Homology, Homotopy and Applications, vol.4(2), 2002, pp.219–225

GROWTH AND LIE BRACKETS IN THE HOMOTOPY LIE ALGEBRA

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(*communicated by Clas Löfwall*)

Abstract

Let L be an infinite dimensional graded Lie algebra that is either the homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ for a finite n-dimensional CW complex X, or else the homotopy Lie algebra for a local noetherian commutative ring R ($UL = Ext_R(I\!k, I\!k)$) in which case put $n =$ (embdim $-$ depth)(R).

Theorem: (i) The integers $\lambda_k = \sum_{q=k}^{k+n-2} \dim L_i$ grow faster than any polynomial in k.

(ii) For some finite sequence $x_1, \dots, x_d$ of elements in L and some N, any $y \in L_{\geqslant N}$ satisfies: some $[x_i, y] \neq 0$.

To Jan–Erik Roos on his sixty–fifth birthday

1. Introduction

Let X be a simply connected CW complex of finite type. Then [**16**] its loop space homology, $H_*(\Omega X;\mathbb{Q})$ is the universal enveloping algebra of the graded Lie algebra $L_X = \{(L_X)_i\}_{i\geqslant 1} = \pi_*(\Omega X)\otimes\mathbb{Q}$, equipped with the Samelson product. Similarly, if R is a commutative local noetherian ring with residue field $I\!k$ then [**1**], [**17**] $Ext_R(I\!k, I\!k)$ is the universal enveloping algebra of a graded Lie algebra $L_R = \{L_R^i\}_{i\geqslant 1}$. We call L_X (L_R) the *homotopy Lie algebra* of X (of R) and call $e_i(X) = \dim (L_X)_i$ (or $e_i(R) = \dim L_R^i$) the i^{th} *deviation* of X (or of R).

For finite complexes X and for all local rings R the hypothesis $\dim L < \infty$ imposes very special conditions (in this case X is called $\mathbb{Q}$-elliptic). For example, if X is $\mathbb{Q}$-elliptic the $H_*(\Omega X;\mathbb{Q})$ is a Poincaré duality algebra [**12**] while if L_R is finite dimensional then R is a complete intersection [**10**], [**11**]. Moreover, it is known (again for finite complexes X and for any R) that

- If $\dim L_X < \infty$ and $\dim L_R < \infty$ then
$$e_i(X) = 0,\ i \geqslant 2\dim X,\ [\mathbf{9}] \quad \text{and} \quad e_i(R) = 0,\ i \geqslant 3,\ [\mathbf{10}], [\mathbf{11}].$$
- If $\dim L_X = \infty$ and $\dim L_R = \infty$ then for some $K > 0$, $C > 1$,

Received January 8, 2001, revised September 17, 2001; published on July 12, 2002.
2000 Mathematics Subject Classification: 55P62, 55P35, 17B70, 16L99.
Key words and phrases: Finite CW complex - Local ring - Homotopy Lie algebra - Depth.

$$\sum_{i=1}^{k} e_i(X) \geqslant KC^k,\ k \geqslant \dim X - 1,\ [\mathbf{3}] \quad \text{and} \quad \sum_{i=1}^{k} e_i(R) \geqslant KC^k,\ k \geqslant 1,\ [\mathbf{2}].$$

- If $\dim L_X = \infty$ and $\dim L_R = \infty$ then
$$\sum_{i=k}^{k+\dim X-2} e_i(X) > 0, \text{ all } k \geqslant 1, [\mathbf{14}] \quad \text{and} \quad e_k(R) > 0, \text{ all } k \geqslant 1, [\mathbf{13}].$$
- If $\dim L = \infty$ ($L = L_X$ or L_R) then for all $x \in L$ of sufficiently large even degree there is some $y = y(x) \in L$ such that $(adx)^k y \neq 0$, $k \geqslant 1$, [**7**].

These results motivate/provide evidence for the two following main conjectures, due to some combination of Avramov - Félix - Halperin - Thomas.

Conjecture 1. If X is finite dimensional, not $\mathbb{Q}$-elliptic, and if R is not a complete intersection then for some $K > 0$, $C > 1$:

$$\textstyle\sum_{i=k}^{k+\dim X-2} e_i(X) \geqslant KC^k, \quad k \geqslant 1, \quad \text{and} \quad e_k(R) \geqslant KC^k, \quad k \geqslant 1.$$

Conjecture 2. If X is finite dimensional, not $\mathbb{Q}$-elliptic, and if R is not a complete intersection then L_X and L_R each contain a free Lie subalgebra on two generators.

This paper makes some progress towards these conjectures. For simplicity we adopt the following notation:

- X is a finite, non $\mathbb{Q}$-elliptic, simply connected CW complex and R is a local noetherian commutative ring that is not a complete intersection.
- L is either L_X or L_R, and L_{even} is the sub Lie algebra of elements of even degree.
- $n = n_X = \dim X$ or $n = n_R = (\operatorname{emb}\dim - \operatorname{depth})(R)$.
- $e_i = e_i(X)$, or $e_i = e_i(R)$.
- $h = h_X = \dim H^*(X;\mathbb{Q})$, or $h = h_R = \dim H_*(K^R)$, K^R denoting the Koszul complex of R.

Then, with the hypotheses and notation above, we establish

Theorem A.

(i) The integers $\lambda_k = \sum_{i=k}^{k+n-2} e_k$ *grow faster than any polynomial in* k. *In particular,*

$$\lambda_k \to \infty \text{ as } k \to \infty.$$

(ii) Moreover, if L_{even} *contains a maximal abelian sub Lie algebra of finite dimension then for some* $K > 0$, $C > 1$,

$$\lambda_k \geqslant KC^k, \quad k \geqslant 1.$$

Theorem B. *There is a finite sequence* $x_1, \ldots, x_d$ *of elements in* L *and an integer* N *such that:*

$$y \in L, \quad \deg y \geqslant N \Rightarrow \text{ some } [x_i, y] \neq 0\,.$$

2. General remarks.

With X is associated the commutative graded differential algebra $A_{PL}(X)$ whose Sullivan minimal model $(\Lambda V, d)$ satisfies [**18**], [**8**]

$$H(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \quad \text{and} \quad e_i(X) = \dim V^{i+1},\ i \geqslant 1.$$

In particular, $H^i(\Lambda V, d) = 0$, $i > n$. Moreover, [**18**], [**8**] the differential $d = \sum_{i\geqslant 2} d_i$, with $d_i : V \to \Lambda^i V$. Finally, $(\Lambda V, d_2) = C^*(L_X)$ where for any graded Lie algebra E over $I\!k$, $C^*(E)$ is the Cartan-Chevalley-Eilenberg-Quillen complex, whose cohomology is $Ext_{UE}(I\!k, I\!k)$.

Similarly, with R is associated its Koszul complex K^R which is connected by quasi-isomorphisms to a commutative graded chain algebra [**2**]. This in turn has a 'Sullivan model' $(\Lambda V, d)$ in which $V = \{V_i\}_{i\geqslant 1}$ and d decreases degrees by 1. Here we have $H_i(\Lambda V, d) = H_i(K^R) = 0$, $i > n$, and

$$e_i(R) = \dim V_{i-1}, \quad i \geqslant 2\,.$$

Moreover $(\Lambda V, d_2) = C^*(L_R^{\geqslant 2})$.

Recall now that the *depth* of an augmented graded algebra A is the least m (or ∞) such that $Ext_A^m(I\!k, A) \neq 0$. We define the *depth of a graded Lie algebra*, E, to be the depth of its universal enveloping algebra (depth E = depth UE) and recall from [**4**] that

$$\text{depth } L_X \leqslant \text{LScat}(X) \leqslant n_X \quad \text{and depth } L_R \leqslant n_R\,. \tag{2.1}$$

We shall make frequent use of the remark [**4**] that if I is an ideal in a graded Lie algebra E then

$$\text{depth } I \leqslant \text{depth } E\,. \tag{2.2}$$

Finally, since in both cases we have $\dim H(\Lambda V, d) = h < \infty$, we can apply a result of Lambrechts:

Lemma 2.3 [**15**]. *For all k sufficiently large, there is some $l \in [k+1, k+n-1]$ such that*
$dim V^l \geqslant dim V^k / hn$.

In fact Lambrechts shows that $\dim V^k \leqslant h \sum_{i=1}^{n-1} \dim V^{k+i} + \dim G_k$, where $G_* \subset L$ is the abelian ideal of Gottlieb elements. As noted in [**4**] this implies that G_* is finite dimensional, and so the inequality of Lemma 2.3 holds for large k.

3. Proof of Theorem A.

(i) We prove this in the case that L_{even} contains an infinite dimensional abelian sub Lie algebra, E, since otherwise (i) will follow from (ii). For convenience, we abuse notation and write the degrees as subscripts.

Note that the sub Lie algebra $F = E_{\leqslant k} \oplus L_{>k}$ has finite codimension in L. Thus we can write $F = I^m \subset I^{m-1} \subset \cdots \subset I^0 = L$ where each I^k is constructed from I^{k+1} by adding a single element of maximal degree. It follows that each I^k is a sub Lie algebra containing I^{k+1} as an ideal. In particular by (2.1) and (2.2).

$$\text{depth } F \leqslant \text{ depth } L \leqslant n\,.$$

On the other hand $F/L_{>k}$ is the abelian Lie algebra $E_{\leqslant k}$ and $UE_{\leqslant k} = I\!k[E_{\leqslant k}]$ is a polynomial algebra. In particular, depth $E_{\leqslant k} = \dim E_{\leqslant k}$, and there are constants $0 < c < C$ such that for any finitely generated $UE_{\leqslant k}$-module M, and some integer $r(M)$,

$$ck^{r(M)} \leqslant \sum_{i \leqslant k} \dim\, M_i \leqslant Ck^{r(M)}, \quad k \text{ sufficiently large.}$$

The integer $r(M)$ is called the *polynomial growth* of M.

Now ([**6**];Theorem 4.1) asserts that for some $q \leqslant n$ and some $\alpha \in Tor_q^{UL_{>k}}(I\!k, I\!k)$ the module $UE_{\leqslant k} \cdot \alpha$ has polynomial growth at least equal to $(\dim E_{\leqslant k}) - n$. But the action of $UE_{\leqslant k}$ in $Tor^{UL_{>k}}(I\!k, I\!k)$ is induced from the adjoint representation of $E_{\leqslant k}$ in the complex $(\Lambda^C(sL_{>k}), \partial)$ dual to $C^*(L_{>k})$; here Λ^C denotes the free co-commutative coalgebra. In particular for some $z \in (\Lambda^C)^q sL_{>k}$, $UE_{\leqslant k} \cdot z$ has polynomial growth at least equal to $(\dim E_{\leqslant k}) - n$.

Since $q \leqslant n$ this implies in turn that for some $y \in L_{>k}$,

$$\text{poly growth } (UE_{\leqslant k} \cdot y) \geqslant \frac{\dim E_{\leqslant k}}{n} - 1\,.$$

Fix some $r > 0$ and choose k so that $\dim E_{\leqslant k} \geqslant (n+1)r$. Then poly growth $(UE_{\leqslant k} \cdot y) \geqslant r$. It follows that there are r elements $x_1, \ldots, x_r \in E_{\leqslant k}$ such that $I\!k[x_1, \ldots, x_r] \xrightarrow{\cong} I\!k[x_1, \ldots, x_r] \cdot y$. Choosing d_i so that the $x_i^{d_i}$ all have the same degree d we see that

$$\dim L_{kd+\deg y} \geqslant \lambda k^r \geqslant \mu((k+1)d + \deg y)^r\,,\ k \geqslant 2\,, \tag{3.1}$$

for some positive constants λ and μ. Now, for k sufficiently large, repeated applications of Lemma 2.3 give an infinite sequence of integers $i_1 < i_2 < ...$ such that $i_1 = kd + \deg y$, and

$$i_{s+1} \leqslant i_s + n - 1 \text{ and } \dim L_{i_s} \geqslant \frac{\mu((k+1)d + \deg y)^r}{(nh)^s}\,, \quad s \geqslant 1\,.$$

It follows at once that (provided k is sufficiently large)

$$\sum_{j=q}^{q+n-2} \dim L_j \geqslant \frac{\mu}{(nh)^d} q^r\,, \quad \deg y + kd \leqslant q \leqslant \deg y + (k+1)d\,.$$

Since both sides of the equation are independent of k this establishes (i) in the presence of an infinite dimensional abelian subalgebra.

(ii) Let $E = \oplus_{i=1}^r I\!k x_i$ be a maximal abelian sub Lie algebra of L_{even}. Give L_{even} the decreasing filtration defined by $F^0 = L_{even}$, and $F^i = \{y \in L_{even} \,|\, [x_j, y] = 0\,, \quad 1 \leqslant j \leqslant i\}$. The maximality of E implies that $F^r = 0$. Choose graded subspaces $V^i \subset$

L_{even} such that $F^{i-1} = V^i \oplus F^i$, and choose integers $d_1, ..., d_r$ so that $d_1 \deg x_1 = ... = d_r \deg x_r = d$. Then for all q and all k

$$(adx_1)^{qd_1} \oplus ... \oplus (adx_r)^{qd_r} : V^1_{2k} \oplus ... \oplus V^r_{2k} \longrightarrow L_{2k+qd}$$

is injective ; i.e. $\dim L_{2k} \leqslant \dim L_{2k+qd}$, $k \geqslant 0$, $q \geqslant 0$.

On the other hand, a simple extension of the argument in ([**8**]; Chapter 33) gives an infinite sequence of even integers $i_1 < i_2 < ...$ such that $i_{s+1} \leqslant n^2 i_s$, $s \geqslant 1$, and constants $a > 0$, $D > 1$ such that $\dim L_{i_s} \geqslant aD^{i_s}$, $s \geqslant 1$. Now application of Lambrecht's lemma 2.3 gives (ii) in the same way it completed the proof of (i).

$\square$

4. Proof of Theorem B.

As recalled in §2, L has finite depth. This means that $Ext_{UL}(I\!k, UL) \neq 0$, and in [**5**] it is shown that for some finitely generated sub Hopf algebra G the restriction $Ext_{UL}(I\!k, UL) \to Ext_G(I\!k, UL)$ is non zero. Suppose G is generated in degrees less than or equal to n, and denote $E = L_{\leqslant n}$. Then the restriction $Ext_{UL}(I\!k, UL) \to Ext_G(I\!k, UL)$ factors through $Ext_{UE}(I\!k, UL)$, and so the restriction $Ext_{UL}(I\!k, UL) \to Ext_{UE}(I\!k, UL)$ is non zero. In particular, E has finite depth. The adjoint action of E in L defines a representation of UE in L, and Theorem B is a corollary of

Theorem C. *For some N and all $y \in L_{\geqslant N}$ the graded vector space $UE \cdot y$ grows faster than any polynomial.*

Proof. Let $Z \subset L$ be the subspace of elements z such that $UE \cdot z$ grows at most polynomially (i.e. for some constant $c > 0$ and some r, $\dim [UE \cdot z]_k \leqslant ck^r$, $k \geqslant 1$. Since $UE \cdot [z, w] \subset [UE \cdot z, UE \cdot w]$ it follows that Z is a sub Lie algebra of L, stable under the adjoint representation of E.

In particular, if $x \in Z \cap E$ then $UE.x$ is an ideal in E of at most polynomial growth. Since depth $UE.x \leqslant$ depth $E < \infty$ (by 2.2) if follows from ([**6**]; Theorem B) that $UE.x$ is finite dimensional . Thus $Z \cap E$ is an ideal in E that is the union of finite dimensional ideals. Since $L = L_{\geqslant 1}$ these finite dimensional ideals are solvable and their sum $Z \cap E$ is then itself finite dimensional [**4**].

Thus $Z_{\geqslant q} \cap E = 0$ (some q) and $E \oplus Z_{\geqslant q}$ is itself a sub Lie algebra of L. Moreover the composite

$$Ext_{UL}(I\!k, UL) \to Ext_{U(E \oplus Z_{\geqslant q})}(I\!k, UL) \to Ext_{UE}(I\!k, UL)$$

is non-zero. But $Ext_{U(E \oplus Z_{\geqslant q})}(I\!k, UL)$ is the cohomology of the complex $(\wedge(sE)^* \otimes \wedge(sZ_{\geqslant q})^* \otimes UL, d)$, and a simple 'filtration argument' shows that the restriction to $(\wedge(sE)^* \otimes UL, d)$ is zero in cohomology unless for some $a \in UL$, $1 \otimes a$ is a cocycle in the quotient complex $(\wedge(sZ_{\geqslant q})^* \otimes UL, d)$. This can only occur when $Z_{\geqslant q}$ is finite dimensional and concentrated in odd degrees [**4**].

Thus Z itself is finite dimensional and it suffices to choose N so that Z is concentrated in degrees $< N$. $\square$

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