proof. Since $(\mathcal{NP}_0(\Omega), \circ)$ is Novikov, we have a homomorphism $f : \mathcal{N}(\Omega) \to (\mathcal{NP}_0(\Omega), \circ)$, which extends the map $a \mapsto a[-1]$, $a \in \Omega$. Since $a[-1] \in \mathcal{NP}_0(\Omega)$, $a \in \Omega$, we have $\mathrm{im}(f) \subseteq \mathcal{NP}_0(\Omega)$. We claim that equality holds. Suppose $p = a_1[-1] \cdots a_n[-1] x_1 \cdots x_m$ is a monomial in $\mathcal{NP}_0(\Omega)$, where x_i has weight $\geqslant 0$. If $n = 1$ and $m = 0$, then $p = a[-1]$ and hence $p \in \mathrm{im}(f)$. We prove by induction firstly on n and secondly on m, that $p \in \mathrm{im}(f)$. If $x_1 = a[0]$, then $p = a[-1] \circ (a_1[-1] \cdots a_n[-1] x_2 \cdots x_m)$ and hence $p \in \mathrm{im}(f)$ by induction. If $x_1 = a[k]$, where $k > 0$, then $n > k$ and

$$p = (a_1[-1] \cdots a_k[-1] a[k-1]) \circ (a_{k+1}[-1] \cdots a_n[-1] x_2 \cdots x_m) - \sum_{i=1}^{k} a_1[-1] \cdots \widehat{a_i[-1]} \cdots a_n[-1] a_i[0] a[k-1] x_2 \cdots x_m,$$

and hence again, $p \in \mathrm{im}(f)$ by induction.

Now the injectivity follows in the same way as in the proof of Theorem 6.3. We may assume that Ω is finite. Let N be the number of Novikov elements of length n. By Proposition 7.4 their residue classes generate the set of elements of length n in $\mathcal{N}(\Omega)$. Also N is the number of monomials of length n and weight -1 in $\mathcal{NP}(\Omega)$. Hence, by the above, we get a surjective R-linear map $R^N \to R^N$ and we may apply Lemma 6.4. $\square$

The algebra $\mathcal{N}(\Omega)$ may also be described in terms of the tree algebra defined in section 3.

Proposition 7.9. *Let $T(\Omega)/\sim$ be defined by the equalities*

$$t(a, x_1, \ldots, x_n, t(b, y_1, \ldots, y_m, z)) = t(b, y_1, \ldots, y_m, t(a, x_1, \ldots, x_n, z)).$$

and let $\mathcal{TN}(\Omega)$ denote the free R-module on $T(\Omega)/\sim$. The operation $\circ$ on $\mathcal{T}(\Omega)$ induces an operation on $\mathcal{TN}(\Omega)$ which makes $\mathcal{TN}(\Omega)$ isomorphic to $\mathcal{N}(\Omega)$.

Proof. We just sketch the proof, since it consists of a number of lengthy but rather straight-forward computations. Firstly one proves that the operation $\circ$ on $\mathcal{T}(\Omega)$ induces an operation on $\mathcal{TN}(\Omega)$, secondly that this right-symmetric algebra is also left-commutative. Thirdly one proves that the set of classes of Novikov elements (defined in the same way as for the r-expressions) generate $\mathcal{TN}(\Omega)$. Then as before we get the isomorphism with $\mathcal{N}(\Omega)$. Observe that the proof of the last step is easier than the proof of Proposition 7.4, since the branches in t-expressions commute. □

8. Lie algebras

The free Lie algebra on a set Ω, denoted $\mathcal{L}(\Omega)$, is the quotient of $\mathcal{M}(\Omega)$ by the ideal generated by all expressions

$$xx \quad \text{and} \quad (xy)z - x(yz) - (xz)y.$$

The first identity implies that $xy = -yx$ in $\mathcal{L}(\Omega)$. In general a Lie algebra is an R-algebra, which satisfies the two identities above. A simple computation shows that any right-symmetric algebra is a Lie algebra under the product $[a, b] = ab - ba$. We will consider the Lie subalgebra generated by Ω in $\mathcal{RS}(\Omega)$ and we will prove that it is free. To do this we will define the set of Hall elements in $M(\Omega)$ as follows, given the total order on $M(\Omega)$ defined before Definition 6.1 .

Definition 8.1. An r-expression $r(a, x_1, \dots, x_n)$ is called a Hall element if $x_1 \leqslant \dots \leqslant x_n$, $a > x_1$ and $r(a, x_1, \dots, x_i) > x_{i+1}$ for $i = 1, \dots, n-1$ and $x_1, \dots, x_n$ are Hall elements. Also, an element in Ω is a Hall element.

It is well-known that the set of Hall elements represents a basis for the free Lie algebra but we will not use this. In fact, this result will be a consequence of the theorem below.

Proposition 8.2. *The residue classes of the set of Hall elements generate $\mathcal{L}(\Omega)$ as an R-module.*

Proof. Using the axiom $xx = 0$ in $\mathcal{L}(\Omega)$, it is enough to consider elements in $M(\Omega)$ of the form (xy), where $x > y$. We will prove by induction over $<$ that any element $p \in M(\Omega)$ of this form may be written as a linear combination of Hall elements $\leqslant p$ modulo the Lie identities. Suppose $p = (uv)$, where $u > v$. By induction we may assume that u, v are Hall elements. If $u \in \Omega$ then p is a Hall element. Suppose $u = (xy)$, where $x > y$. Then p is a Hall element if also $y \leqslant v$. If $y > v$ we may replace p by $(xv)y + x(yv)$. Since $x > y$, we have $|xv| > |y|$ and since $v < y$ we have $xv < xy$. Hence $(xv)y < (xy)v$ and by induction $(xv)y$ may be replaced by a linear combination of Hall elements $< (xy)v$.

We have $x(yv) < (xy)v$ and $(yv)x < (xy)v$ and hence, by induction, $x(yv)$ may also be replaced by a linear combination of Hall elements $< (xy)v$ modulo the Lie identities and we are done.

□

Theorem 8.3. *For any set Ω, the Lie subalgebra of $\mathcal{RS}(\Omega)$ generated by Ω is isomorphic to $\mathcal{L}(\Omega)$.*

Proof. We use Proposition 6.3 and consider $\mathcal{RS}(\Omega)$ as $(\mathcal{T}(\Omega), \circ)$. We have a Lie homomorphism $\varphi : \mathcal{L}(\Omega) \to \mathcal{T}(\Omega)$ whose image is the Lie subalgebra of $\mathcal{T}(\Omega)$ generated by Ω under the product $[x, y] = x \circ y - y \circ x$. We will now use the well-order $\prec_{revlex}$ (written $<$ in this proof) on $T(\Omega)$ defined in Definition 5.3. Consider the map $x \mapsto x^t$ from $M(\Omega)$ to $T(\Omega)$, where x^t is obtained from x by replacing all r by t. Then if x, y are Hall elements in $M(\Omega)$, we have $x < y \Leftrightarrow x^t < y^t$.

We prove the following two statements.

1. $lead([x,y]) = lead(x) \bullet lead(y), \quad \text{where} \quad x > y$
2. $lead(\varphi(x)) = x^t \quad \text{if } x \text{ is a Hall element}$

Since $<$ is $\circ$-leading, we have for $x, y \in T(\Omega)$ that $lead([x,y]) = \max(x \bullet y, y \bullet x)$ if $x \bullet y \neq y \bullet x$. Suppose

$$y < x \quad \text{and} \quad x = t(a, x_1, \dots, x_n), \; y = t(b, y_1, \dots, y_m), \quad \text{where} \quad x_1 \leqslant \dots \leqslant x_n \quad \text{and} \quad y_1 \leqslant \dots \leqslant y_m.$$

We have $y \bullet x = t(b, y_1, \dots, y_m, x)$ and hence by definition $y \bullet x < x \bullet y$ if $|y| < |x|$, since $|x| > \max(|x_n|, |y|)$. If $y < x$ and $|y| = |x|$, then $x \bullet y = t(a, x_1, \dots, x_n, y)$ and again by definition $y \bullet x < x \bullet y$. Hence we have proved that $lead([x,y]) = x \bullet y$ if $x, y \in T(\Omega)$ and $y < x$. By Proposition 5.2, we get

$$lead([x,y]) = lead(x) \bullet lead(y) \quad \text{if} \quad x, y \in \mathcal{T}(\Omega) \setminus \{0\} \quad \text{and} \quad y < x$$

and hence the first claim is proved. The second claim is proved by induction on the length of x, where $x = r(a, x_1, \dots, x_n)$. By the induction hypothesis we have

$$lead(\varphi(r(a, x_1, \dots, x_{n-1}))) = t(a, x_1^t, \dots, x_{n-1}^t)$$

and $lead(\varphi(x_n)) = x_n^t$. Since the map $x \to x^t$ is order-preserving on Hall elements, we have $x_n^t < t(a, x_1^t, \dots, x_{n-1}^t)$. Hence from above we get

$$\begin{aligned} lead(\varphi(x)) &= lead([\varphi(r(a, x_1, \dots, x_{n-1})), \varphi(x_n)]) \\ &= lead(\varphi(r(a, x_1, \dots, x_{n-1}))) \bullet lead(\varphi(x_n)) \\ &= t(a, x_1^t, \dots, x_{n-1}^t) \bullet x_n^t = t(a, x_1^t, \dots, x_n^t) = x^t. \end{aligned}$$

Now the injectivity of φ easily follows, since the set of all x^t, where x is a Hall element, is linearly independent by construction of $\mathcal{T}(\Omega)$. We may also use Lemma 6.4 and argue as in the proof of Proposition 6.3. □

9. A non-archimedian norm

Let Ω be a set. Consider the compatible order $\prec_{revlex}$ on $\mathcal{T}(\Omega)$. By Proposition 6.6 this transforms to a compatible order on $\mathcal{RS}(\Omega)$, which we will write as $<$. Let $RS(\Omega)$ be the corresponding basis of $\mathcal{RS}(\Omega)$; i.e., the set of right-symmetric r-elements $x = r(a, x_1, \dots, x_k) \in M(\Omega)$, such that $a \in \Omega$, $x_1 \leqslant x_2 \leqslant \dots \leqslant x_k$ and $x_1, \dots, x_k \in RS(\Omega)$. Furthermore, for $x \in RS(\Omega)$, let $|x|$ denote the length of x.

Definition 9.1. Define maps $m^+,\ l^+ : \mathcal{RS}(\Omega) \to \mathbf{N} \cup \{\infty\}$ by

- $m^+(0) = l^+(0) = \infty$
- $m^+(a) = 0,\ l^+(a) = 1,\ \text{if } a \in \Omega$
- $m^+(x) = |x_k|,\ l^+(x) = |x|,\ \text{if } x = r(a, x_1, \dots, x_k) \in RS(\Omega)$
- $m^+(x) = \min_{\lambda_{x'} \neq 0} m^+(x'),\ l^+(x) = \min_{\lambda_{x'} \neq 0} l^+(x')$,
  if $x = \sum_{x' \in RS(\Omega)} \lambda_{x'} x' \in \mathcal{RS}(\Omega)$

In terms of trees $m^+(t)$ is the number of vertices in the maximal branch of t, if $t \in T(\Omega)$.

Proposition 9.2. *For any $x, y \in \mathcal{RS}(\Omega)$,*

$$\begin{aligned} m^+(x+y) &\geqslant \min\{m^+(x), m^+(y)\} \\ l^+(x+y) &\geqslant \min\{l^+(x), l^+(y)\} \\ m^+(\lambda x) &\geqslant m^+(x) \\ l^+(\lambda x) &\geqslant l^+(x) \\ l^+(x) &\geqslant m^+(x) \end{aligned}$$

Proof. Evident. □

Let $\eta_i \in \mathcal{RS}(\Omega)$, $i = 1, 2, \ldots$. Any element of $\mathcal{RS}(\Omega)$ can be viewed as a right-symmetric polynomial on variables $a_1, a_2, \ldots$. For $f(a_1, a_2, \ldots) \in \mathcal{RS}(\Omega)$ denote by $\eta f \in \mathcal{RS}(\Omega)$ the polynomial obtained from f by the substitution $a_i \mapsto \eta_i$.

Proposition 9.3. *Any endomorphism η of $\mathcal{RS}(\Omega)$ (i.e., a homomorhism of $\mathcal{RS}(\Omega)$ to itself as an algebra) is uniquely defined by a substitution $a \mapsto \eta(a) \in \mathcal{RS}(\Omega)$, $a \in \Omega$.*

Proof. Since $\mathcal{RS}(\Omega)$ is free right-symmetric, any map $\Omega \to \mathcal{RS}(\Omega)$ can be prolongated in a unique way to a homomorphism of right-symmetric algebras $\mathcal{RS}(\Omega) \to \mathcal{RS}(\Omega)$. Inversely, any endomorphsim $\eta : \mathcal{RS}(\Omega) \to \mathcal{RS}(\Omega)$ is induced by its restriction $\Omega \to \mathcal{RS}(\Omega)$. □

Definition 9.4. An ideal in a free algebra (such as $\mathcal{M}(\Omega)$ or $\mathcal{RS}(\Omega)$) is called a T-ideal if it is invariant under all endomorphisms of the algebra.

Example 9.5. The ideal in $\mathcal{M}(\Omega)$ which defines $\mathcal{RS}(\Omega)$) is a T-ideal. More generally, if $f_1, \ldots f_k \in \mathcal{M}(\Omega)$, then the free algebra on Ω with axioms $f_1 = 0, \ldots, f_k = 0$ is equal to $\mathcal{M}(\Omega)$ modulo the T-ideal generated by $f_1, \ldots f_k$. If A is any right-symmetric algebra, then $\{f \in \mathcal{RS}(\Omega);\ f = 0 \text{ is an identity on } A\}$ is a T-ideal in $\mathcal{RS}(\Omega)$.

It follows from Proposition 9.2 that $\{x \in \mathcal{RS}(\Omega);\ m^+(x) \geqslant q\}$ and $\{x \in \mathcal{RS}(\Omega);\ l^+(x) \geqslant q\}$ are ideals in $\mathcal{RS}(\Omega)$. We will now prove that they are also T-ideals.

Lemma 9.6. *For any $x, y, x_1, \ldots x_k \in RS(\Omega)$ and $a \in \Omega$, we have*

$$(i) \qquad m^+(r(a, x_1, \ldots, x_k)) = \max_{1 \leqslant i \leqslant k} |x_i|$$

$$(ii) \qquad m^+(x * y) = \max(m^+(x), |y|)$$

For any $x, y \in \mathcal{RS}(\Omega)$ and for any endomorphism η of $\mathcal{RS}(\Omega)$, we have

$$(iii) \qquad m^+(x * y) \geqslant \max(m^+(x), l^+(y))$$

$$(iv) \qquad l^+(\eta(x)) \geqslant l^+(x)$$

Proof. By Proposition 5.4 and 6.6 we have

$$r(a, x_1, \ldots, x_k) = r(a, x_{i_1}, \ldots, x_{i_k}) + \sum y_i$$

where $x_{i_1} \leqslant \ldots \leqslant x_{i_k}$, $y_i \prec_{revlex} r(a, x_{i_1}, \ldots, x_{i_k})$ and $|y_i| = |r(a, x_1, \ldots, x_k)|$. But, by definition of $\prec_{revlex}$, we have that $m^+(y) \geqslant m^+(x)$ if $|x| = |y|$ and $y \prec_{revlex} x$. Hence,

$$m^+(r(a, x_1, \ldots, x_k)) = m^+(r(a, x_{i_1}, \ldots, x_{i_k})) = |x_{i_k}| = \max_{1 \leqslant i \leqslant k} |x_i|\,,$$

which proves (i). Now (ii) follows directly, since

$$r(a, x_1, \ldots, x_k) * y = r(a, x_1, \ldots, x_k, y)\,.$$

To prove (iii), suppose $x = \sum \lambda_i x_i$ and $y = \sum \lambda'_j y_j$, where $\lambda_i, \lambda'_j \neq 0$ for all i, j. Then

$$\begin{aligned} m^+(x * y) &= \min_{i,j}(m^+(x_i * y_j)) \geqslant \min_{i,j}(\max(m^+(x_i), l^+(y_j))) \\ &\geqslant \max(\min_i(m^+(x_i)), \min_j(l^+(y_j))) = \max(m^+(x), l^+(y))\,. \end{aligned}$$

By Proposition 9.2, it is enough to prove (iv) when $x \in RS(\Omega)$. Consider x as a polynomial, $x = f(a_0, a_1, \ldots)$, where $a_i \in \Omega$. Then $\eta(x)$ is a linear combination of terms $y = f(x_0, x_1, \ldots)$, where $x_i \in RS(\Omega)$, and hence $|y| \geqslant |x|$ for all nonzero terms y. Hence

$$l^+(\eta(x)) \geqslant |x| = l^+(x)\,.$$

□

Theorem 9.7. *For any $x, y \in \mathcal{RS}(\Omega)$ and for any endomorphism η of $\mathcal{RS}(\Omega)$, we have*

$$\begin{aligned} &(i) && m^+(x * y) \geqslant \max(m^+(x), m^+(y)) \\ &(ii) && m^+(\eta(x)) \geqslant m^+(x)\,. \end{aligned}$$

Proof. The first claim follows from Lemma 9.6 (iii) and Proposition 9.2. It is enough to prove the second claim when $x \in RS(\Omega)$. We prove the inequality by induction over the length of x. If $x \in \Omega$, we have $m^+(\eta(x)) \geqslant 0 = m^+(x)$. Suppose $x, y \in RS(\Omega)$ and that the claim is true for x. We have

$$\begin{aligned} m^+(\eta(x * y)) &= m^+(\eta(x) * \eta(y)) \geqslant \max(m^+(\eta(x)), l^+(\eta(y))) \\ &\geqslant \max(m^+(x), l^+(y)) = \max(m^+(x), |y|) = m^+(x * y)\,, \end{aligned}$$

where the first inequality follows from Lemma 9.6 (iii), the second inequality follows by induction and Lemma 9.6 (iv) and the last equality follows from Lemma 9.6 (ii). □

Corollary 9.8. *For any $q \in \mathbf{N}$, the set $\mathcal{RS}_q(\Omega) = \{x \in \mathcal{RS}(\Omega);\ m^+(x) \geqslant q\}$ and the set $\{x \in \mathcal{RS}(\Omega);\ l^+(x) \geqslant q\}$ are T-ideals in $\mathcal{RS}(\Omega)$.*

Proof. This follows from Proosition 9.2 and Theorem 9.7. □

So, the system of T-ideals $\{\mathcal{RS}_q(\Omega)\}$, $q \geqslant 0$, endows $\mathcal{RS}(\Omega)$ with a filtration,

$$\mathcal{RS}(\Omega) = \mathcal{RS}_0(\Omega) \supset \mathcal{RS}_1(\Omega) \supset \mathcal{RS}_2(\Omega) \supset \cdots$$

$$\mathcal{RS}_p(\Omega) * \mathcal{RS}_q(\Omega) \subseteq \mathcal{RS}_{max(p,q)}(\Omega)\,,$$

which defines a topology on $\mathcal{RS}(\Omega)$, such that $*$ is continuous. As usual this topology is metrizable with a non-archimedean metric, which may be defined in the following way.

Definition 9.9. Let $\| \, \| : \mathcal{RS}(\Omega) \to \mathbf{Q}$ be the map defined by

- $\|0\| = 0$
- $\|x\| = 1/(m^+(x) + 1)$, if $x \neq 0$

The map $\| \, \| : \mathcal{RS}(\Omega) \to \mathbf{Q}$ has the following properties.

Theorem 9.10. *For any $x, y \in \mathcal{RS}(\Omega)$, $\lambda \in k$ and any endomorphism η of $\mathcal{RS}(\Omega)$, we have*

$$\begin{aligned}
&0 \leqslant \|x\| \leqslant 1 \text{ and } \|x\| = 0 \Leftrightarrow x = 0 \\
&\|\lambda x\| = \|x\|, \text{ if } \lambda \neq 0 \\
&\|x + y\| \leqslant max(\|x\|, \|y\|) \\
&\|x * y\| \leqslant min(\|x\|, \|y\|) \\
&\|\eta(x)\| \leqslant \|x\|.
\end{aligned}$$

Proof. It follows from Theorem 9.7 and Proposition 9.2. □

Let $B_q(\Omega) = \{x \in \mathcal{RS}(\Omega) : \|x\| \leqslant q\}$ be the ball of radius q in $\mathcal{RS}(\Omega)$. We have

$$B_p(\Omega) * B_q(\Omega) \subseteq B_{\min(p,q)}$$

and from Theorem 9.10 we obtain the following statement.

Corollary 9.11. *For any $q \leqslant 1$, the ball B_q is a T-ideal of $\mathcal{RS}(\Omega)$ and hence, for any T-ideal J generated by elements $s_1, \ldots, s_k \in B_q$ we have $J \subseteq B_q$.*

10. On the T-ideal generated by right-bracketed polynomials

Let f and g be right-symmetric polynomials in $\mathcal{RS}(\Omega)$. We say that the identity $g = 0$ follows from the identity $f = 0$, and write

$$f = 0 \Rightarrow g = 0,$$

if any right-symmetric algebra A that satisfies the identity $f = 0$ also satisfies the identity $g = 0$. The condition $f = 0 \Rightarrow g = 0$ is equivalent to the condition that the T-ideal generated by f contains g. In other words, the element g of the free algebra $\mathcal{RS}(\Omega)$ can be obtained from $f(a_1, \ldots, a_k)$ using the following operations:

- substitution of a_i by any element in the free algebra $\mathcal{RS}(\Omega)$
- multiplying f with any element of $\mathcal{RS}(\Omega)$ from right or left side (or both)
- taking linear combinations of obtained elements.

Details about general facts on polynomial identities, see for example, [**14**].

Let s^l_{q+1} be the left standard polynomial of degree $q+1$ in $\mathcal{RS}(\Omega)$. It is a skew-symmetric polynomial in the last q arguments and defined by

$$s^l_{q+1}(a_0, a_1, \dots, a_q) = \sum_{\sigma \in Sym_q} \operatorname{sign}(\sigma) r(a_{\sigma(1)}, r(a_{\sigma(2)}, \dots, r(a_{\sigma(q)}, a_0) \cdots).$$

Let s^r_q be the right standard polynomial of degree q in $\mathcal{RS}(\Omega)$. It is a skew-symmetric polynomial and defined by

$$s^r_q(a_1, \dots, a_q) = \sum_{\sigma \in Sym_q} \operatorname{sign}(\sigma) r(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(q)}).$$

Lemma 10.1. *For any $q > 0$, we have the following identities,*

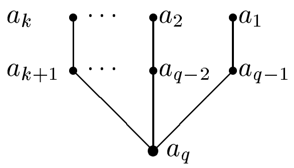
$$s^r_q(a_1, \dots, a_q) = \sum_i (-1)^{i+q} s^r_q(a_1, \dots, \hat{a}_i, \dots, a_q) * a_i$$

$$s^r_{q+2}(a_1, \dots, a_{q+2}) = \sum_{i<j} (-1)^{i+j+1} s^r_q(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{q+2}) * s^r_2(a_i, a_j)$$

Proof. The first equality is obvious and the second follows easily from the right-symmetric identity. □

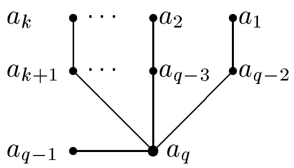
Let $L_q \in \mathcal{RS}(\Omega)$ be defined by the following pictures of the corresponding elements in $T(\Omega)$.

$L_q =$



if $q = 2k+1$ is odd, where $k \geqslant 0$, and

$-L_q =$



if $q = 2(k+1)$ is even, where $k \geqslant 0$. E.g., $L_6 = -r(a_6, a_5, r(a_3, a_2), r(a_4, a_1))$. For $n \in \mathbf{Z}$, let $\lfloor n/2 \rfloor$ denote the greatest integer no more than $n/2$. Consider, as in section 9, the compatible order $\prec_{revlex}$ on $\mathcal{RS}(\Omega)$, which we will write as $<$.

Lemma 10.2. *If $a_1, \dots, a_q \in \Omega$ and $a_1 \leqslant \dots \leqslant a_q$, then the leading term of $s^r_q(a_1, \dots, a_q)$ is equal to $(-1)^k k! L_q$, where $k = \lfloor (q-1)/2 \rfloor$.*

Proof. We use induction on k. The statement is trivial for $k = 0$ (i.e., for $q = 1$ and $q = 2$).

Suppose $q \geqslant 3$ and that our statement is true for $k - 1$. Then, by lemma 10.1,

$$X = s^r_q(a_1, \ldots, a_q) = \sum_{i<j} X_{i,j},$$

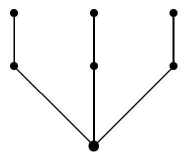
where

$$X_{i,j} = (-1)^{i+j+1} s^r_q(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_q) * (a_i * a_j - a_j * a_i).$$

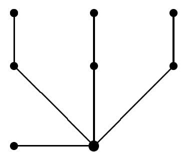
To obtain the leading term of X, we should find the maximum of the leading term of $X_{i,j}$ and add these maxima for $i < j$. If the result is non-zero, this gives the leading term of X. Since the order is compatible and $lead(x \circ y) = x \bullet y$, the leading term of $X_{i,j}$ is equal to the $\bullet$-product of the leading term of

$$(-1)^{i+j+1} s^r_q(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_q)$$

and $-r(a_j, a_i)$. Therefore, by the induction hypothesis, the leading term of $X_{i,j}$ as an unlabeled tree (up to a scalar multiple) has the following form



if $q = 2k + 1$, and



if $q = 2(k + 1)$.

Note that L_q is the maximal element among all normal form elements, such that the underlying unlabeled tree has the form above. Thus, the leading term of $X_{i,j}$ is L_q up to a scalar multiple, if and only if $i + j = q$, in case of $q = 2k + 1$ and $i + j = q - 1$, in case of $q = 2(k + 1)$. Also, the scalar multiple is $(-1)^q(-1)^{k-1}(k - 1)! = (-1)^k(k-1)!$, in case of $q = 2k+1$ and $(-1)^{q-1}(-1)^{k-1}(k-1)! = (-1)^k(k-1)!$, in case of $q = 2(k + 1)$. There are exactly k such pairs (i, j), and hence we get that the sum of these terms is $(-1)^k k! L_q$ (in both cases). Since this is non-zero, it must be the leading term of X, which completes the induction step. $\square$

Theorem 10.3. *i) The T-ideal of the free right-symmetric algebra $\mathcal{RS}(\Omega)$ generated by s^l_3 contains s^r_q for all $q \geqslant 3$.*

ii) The T-ideal of the free right-symmetric algebra $\mathcal{RS}(\Omega)$ generated by all the polynomials s^l_k for $k > 3$ does not contain s^r_q for any q.

Corollary 10.4. *The polynomial s^l_{2n+1}, where $n > 1$, generates a T-ideal which does not contain s^r_q for any q. Hence $s^l_{2n+1} = 0 \not\Rightarrow s^r_q = 0$ for any q.*

Corollary 10.5. *The Witt algebra W_n^{rsym} has at least two independent polynomial identities when $n > 1$.*

Proof. Proof of Theorem 10.3.

i) Let J be T-ideal generated by s_3^l. Then by the right-symmetric identity the computation below proves that $s_3^r \in J$.

$$\begin{aligned} s_3^r(a_1,a_2,a_3) &= (a_1 * a_2) * a_3 - (a_1 * a_3) * a_2 + (a_2 * a_3) * a_1 \\ &\quad - (a_2 * a_1) * a_3 + (a_3 * a_1) * a_2 - (a_3 * a_2) * a_1 \\ &= a_1 * (a_2 * a_3) - a_1 * (a_3 * a_2) + a_2 * (a_3 * a_1) \\ &\quad -a_2 * (a_1 * a_3) + a_3 * (a_1 * a_2) - a_3 * (a_2 * a_1) \\ &= a_1 * (a_2 * a_3) - a_2 * (a_1 * a_3) - (a_1 * (a_3 * a_2) \\ &\quad -a_3 * (a_1 * a_2)) + (a_2 * (a_3 * a_1) - a_3 * (a_2 * a_1)) \in J. \end{aligned}$$

By lemma 10.1, we have $s_3^r \in J \Rightarrow s_q^r \in J$, for any $q > 3$.

ii) As an element of $\mathcal{RS}(\Omega)$, $s_k^l(a_1,\dots,a_k)$ is a sum of r-elements of the form $r(b_1, r(b_2, \dots, r(b_{k-1}, b_k)\cdots))$, where $b_i \in \Omega$. Therefore,

$$\|s_k^l(a_1,\dots,a_k)\| = 1/k \leqslant 1/4,$$

if $k > 3$. By corollary 9.11, the T-ideal generated by all s_k^l for $k > 3$ lies in the ball of radius $1/4$:

$$J \subseteq B_{1/4}.$$

According to lemma 10.2, the leading term of $s_q^r(a_1,\dots,a_q)$ has a maximal branch with at most 2 vertices. Therefore,

$$||s_q^r(a_1,\dots,a_q)|| \geqslant 1/3.$$

Hence,

$$s_q^r(a_1,\dots,a_q) \notin B_{1/4}$$

and we get,

$$s_q^r(a_1,\dots,a_q) \notin J$$

for any q.

□

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