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WEAK (CO)FIBRATIONS IN CATEGORIES OF (CO)FIBRANT OBJECTS

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Abstract

We introduce a fibre homotopy relation for maps in a category of cofibrant objects equipped with a choice of cylinder objects. Weak fibrations are defined to be those morphisms having the weak right lifting property with respect to weak equivalences. We prove a version of Dold's fibre homotopy equivalence theorem and give a number of examples of weak fibrations. If the category of cofibrant objects comes from a model category, we compare fibrations and weak fibrations, and we compare our fibre homotopy relation, which is defined in terms of left homotopies and cylinders, with the fibre homotopy relation defined in terms of right homotopies and path objects. We also dualize our notion of weak fibration in a category of cofibrant objects to a notion of weak cofibration in a category of fibrant objects, and give examples of these weak cofibrations. A section is devoted to the case of chain complexes in an abelian category.

0. Introduction

The fibre homotopy equivalence theorem of Dold [**Dol63**, Theorem 6.1] in Top has been generalized by various authors. Besides the original work by Dold, the book [**DKP70**] of tom Dieck-Kamps-Puppe gives an exposition on weak fibrations (*h-Faserungen* in Top). Some of the generalizations consider maps which are simultaneously over a given space and under a given space. Booth [**Boo93**] also obtains versions of Dold's theorem, using suitably defined generalizations of the covering homotopy property. In other cases the fibre homotopy equivalences were studied in a categorical setting, as for example in the paper [**HKK96**] by Hardie-Kamps-Kieboom and the book [**KP97**] of Kamps-Porter. Homotopy structure can be imposed on an appropriate category in several ways. In [**HKK96**] and [**KP97**]

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the basic assumption is that the category has some cylinder functor. In the article [**Kam72**], Kamps uses cylinder functors to define a notion of weak fibration. A model category structure, a concept due to Quillen [**Qui67**], is another way of introducing a homotopy relation in a category. In fact in a model category there are two dual ways of defining homotopy of maps: *left homotopies*, defined in terms of *cylinder objects*, and *right homotopies*, defined in terms of *cocylinder objects*. These two methods feature in categories of cofibrant objects and, respectively, categories of fibrant objects. Of these two notions, the latter was introduced by K. S. Brown [**Bro73**] in 1973 and dualized into the former by Kamps and Porter (see [**KP97**]). We consider a notion of weak fibration in the context of a category of cofibrant objects with a *cylinder object choice*, i.e., a chosen cylinder object for every object of the category. Our weak fibrations, and their properties, depend on this cylinder object choice. In case this choice comes from a cylinder functor satisfying certain Kan filler conditions, our fibre homotopy relation coincides with the one used in [**KP97**]. This makes it possible to compare our weak fibrations with Kamps's.

The aim of this article is to study fibre homotopies and weak fibrations in a category of cofibrant objects and, dually, relative homotopies and weak cofibrations in a category of fibrant objects. The presentation is as follows. In Section 1 we recall the axioms of a category of cofibrant objects and introduce the notion of cylinder object choice. The definition of fibre homotopy from [**KP97**] is adapted to our context. Based on one of the equivalent formulations—due to Kieboom [**Kie87**]—of the concept of weak fibration in the topological case, for a category of cofibrant objects we define the concept of weak fibration in terms of the so-called *weak right lifting property*. Depending on properties of the cylinder object choice, we give alternative characterisations of the notion of weak fibration and we show that the class of weak fibrations is closed with respect to composition. We show that weak fibrations are preserved by pullback if the pullback exists, and that in case the category of cofibrant objects comes from a model category, every fibration between cofibrant objects is a weak fibration. In Section 2 we look at fibre homotopy equivalences. We prove a version of Dold's fibre homotopy equivalence theorem, as well as a theorem regarding stability under fibre homotopy dominance of weak fibrations. Section 3 treats some examples of categories of cofibrant objects and their weak fibrations. In Section 4, we show that, when working over a fibration in a model category, the fibre homotopy relation as defined in Section 1 is equivalent to the right homotopy relation over the given fibration, Theorem 4.4. In Section 5 we consider the dual situation: weak cofibrations in a category of fibrant objects equipped with a suitable cocylinder functor. We dualize the theorems and notions from the preceding sections. Section 6 is devoted to some examples of categories of fibrant objects and their weak cofibrations. Finally, in Section 7, we describe the weak fibrations and weak cofibrations that arise when considering model structures on the category of chain complexes in an abelian category, which were recently introduced by Christensen and Hovey [**CH02**].

For the basics on categories of cofibrant objects, cylinders and Kan conditions we refer to the book [**KP97**] of Kamps and Porter. The foundational work on model categories appears in the book [**Qui67**] of Quillen. Hovey's book [**Hov99**] provides an excellent introduction to model categories. There is also the introductory paper [**DS95**] by Dwyer and Spalinski, and Baues's book [**Bau89**] that cover most of the necessary material. The book [**Jam84**] of James has a fairly comprehensive treatment of fibrewise topology and homotopy theory.

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1. Fibre homotopy and weak fibrations

For the definition of model category we refer to [**Hov99**], which uses a slightly different definition from Quillen's original one. A model category is denoted

$$(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we}),$$

where *fib* is the class of fibrations, *cof* is the class of cofibrations and *we* is the class of weak equivalences. A cofibrant object is an object for which the unique morphism from an initial object to it is a cofibration. Dually, an object is fibrant if the unique morphism to a terminal object is a fibration. From now on, morphisms of a category $\mathcal{C}$ will also be called maps in $\mathcal{C}$.

We recall the axioms of a category of cofibrant objects.

Definition 1.1. Consider a triple $(\mathcal{C}, \mathit{cof}, \mathit{we})$, where $\mathcal{C}$ is a category with binary coproducts and an initial object e, and where *cof* and *we* are two classes of maps of $\mathcal{C}$. Maps in *cof*, *we* and $\mathit{cof} \cap \mathit{we}$ are respectively called *cofibrations*, *weak equivalences* and *trivial cofibrations*.

Let X be an object of $\mathcal{C}$ and let $\nabla_X = 1_X + 1_X : X \sqcup X \longrightarrow X$ denote the folding map (codiagonal morphism). A *cylinder object* $(X \times I, e_0, e_1, \sigma)$ on X consists of an object $X \times I$ of $\mathcal{C}$ and maps

$$e_0, e_1 : X \longrightarrow X \times I, \qquad \sigma : X \times I \longrightarrow X$$

such that the sum $e_0 + e_1 : X \sqcup X \longrightarrow X \times I$ is a cofibration, σ is a weak equivalence and $\sigma \circ (e_0 + e_1) = \nabla_X$.

A triple $(\mathcal{C}, \mathit{cof}, \mathit{we})$ is called a *category of cofibrant objects* if the following axioms hold.

C1 Any isomorphism is a weak equivalence. For two maps f and g in $\mathcal{C}$ such that $g \circ f$ exists, if any two out of three maps f, g and $g \circ f$ are weak equivalences, then so is the third.

C2 Any isomorphism is a cofibration and the class *cof* is closed under composition.

C3 Given any pair of maps $i : A \longrightarrow X$, $u : A \longrightarrow B$ with $i \in cof$ the pushout

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d, "\bar{\imath}"] \\ X \arrow[r, "\bar{u}"'] & X \sqcup_A B \end{tikzcd}$$

exists and $\bar{\imath}$ is a cofibration. If i is trivial, so is $\bar{\imath}$.

C4 For any object X of $\mathcal{C}$ there is a cylinder object $(X \times I, e_0, e_1, \sigma)$.

C5 For any object X of $\mathcal{C}$ the unique map $e \longrightarrow X$ is a cofibration.

Note that for any cylinder object $(X \times I, e_0, e_1, \sigma)$ on an object X of $\mathcal{C}$, the maps e_0 and e_1 are trivial cofibrations. We say that e_0, e_1 are *cylinder cofibrations* and that σ is a *cylinder retraction*.

Note that for any model category $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$, the full subcategory $\mathcal{M}_c$ of all cofibrant objects, together with the classes $\mathit{cof} \cap \mathcal{M}_c$ and $\mathit{we} \cap \mathcal{M}_c$ of cofibrations, resp. weak equivalences between cofibrant objects, forms a category of cofibrant objects $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$. Hovey's notion of cylinder object in a model category (see Definition 1.2.4 of [**Hov99**]) is essentially the same as the one defined above. (The notion of cylinder object used in [**DS95**] is weaker, in the sense that they only require σ to be a weak equivalence. When also $e_0 + e_1$ is a cofibration, they speak of a *good* cylinder object.) Consequently, the cylinder objects of $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$ are exactly the cylinder objects on cofibrant objects of $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$.

In order to define our notion of fibre homotopy in a category of cofibrant objects $(\mathcal{C}, \mathit{cof}, \mathit{we})$, we require that a cylinder object is chosen for each object $X \in |\mathcal{C}|$:

Definition 1.2. If $(\mathcal{C}, \mathit{cof}, \mathit{we})$ is a category of cofibrant objects, then a *cylinder object choice* $\mathfrak{I}$ is a family

$$(X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{C}|},$$

where for each object X of $\mathcal{C}$, $(X \times I, e_0(X), e_1(X), \sigma(X))$ is a cylinder object on X.

Example 1.3. Let $\mathcal{C}$ be a category. A *cylinder* or *cylinder functor*

$$\mathbf{I} = ((\cdot) \times I, e_0, e_1, \sigma)$$

on $\mathcal{C}$ is a functor

$$(\cdot) \times I : \mathcal{C} \longrightarrow \mathcal{C}$$

together with natural transformations

$$e_0, e_1 : 1_{\mathcal{C}} \Longrightarrow (\cdot) \times I, \qquad \sigma : (\cdot) \times I \Longrightarrow 1_{\mathcal{C}}$$

such that $\sigma e_0 = \sigma e_1 = 1_{1_{\mathcal{C}}}$. Let $(\mathcal{C}, \mathit{cof}, \mathit{we})$ be a category of cofibrant objects. A cylinder $((\cdot) \times I, e_0, e_1, \sigma)$ on $\mathcal{C}$ is called *suitable* if $(X \times I, e_0(X), e_1(X), \sigma(X))$ is a cylinder object on X for all $X \in |\mathcal{C}|$. Let $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$ be a model category. A cylinder $((\cdot) \times I, e_0, e_1, \sigma)$ on $\mathcal{M}$ is called *suitable* if $(X \times I, e_0(X), e_1(X), \sigma(X))$ is a cylinder object (in the sense of [**Hov99**]) on X for all $X \in |\mathcal{M}|$. If $\mathbf{I}$ is a suitable cylinder on $(\mathcal{C}, cof, we)$, then $\mathbf{I}$ evidently induces a cylinder object choice on $(\mathcal{C}, cof, we)$. Furthermore, note that if $(\mathcal{C}, cof, we)$ is a category of cofibrant objects generated by a cylinder $\mathbf{I}$—see [**KP97**], Definition II.1.5—then $\mathbf{I}$ is automatically suitable.

Recall that in a category with a cylinder functor there is a notion of homotopy over a certain object—cf. [**KP97**], Definition I.6.1(b). The following definition introduces a similar concept for categories of cofibrant objects equipped with a cylinder object choice.

Definition 1.4. Let $(\mathcal{C}, cof, we)$ be a category of cofibrant objects equipped with a cylinder object choice $\mathfrak{I} = (X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{C}|}$, and let $p : E \longrightarrow B$ be a map in $\mathcal{C}$. Suppose further that we have a commutative diagram as follows:

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "p\circ f = p\circ g"'] & & E \arrow[dl, "p"] \\ & B. & \end{tikzcd}$$

Then we say that f is *homotopic to g over p (with respect to $\mathfrak{I}$)*, and we write $f \simeq_p g$, if there is a map $H : X \times I \longrightarrow E$ such that

$$\begin{cases} H \circ e_0(X) = f \\ H \circ e_1(X) = g \\ p \circ H = p \circ f \circ \sigma(X) = p \circ g \circ \sigma(X). \end{cases}$$

The map H is said to be a *fibre homotopy (over p) from f to g*.

If $f : X \longrightarrow E$ and $p : E \longrightarrow B$ are maps in $\mathcal{C}$, then being fibre homotopic over p is a relation on the set $[f]_p$ of all maps $\overline{f} : X \longrightarrow E$ such that $p \circ \overline{f} = p \circ f$.

It is important to keep in mind that the notion of fibre homotopy depends on the cylinder object choice $\mathfrak{I}$ on $(\mathcal{C}, cof, we)$.

Example 1.5. Choosing *cof* and *we* to be all functions and *fib* to be all isomorphisms between sets, defines a model structure $(\mathsf{Set}, fib, cof, we)$ on Set. On the induced category of cofibrant objects (Set, cof, we), we consider the following two cylinder object choices: $\mathfrak{I}$ maps a set X to the cylinder object $(X, 1_X, 1_X, 1_X)$; $\mathfrak{I}'$ maps a set X to the cylinder object $(X \sqcup X, \mathrm{in}_0(X), \mathrm{in}_1(X), \nabla_X)$, where $\mathrm{in}_0(X)$ and $\mathrm{in}_1(X)$ denote the two canonical injections of X into the coproduct $X \sqcup X$.

Now let f, g and p be maps such as in Definition 1.4 above. Then $f \simeq_p g$ with respect to $\mathfrak{I}$ if and only if f equals g, but unless p is an injection, f can be fibre homotopic to g over p with respect to $\mathfrak{I}'$ without f and g being equal. In the extremal case of B being a terminal object of Set, we even have that $f \simeq_p g$ with respect to $\mathfrak{I}'$ for any two maps f and g from X to E.

However note that if p is a fibration, then the fibre homotopy relations with respect to $\mathfrak{I}$ and $\mathfrak{I}'$ do coincide. That this holds true in general is proved in Theorem 4.4.

Proposition 1.6. *Let $p : E \longrightarrow B$ and $f, g : X \longrightarrow E$ be maps of $\mathcal{C}$ such that $f \simeq_p g$. Then $f \in we$ if and only if $g \in we$.*

Proof. This follows immediately from **C1** and the fact that for each cylinder object $(X \times I, e_0, e_1, \sigma)$ on X, the cylinder cofibrations e_0 and e_1 are weak equivalences. $\square$

Proposition 1.7. *Let $(\mathcal{C}, cof, we)$ be a category of cofibrant objects equipped with a cylinder object choice $\mathfrak{I} = (X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{C}|}$ and let $p : E \longrightarrow B$ be a map in $\mathcal{C}$. Then the following properties hold:*

1. *For each map $f : X \longrightarrow E$ in $\mathcal{C}$, we have that $f \simeq_p f$.*
2. *Suppose that we have the following commutative diagram.*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr] & E \arrow[r, "h"] \arrow[d, "p"] & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

If $f \simeq_p g$ then $h \circ f \simeq_{p'} h \circ g$.

Proof. We give a proof of (2): if $H : X \times I \longrightarrow E$ is a fibre homotopy over p from f to g, then $h \circ H$ is a fibre homotopy over p' from $h \circ f$ to $h \circ g$. $\square$

From now on, unless mentioned otherwise, we will suppose that we work in a category of cofibrant objects $(\mathcal{C}, cof, we)$ equipped with a cylinder object choice

$$\mathfrak{I} = (X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{C}|}.$$

Definition 1.8. (cf. [**Kie87**]) Suppose that $i : A \longrightarrow X$ and $p : E \longrightarrow B$ are maps in $\mathcal{C}$. We say that p has the *weak right lifting property* (WRLP) with respect to i if whenever we have a commutative square as below,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd} \tag{A}$$

there exists a map $h : X \longrightarrow E$ such that $p \circ h = g$ and $h \circ i \simeq_p f$. A map $p : E \longrightarrow B$ in $\mathcal{C}$ is said to be a *weak fibration* if it has the WRLP with respect to all weak equivalences $i : A \longrightarrow X$.

The following result (cf. [**Dol63**, 5.13]) follows easily from Definition 1.8, since for any cylinder object $(X \times I, e_0, e_1, \sigma)$ on an object X, the map $e_0 : X \longrightarrow X \times I$ is a weak equivalence.

Proposition 1.9. *Consider a commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_0(X)"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] & B \end{tikzcd}$$

in which p is a weak fibration. Then there is a homotopy $\overline{H} : X \times I \longrightarrow E$ such that $p \circ \overline{H} = H$ and $\overline{H} \circ e_0(X) \simeq_p f$.

These two lifting properties, i.e. the WRLP and the homotopy lifting property from Proposition 1.9, will not be equivalent in an arbitrary category of cofibrant objects. Yet we will be able to prove them to be equivalent in case the category of cofibrant objects comes from a model category, and if moreover it is equipped with a cylinder **I** that is generating and satisfies the Kan filler conditions DNE(2) and E(3); see Proposition 2.9. This means that under these assumptions, our notion of weak fibration coincides with Kamps's notion of *h-Faserung*, as defined in the article [**Kam72**].

The following construction, known as the *mapping cylinder factorisation* (see [**KP97**], page 9), simplifies some arguments regarding composition of weak fibrations and their behaviour with respect to pullbacks.

Definition 1.10. Let $f : X \longrightarrow Y$ a map in $\mathcal{C}$. A *mapping cylinder* of f is a triple (M_f, π_f, j_f) (sometimes denoted shortly M_f) with $M_f \in |\mathcal{C}|$, and $\pi_f : X \times I \longrightarrow M_f$ and $j_f : Y \longrightarrow M_f$ maps in $\mathcal{C}$, such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_0(X)"'] & Y \arrow[d, "j_f"] \\ X \times I \arrow[r, "\pi_f"'] \arrow[ur, phantom, "\ulcorner"] & M_f \end{tikzcd}$$

is a pushout in $\mathcal{C}$.

If $f : X \longrightarrow Y$ is a map in $\mathcal{C}$, then a mapping cylinder of f always exists by **C3**. Being a mapping cylinder of f depends on the cylinder object choice $\mathfrak{I}$. The map j_f is a trivial cofibration since $e_0(X)$ is. We shall refer to the map $k_f = \pi_f \circ e_1(X) : X \longrightarrow M_f$ as the *mapping cylinder cofibration.* If $f \in we$, then k_f is a trivial cofibration.

Definition 1.11. Let $f : X \longrightarrow Y$ be a map in $\mathcal{C}$ and (M_f, π_f, j_f) a mapping cylinder of f. Due to pushout properties there is a unique map $q_f : M_f \longrightarrow Y$, which we call the *mapping cylinder projection*, such that $q_f \circ j_f = 1_Y$ and $q_f \circ \pi_f = f \circ \sigma(X)$. Thus we obtain a factorisation $f = q_f \circ k_f$ of f as a cofibration followed by a weak equivalence.

Proposition 1.12. *Let $p : E \longrightarrow B$ and $i : A \longrightarrow X$ be any maps in $\mathcal{C}$, then the following conditions are equivalent:*

1. *the map p has the* WRLP *with respect to i,*
2. *given the diagram of solid arrows below, then for any mapping cylinder*

$$(M_i, \pi_i, j_i)$$

for i there exists a map $H : M_i \longrightarrow E$ such that diagram **B** *below commutes,*

3. *given the diagram of solid arrows below, there exists a mapping cylinder*

$$(M_i, \pi_i, j_i)$$

for i and a map $H : M_i \longrightarrow E$ such that diagram **B** *below commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "i"'] \arrow[dr, dotted, "k_i"] & & E \arrow[dd, "p"] \\ & M_i \arrow[ur, dotted, "H"'] \arrow[dl, dotted, "g_i"] & \\ X \arrow[rr, "g"'] & & B \end{tikzcd} \tag{B}$$

Proof. Suppose that $f : A \longrightarrow E$ and $g : X \longrightarrow B$ are any maps such that $p \circ f = g \circ i$. If we have H as in condition (3), then a map h as in Definition 1.8 is defined by $h = H \circ j_i$. Indeed, $p \circ h = g$, and $H \circ \pi_i$ is a fibre homotopy $h \circ i \simeq_p f$. Thus condition (1) follows.

Condition (2) obviously implies condition (3).

Now suppose that condition (1) holds and take a mapping cylinder (M_i, π_i, j_i) for i. Then there is a map $h : X \longrightarrow E$ and a fibre homotopy $F : A \times I \longrightarrow E$ over p from $h \circ i$ to f. In the commutative diagram of solid arrows below, we have in particular the pushout square that defines M_i.

$$\begin{tikzcd} & X \arrow[dr, "j_i"] \arrow[drr, bend left, "h"] & & \\ A \arrow[ur, "i"] \arrow[dr, "e_0(A)"'] & & M_i \arrow[r, dotted, "H"] & E \\ & A \times I \arrow[ur, "\pi_i"] \arrow[urr, bend right, "F"'] & & \end{tikzcd}$$

The universal property of pushouts yields a unique map $H : M_i \longrightarrow E$ such that $H \circ \pi_i = F$ and $H \circ j_i = h$. Now

$$H \circ k_i = H \circ \pi_i \circ e_0(A) = F \circ e_0(A) = f,$$

i.e., the upper middle triangle in **B** commutes.

We now prove that the lower right triangle in **B** commutes. The universal property of pushouts yields a unique map $k : M_i \longrightarrow B$ satisfying the conditions

$$k \circ \pi_i = p \circ F \qquad \text{and} \qquad k \circ j_i = g.$$

Now we show that both of the maps $g \circ q_i$ and $p \circ H$ can fulfill the role of k:

$$g \circ q_i \circ \pi_i = p \circ h \circ i \circ \sigma(A)$$

and $g \circ q_i \circ j_i = g$; $p \circ H \circ \pi_i = p \circ F$ and $p \circ H \circ j_i = p \circ h = g$. Thus it follows that $p \circ H = k = g \circ q_i$, and condition (2) follows. □

Proposition 1.13. *Suppose that we have a pullback square as below. Let $i : A \longrightarrow X$ be a map such that p has the* WRLP *with respect to i. Then p' has the* WRLP *with respect to i.*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"'] & B \end{tikzcd} \tag{C}$$

Proof. Let f_1 and g_1 be maps such that the following square, on the left below, is commutative. Since p has the WRLP with respect to i, there exists a mapping cylinder M_i of i and a map $h : M_i \longrightarrow E$ such that the diagram, on the right hand side below, is commutative.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "i"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "g_1"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "k_i"'] & E' \arrow[r, "f"] & E \arrow[d, "p"] \\ M_i \arrow[rru, "h"] \arrow[r, "g_1 \circ q_i"'] & B' \arrow[r, "g"'] & B \end{tikzcd}$$

Since we have a pullback square **C**, there exists a map $h' : M_i \longrightarrow E'$ such that $p' \circ h' = g_1 \circ q_i$ (i.e., the lower right triangle in the following diagram commutes) and $f \circ h' = h$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "k_i"'] & E' \arrow[d, "p'"] \\ M_i \arrow[ru, "h'"'] \arrow[r, "g_1 \circ q_i"'] & B' \end{tikzcd}$$

Furthermore, the universal property of pullbacks yields a unique map $\overline{f} : X \longrightarrow E'$ such that $p' \circ \overline{f} = g_1 \circ i$ and $f \circ \overline{f} = f \circ f_1$. Obviously then $f_1 = \overline{f}$. But also the map $h' \circ k_i$ fulfils the conditions defining $\overline{f}$. Thus $h' \circ k_i = f_1$. Therefore, also the upper left triangle in the last diagram is commutative. The result follows. □

Corollary 1.14. *If, in the pullback square* **C**, *the map p is a weak fibration, then p' is a weak fibration.*

In homotopy theory, we often need that the fibre homotopy relation over a map $p : E \longrightarrow B$ yields an equivalence relation on the set $[f]_p$ for each map $f : X \longrightarrow E$, and that it is stable under precomposition. Therefore we restrict the class of cylinder object choices in the following way:

Definition 1.15. Let $(\mathcal{C}, cof, we)$ be a category of cofibrant objects. Then a cylinder object choice $\mathfrak{I}$ is called *nice* if, for each map $p : E \longrightarrow B$ in $\mathcal{C}$, we have the following properties.

1. For each map $f : X \longrightarrow E$ in $\mathcal{C}$, the relation $\simeq_p$ is an equivalence relation on $[f]_p$.

2. Suppose that we have the following commutative diagram.

$$\begin{tikzcd} E' \arrow[r, "h"] \arrow[dr] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

If $f \simeq_p g$, then $f \circ h \simeq_p g \circ h$.

The following proposition gives an interesting situation in which a category of cofibrant objects can be equipped with a nice cylinder object choice. Namely, this is the case for a cylinder object choice induced by a suitable cylinder $\mathbf{I}$ which satisfies the so-called *Kan filler condition* DNE$(2,1,1)$ (see [**KP97**], p. 27). In [**Kam72**], [**HKK96**] and [**KP97**], homotopy theory is discussed in the context of a category equipped with a cylinder $\mathbf{I}$. For the resulting homotopy relation to have suitable properties, such a cylinder must satisfy certain conditions. The Kan filler condition DNE$(2,1,1)$ is a sufficient condition for the homotopy relation over a certain object to be an equivalence relation. It is fulfilled in many cases; see Section 3.

Proposition 1.16. *Let $(\mathcal{C}, cof, we)$ be a category of cofibrant objects equipped with a suitable cylinder $\mathbf{I} = ((\cdot) \times I, e_0, e_1, \sigma)$ which satisfies the Kan filler condition* DNE$(2,1,1)$*. Then the cylinder object choice induced by $\mathbf{I}$ is nice.*

Proof. Condition 1. in Definition 1.15 is just Proposition I.6.2 in the book [**KP97**] of Kamps and Porter. The second condition follows from the functoriality of $(\cdot) \times I$. ☐

For the remaining part of this section, we suppose that the category of cofibrant objects $(\mathcal{C}, cof, we)$ is equipped with a nice cylinder object choice.

Proposition 1.17. *Let $p : E \longrightarrow B$ be a map in $\mathcal{C}$, then the following conditions are equivalent:*

1. *p has the* WRLP *with respect to all $i \in we$,*
2. *p has the* WRLP *with respect to all $i \in cof \cap we$.*

Proof. Suppose that we have a commutative square such as $\mathbf{A}$ above, where i is a weak equivalence, and suppose that condition (2) holds. The mapping cylinder factorisation of $i : A \longrightarrow X$ yields a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k_i"'] & E \arrow[d, "p"] \\ M_i \arrow[r, "g \circ q_i"'] & B, \end{tikzcd}$$

where the mapping cylinder cofibration k_i is trivial. We get a map $\overline{h} : M_i \longrightarrow E$ such that $p \circ \overline{h} = g \circ q_i$ and $\overline{h} \circ k_i \simeq_p f$. Put $h = \overline{h} \circ j_i : X \longrightarrow E$, then $p \circ h = g \circ q_i \circ j_i = g$. Now we only need a fibre homotopy $H : h \circ i \simeq_p \overline{h} \circ k_i$ to prove that $h \circ i \simeq_p f$.

Clearly, the triangle

$$\begin{tikzcd} A \arrow[rr, shift left, "h\circ i"] \arrow[rr, shift right, "\overline{h}\circ k_i"'] \arrow[dr, "g\circ i"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

commutes. Put $H = \overline{h} \circ \pi_i : A \times I \longrightarrow E$ to get

$$\begin{cases} H \circ e_0(A) = \overline{h} \circ \pi_i \circ e_0(A) = \overline{h} \circ j_i \circ i = h \circ i \\ H \circ e_1(A) = \overline{h} \circ \pi_i \circ e_1(A) = \overline{h} \circ k_i \\ p \circ H = p \circ \overline{h} \circ \pi_i = g \circ q_i \circ \pi_i = g \circ i \circ \sigma(A). \end{cases}$$

Thus condition (1) holds. □

This gives us the following characterisation of weak fibrations.

Proposition 1.18. *Let $p : E \longrightarrow B$ be a map in $\mathcal{C}$. The following conditions are equivalent:*

1. *p is a weak fibration,*
2. *p has the* WRLP *with respect to all trivial cofibrations.*

The following two corollaries will make clear why the name weak fibration is well-chosen: in case the category of cofibrant objects $(\mathcal{C}, cof, we)$ arises from a model category, a map of $\mathcal{C}$ which is a fibration in the model category is always a weak fibration in the category of cofibrant objects.

Corollary 1.19. *Let $(\mathcal{M}, fib, cof, we)$ be a model category and $(\mathcal{M}_c, cof \cap \mathcal{M}_c, we \cap \mathcal{M}_c)$ the associated category of cofibrant objects. Let $(\mathcal{M}_c, cof \cap \mathcal{M}_c, we \cap \mathcal{M}_c)$ be equipped with a nice cylinder object choice $\mathfrak{I}$. If $p : E \longrightarrow B$ is a map of $\mathcal{M}_c$ such that $p \in fib$, then p is a weak fibration.*

Proof. p, considered as a map in $\mathcal{M}$, has the right lifting property with respect to all maps in $cof \cap we$. Thus it also has the right lifting property, and a fortiori the WRLP, with respect to all trivial cofibrations of $\mathcal{M}_c$. But then p is a weak fibration by Proposition 1.18. □

Corollary 1.20. *Let $(\mathcal{M}, fib, cof, we)$ be a model category equipped with a suitable cylinder $\mathbf{I}$ satisfying* DNE$(2,1,1)$*. Let $(\mathcal{M}_c, cof \cap \mathcal{M}_c, we \cap \mathcal{M}_c)$ be the associated category of cofibrant objects. If $p : E \longrightarrow B$ is a map of $\mathcal{M}_c$ such that $p \in fib$, then p is a weak fibration.*

Proof. We only need to show that restricting the functor $(\cdot) \times I : \mathcal{M} \longrightarrow \mathcal{M}$ to $\mathcal{M}_c$ also corestricts it to $\mathcal{M}_c$. Indeed, for X a cofibrant object of $\mathcal{M}$, $X \times I$ is cofibrant in $\mathcal{M}$ as well: the unique map $\varnothing \longrightarrow X \times I$ is a cofibration, since it can be factorised as

$$\varnothing \longrightarrow X \sqcup X \xrightarrow{e_0+e_1} X \times I.$$

The left map is a cofibration since $X \sqcup X$ is cofibrant due to **C3**, and the right map is a cofibration since $(X \times I, e_0(X), e_1(X), \sigma(X))$ is a cylinder object of $(\mathcal{M}_c, cof \cap$

$\mathcal{M}_c, we \cap \mathcal{M}_c$). Using techniques from [**KP97**], one shows that this restriction also satisfies DNE$(2,1,1)$. Hence it induces a nice cylinder object choice on $(\mathcal{M}_c, cof \cap \mathcal{M}_c, we \cap \mathcal{M}_c)$. ☐

Proposition 1.21. *Suppose that $q : D \longrightarrow E$ and $p : E \longrightarrow B$ are weak fibrations. Then $p \circ q$ is a weak fibration.*

Proof. Consider a commutative diagram of solid arrows

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & D \arrow[d, "p\circ q"] \\
X \arrow[r, "g"'] \arrow[ur, dotted, "h"'] & B,
\end{tikzcd}$$

where i is a weak equivalence. We must construct an arrow $h : X \longrightarrow D$ such that $p \circ q \circ h = g$ and $h \circ i \simeq_{p\circ q} f$. Now p being a weak fibration implies that there is a mapping cylinder M_i for i and a map $H : M_i \longrightarrow E$ such that the diagram below commutes.

$$\begin{tikzcd}
A \arrow[rr, "q\circ f"] \arrow[dd, "i"'] \arrow[dr, "k_i"] & & E \arrow[dd, "p"] \\
& M_i \arrow[ur, "H"] \arrow[dl, "q_i"] & \\
X \arrow[rr, "g"'] & & B
\end{tikzcd}$$

The construction of a mapping cylinder M_{k_i} for k_i gives rise to a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[dr, "k_{k_i}"] \arrow[ddr, bend right, "k_i"'] & A \times I \arrow[d, "\pi_{k_i}"] & A \arrow[l, "e_0"'] \arrow[r, "f"] \arrow[d, "k_i"'] & D \arrow[d, "q"] \\
& M_{k_i} \arrow[d, "q_{k_i}"'] & M_i \arrow[l, "j_{k_i}"'] \arrow[r, "H"'] \arrow[dl, equal] & E \\
& M_i & &
\end{tikzcd}$$

The mapping cylinder cofibration k_i is a weak equivalence; hence, q being a weak fibration implies that there is a map $K : M_{k_i} \longrightarrow D$ such that the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "k_i"'] \arrow[dr, "k_{k_i}"] & & D \arrow[dd, "q"] \\
& M_{k_i} \arrow[ur, "K"] \arrow[dl, "q_{k_i}"] & \\
M_i \arrow[rr, "H"'] & & E
\end{tikzcd}$$

commutes. Put $h = K \circ j_{k_i} \circ j_i : X \longrightarrow D$. We now show that h is indeed the needed map.

The equality $p \circ q \circ h = g$ follows by straightforward calculation. One also easily verifies that $K \circ j_{k_i} \circ \pi_i$ is a fibre homotopy

$$K \circ j_{k_i} \circ \pi_i \circ e_0 \simeq_{p \circ q} K \circ j_{k_i} \circ \pi_i \circ e_1$$

and that $K \circ \pi_{k_i}$ is a fibre homotopy $K \circ \pi_{k_i} \circ e_0 \simeq_{p \circ q} K \circ \pi_{k_i} \circ e_1$. It follows that

$$\begin{array}{llllll} h \circ i & = & K \circ j_{k_i} \circ j_i \circ i & = & K \circ j_{k_i} \circ \pi_i \circ e_0 & \simeq_{p \circ q} \\ K \circ j_{k_i} \circ \pi_i \circ e_1 & = & K \circ j_{k_i} \circ k_i & = & K \circ \pi_{k_i} \circ e_0 & \simeq_{p \circ q} \\ K \circ \pi_{k_i} \circ e_1 & = & K \circ k_{k_i} & = & f, & \end{array}$$

which proves the assertion. □

2. Fibre homotopy equivalence

Throughout this section, unless mentioned otherwise, we assume that we work in a category of cofibrant objects $(\mathcal{C}, cof, we)$ equipped with a nice cylinder object choice $\mathfrak{I} = (X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{C}|}$.

Definition 2.1. Suppose that we have a commutative triangle **D**. Note that f can be regarded as a morphism, in the category $\mathcal{C}/_B$ of objects over B, from p to p'.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd} \qquad (\mathbf{D})$$

A morphism $g : p' \longrightarrow p$ is said to be a *fibre homotopy inverse* for $f : p \longrightarrow p'$ if $g \circ f \simeq_p 1_E$ and $f \circ g \simeq_{p'} 1_{E'}$. The map f is said to be a *fibre homotopy equivalence (between p and p')* if a fibre homotopy inverse for f does exist. The maps p and p' are called *fibre homotopy equivalent* if a fibre homotopy equivalence $f : p \longrightarrow p'$ exists.

Proposition 2.2. *Let B be a $\mathcal{C}$-object. The relation on $|\mathcal{C}/_B|$ of being fibre homotopy equivalent is an equivalence relation.*

The following theorem is a categorical version of the fibre homotopy equivalence theorem [**Dol63**, Theorem 6.1] of Dold.

Theorem 2.3. *Suppose that in the commutative triangle of diagram **D**, p and p' are weak fibrations. If $f : E \longrightarrow E'$ is a weak equivalence then $f : p \longrightarrow p'$ is a fibre homotopy equivalence.*

Proof. Given diagram **D**, we consider the following commutative square.

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "f"'] & E \arrow[d, "p"] \\ E' \arrow[r, "p'"'] & B \end{tikzcd}$$

Since p is a weak fibration and f a weak equivalence, there exists a map $g : E' \longrightarrow E$ such that $p \circ g = p'$ and $g \circ f \simeq_p 1_E$. So it suffices to show that $f \circ g \simeq_{p'} 1_{E'}$. Now Proposition 1.6 implies that $g \circ f$ is a weak equivalence. Furthermore, f is a weak equivalence, and consequently, g is a weak equivalence. For the following commutative square, there exists a map $k : E \longrightarrow E'$ such that $p' \circ k = p$ and $k \circ g \simeq_{p'} 1_{E'}$.

$$\begin{tikzcd} E' \arrow[r, equal] \arrow[d, "g"'] & E' \arrow[d, "p'"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

But then,

$$f \circ g \simeq_{p'} (k \circ g) \circ f \circ g = k \circ (g \circ f) \circ g \simeq_{p'} k \circ 1_E \circ g = k \circ g \simeq_{p'} 1_{E'},$$

and this completes the proof of the theorem. □

The relative simplicity of the proof of Theorem 2.3, and also of Theorem 2.4 below, is the result of the particular choice of the equivalences in [**Kie87**], to model our categorical definition of weak fibration.

Weak fibrations are stable under fibre homotopy dominance, as states the following theorem:

Theorem 2.4. *Suppose that we have a commutative diagram as below, where $g \circ f \simeq_p 1_E$.*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "p"'] & E' \arrow[r, "g"] \arrow[d, "p'"] & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

If p' is a weak fibration then p is a weak fibration.

Proof. Suppose that we have a commutative diagram of solid arrows

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & E \arrow[r, shift left, "f"] \arrow[d, "p"'] & E' \arrow[l, dotted, shift left, "g"] \arrow[dl, "p'"] \\ X \arrow[r, "k"'] & B & \end{tikzcd}$$

where i is a weak equivalence. p' being a weak fibration yields a map $l' : X \longrightarrow E'$ such that $p' \circ l' = k$ and $l' \circ i \simeq_{p'} f \circ h$. Put $l = g \circ l' : X \longrightarrow E$. Then

$$p \circ l = p \circ g \circ l' = p' \circ l' = k$$

and

$$l \circ i = g \circ l' \circ i \simeq_p g \circ f \circ h \simeq_p 1_E \circ h = h.$$

Thus p has the WRLP with respect to i. □

In case the category of cofibrant objects comes from a model category, the previous proposition implies that a map between cofibrant objects is a weak fibration exactly when it is fibre homotopy equivalent to a fibration (which must of course also be a map between cofibrant objects). This is a categorical version of a result in [**DKP70**]. In the book of James [**Jam84**], weak fibrations are defined as maps of topological spaces which are fibre homotopy equivalent to fibrations.

Theorem 2.5. *Let $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$ be a model category and $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$ the associated category of cofibrant objects. Then a map in $\mathcal{M}_c$ is a weak fibration if and only if it is fibre homotopy equivalent to a fibration.*

Proof. Let E, E' and B be cofibrant objects of $\mathcal{M}$. If a map $p : E \longrightarrow B$ is fibre homotopy equivalent to a fibration $p' : E' \longrightarrow B$, then in particular it is dominated by it. But by Corollary 1.19, p' is a weak fibration, and so the assumptions of Theorem 2.4 hold. Hence p is a weak fibration.

Now we prove the converse. Suppose that $p : E \longrightarrow B$ is a weak fibration in $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$. Then p can be factorized in $\mathcal{M}$ as a trivial cofibration $f : E \longrightarrow E'$ followed by a fibration $p' : E' \longrightarrow B$. Since f is a cofibration and E is cofibrant, it follows that E' is cofibrant. Now Corollary 1.19 implies that p' is a weak fibration; thus, Dold's Theorem 2.3 applies, and the weak equivalence f is a fibre homotopy equivalence between p and p'. □

The next proposition is a categorical version of [**Kie87**], Theorem 2, and at the same time of [**DKP70**], Satz 6.26: a characterisation of those weak fibrations that are also weak equivalences. It brings into consideration a notion of *closed* category of cofibrant objects, after Quillen's notion of *closed* model category (see [**Qui67**], I.5 and [**Bro73**], I.6). This would be a category of cofibrant objects such that the class of weak equivalences (and possibly also the class of cofibrations) is closed under retracts.

Definition 2.6. (cf. [**DKP70**], Definition 6.24) A map $p : E \longrightarrow B$ of $\mathcal{C}$ is called *shrinkable (schrumpfbar)* when there exists a map $s : B \longrightarrow E$ such that $p \circ s = 1_B$ and $s \circ p \simeq_p 1_E$.

Proposition 2.7. *Let $p : E \longrightarrow B$ be a map of $\mathcal{C}$.*

1. *p is a weak fibration and a weak equivalence,*
2. *p is shrinkable,*
3. *p has the* WRLP *with respect to all maps $i : A \longrightarrow X$ in $\mathcal{C}$.*

*The implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) always hold and (1) $\Leftarrow$ (2) holds as soon as the class we of weak equivalences is closed under retracts (see [**Qui67**], I.5 and [**Bro73**], I.6).*

Proof. First suppose that (1) holds, and consider the commutative triangle

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p"'] & & B \arrow[dl, "1_B"] \\ & B. & \end{tikzcd}$$

Both p and 1_B are weak fibrations and p is a weak equivalence; thus, Dold's Theorem 2.3 implies that p is a fibre homotopy equivalence between p and 1_B. We get a map $s : B \longrightarrow E$ such that $p \circ s = 1_B$ and $s \circ p \simeq_p 1_E$, and p is shrinkable.

Now suppose that p is shrinkable and consider a commutative square as in **A** above. Put $h = s \circ g : X \longrightarrow E$. Then $p \circ h = p \circ s \circ g = g$ and

$$h \circ i = s \circ g \circ i = s \circ p \circ f \simeq_p 1_E \circ f = f :$$

condition (3) holds.

Next suppose that (3) holds. Then p has the WRLP with respect to itself. Thus, for the commutative square of unbroken arrows

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dotted, "s"] & B, \end{tikzcd}$$

there exists a map $s : B \longrightarrow E$ such that $p \circ s = 1_B$ and $s \circ p \simeq_p 1_E$. This already proves that (2) and (3) are equivalent.

Finally suppose that (2) and (3) hold. To prove (1) we only need to show that p is a weak equivalence. There is a map $s : B \longrightarrow E$ such that $p \circ s = 1_B$ and $s \circ p \simeq_p 1_E$. The diagram

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "s"'] & E \arrow[r, "p"] \arrow[d, "s \circ p"'] & B \arrow[d, "s"] \\ E \arrow[r, equal] & E \arrow[r, equal] & E \end{tikzcd}$$

shows s as a retract of $s \circ p$. But 1_E is a weak equivalence, so Proposition 1.6 implies that $s \circ p$ is a weak equivalence. By hypothesis then also s is a weak equivalence. Thus, p is a weak equivalence. □

Corollary 2.8. *Let $(\mathcal{C}, cof, we)$ be a category of cofibrant objects such that we is closed under retracts. If, in the pullback square* **C***, p is a weak fibration and weak equivalence, then p' is a weak fibration and weak equivalence.*

Proof. This is an immediate consequence of Proposition 2.7 and Proposition 1.13. □

To end this section we prove that sometimes our notion of weak fibration coincides with Kamps's notion of h-Faserung, defined in the article [**Kam72**]. That these notions do not always coincide will be shown in Example 6.4.

Proposition 2.9. *Let $(\mathcal{M}_c, cof \cap \mathcal{M}_c, we \cap \mathcal{M}_c)$ be a category of cofibrant objects coming from a model category $(\mathcal{M}, fib, cof, we)$, such that its cylinder* **I** *is generating and satisfies* DNE(2) *and* E(3) *(see [**KP97**]). Then the converse of Proposition 1.9 holds: any map $p : E \longrightarrow B$ of $\mathcal{M}_c$ which has the* WRLP *with respect to all maps $e_0(X) : X \longrightarrow X \times I$ is a weak fibration.*

Proof. Let $p : E \longrightarrow B$ be a map of $\mathcal{M}_c$ which has the WRLP with respect to all maps $e_0(X) : X \longrightarrow X \times I$. Then p is a h-Faserung as in [**Kam72**], Definition 1.7.

Because of Proposition 1.17, we only need to prove it has the WRLP with respect to all trivial cofibrations of $\mathcal{M}_c$. Now consider (in the category $\mathcal{M}_c$) a commutative square $\mathbf{A}$ in which i is a trivial cofibration. Then p is also a map of $\mathcal{M}$; thus it can be factored into a trivial cofibration $\overline{\imath} : E \longrightarrow P$ followed by a fibration $\overline{p} : P \longrightarrow B$. Now $\overline{\imath}$ being a cofibration implies that P is a cofibrant object of $\mathcal{M}$, and therefore $\overline{\imath}$ and $\overline{p}$ are maps of $\mathcal{M}_c$. As maps of $\mathcal{M}$, $\overline{p}$ has the right lifting property with respect to i. Let $\overline{h} : X \longrightarrow P$ denote a lifting in the square

$$\begin{tikzcd} A \arrow[r, "\overline{\imath}\circ f"] \arrow[d, "i"'] & P \arrow[d, "\overline{p}"] \\ X \arrow[r, "g"'] \arrow[ur, dotted, "\overline{h}"] & B. \end{tikzcd}$$

The $\mathcal{M}_c$-morphism $\overline{p}$ is a fibration of $\mathcal{M}$, hence (by Corollary 1.19) a weak fibration of $\mathcal{M}_c$. But then Proposition 1.9 implies that $\overline{p}$ has the WRLP with respect to all maps $e_0(X) : X \longrightarrow X \times I$, and $\overline{p}$ is a h-Faserung in the sense of Kamps, [**Kam72**].

We get the commutative diagram of solid arrows

$$\begin{tikzcd} E \arrow[rr, shift left, "\overline{\imath}"] \arrow[dr, "p"'] & & P \arrow[ll, dotted, shift left, "\widetilde{\imath}"] \arrow[dl, "\overline{p}"] \\ & B. & \end{tikzcd}$$

The cylinder $\mathbf{I}$ is generating, which means that in particular $\overline{\imath}$ is a homotopy equivalence (*h-Äquivalenz*) in the sense of [**Kam72**], Definition 1.5. Thus Kamps's version of Dold's theorem ([**Kam72**], Satz 6.1) applies and gives a fibre homotopy inverse $\widetilde{\imath} : P \longrightarrow E$. Note that in this category of cofibrant objects his notion of fibre homotopy equivalence and ours coincide, so we can write $\widetilde{\imath}\circ\overline{\imath} \simeq_p 1_E$ and $\overline{\imath}\circ\widetilde{\imath} \simeq_{\overline{p}} 1_P$.

Put $h = \widetilde{\imath} \circ \overline{h} : X \longrightarrow E$, then h is a weak lifting for the square $\mathbf{A}$: $p \circ h = p \circ \widetilde{\imath} \circ \overline{h} = \overline{p} \circ \overline{h} = g$ and

$$h \circ i = \widetilde{\imath} \circ \overline{h} \circ i = \widetilde{\imath} \circ \overline{\imath} \circ f \simeq_p 1_E \circ f = f.$$

This proves that p is weak fibration. □

3. Examples of weak fibrations

Example 3.1. For the topological case we first consider the structure of category of cofibrant objects on Top induced by the model structure, originally described by Strøm in [**Str72**]; see for instance Example 3.6 of [**DS95**]. Its cofibrations (usually called *Hurewicz-cofibrations*) are closed continuous maps which have the homotopy extension property and its weak equivalences are homotopy equivalences. The standard cylinder

$$(\cdot) \times I : \mathsf{Top} \longrightarrow \mathsf{Top} : X \longmapsto X \times [0,1]\,,$$

which maps a space to a product with the unit interval $[0,1]$, together with the obvious natural transformations, satisfies the Kan condition DNE(n) for all n, so it satisfies DNE$(2,1,1)$, and it is clearly suitable.

Hence it induces a nice cylinder object choice such that two maps are fibre homotopic if and only if they are fibre homotopic in the usual, topological sense (see for instance [**DKP70**, Definition 0.22]). A continuous map has the homotopy lifting property mentioned above in Proposition 1.9, precisely when it has the WRLP with respect to homotopy equivalences (see [**Kie87**]). Thus the categorical definition coincides with the definition of weak fibration as given by [**Dol63**], or h-Faserung as in [**DKP70**].

Example 3.2. Now we consider the other standard model structure on Top, the one first described by Quillen in [**Qui67**]. Alternatively, a detailed description of this model structure can be found in [**Hov99**] and [**DS95**]. For us, its most important characteristics are that every object is fibrant and every CW-complex cofibrant, and that a continuous map $f : X \longrightarrow Y$ is a weak equivalence if and only if the induced map

$$\pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $n \geqslant 0$ and $x \in X$. We use this model structure to formulate Whitehead's Theorem (see, for instance, [**Mau70**], Theorem 7.5.4) and prove it as a result of Dold's Theorem.

Theorem 3.3. *(cf. [**Mau70**], Theorem 7.5.4) Let X and Y be CW-complexes and $f : X \longrightarrow Y$ a continuous map. If f is a weak equivalence then f is a homotopy equivalence.*

Proof. Let $*$ denote a one-point topological space, a terminal object of Top. X and Y are fibrant objects; hence, the unique maps p' and p in the commutative diagram below are fibrations.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p'"'] & & Y \arrow[dl, "p"] \\ & * & \end{tikzcd}$$

Now X, Y and $*$ are CW-complexes. This means that the diagram above is a diagram in the category of cofibrant objects (Top_c, $cof \cap \mathsf{Top}_c$, $we \cap \mathsf{Top}_c$) associated with Quillen's model structure on Top. According to Quillen ([**Qui67**], "Remarks" below Definition I.1.4), the restriction of the cylinder from Example 3.1 to Top_c is suitable. But then Corollary 1.19 implies that p and p' are weak fibrations, and Dold's Theorem 2.3 implies that f is a fibre homotopy equivalence. In particular, f is a homotopy equivalence. $\square$

Of course this proof does not only work for maps between CW-complexes but, more generally, also for maps between cofibrant objects. Consequently, in the category of cofibrant objects (Top_c, $cof \cap \mathsf{Top}_c$, $we \cap \mathsf{Top}_c$), a map is a weak equivalence if and only if it is a homotopy equivalence. Thus, a map between cofibrant objects is a weak fibration of (Top_c, $cof \cap \mathsf{Top}_c$, $we \cap \mathsf{Top}_c$) exactly when it is a weak fibration in the sense of Example 3.1.

Example 3.4. Let Gpd denote the category of groupoids (i.e., small categories in which every morphism is an isomorphism) and functors between them. The following choice of classes *fib*, *cof* and *we* defines a model structure on Gpd: the weak equivalences are equivalences of categories, the cofibrations are functors which are injective on objects and the fibrations are functors $p : \mathcal{E} \longrightarrow \mathcal{B}$ such that for any object e of $\mathcal{E}$ and any map $\beta : p(e) \longrightarrow b$ in $\mathcal{B}$ there exists a map $\epsilon : e \longrightarrow e'$ in $\mathcal{E}$ such that $p(\epsilon) = \beta$. Every object is fibrant and cofibrant.

Let $\mathcal{I}$ be the category with two objects 0, 1 and two non-identity morphisms $\iota : 0 \longrightarrow 1$ and $\iota^{-1} : 1 \longrightarrow 0$.

$$\begin{tikzcd} 0 \arrow[loop left, "1_0"] \arrow[r, bend left, "\iota"] & 1 \arrow[l, bend left, "\iota^{-1}"] \arrow[loop right, "1_1"] \end{tikzcd}$$

Let $(\cdot) \times I : \mathsf{Gpd} \longrightarrow \mathsf{Gpd}$ be the functor defined by $\mathcal{X} \times I = \mathcal{X} \times \mathcal{I}$ on groupoids $\mathcal{X}$ and

$$f \times I = f \times 1_{\mathcal{I}} : \mathcal{X} \times \mathcal{I} \longrightarrow \mathcal{Y} \times \mathcal{I}$$

on functors $f : \mathcal{X} \longrightarrow \mathcal{Y}$ between groupoids $\mathcal{X}$ and $\mathcal{Y}$. The equations $\epsilon_0(\mathcal{X})(x) = (x, 0)$, $\epsilon_0(\mathcal{X})(\xi) = (\xi, 1_0)$, $\epsilon_1(\mathcal{X})(x) = (x, 1)$ and $\epsilon_1(\mathcal{X})(\xi) = (\xi, 1_1)$ for x an object and ξ a map of $\mathcal{X}$ define natural transformations $e_0, e_1 : 1_{\mathsf{Gpd}} \Longrightarrow (\cdot) \times I$. Let $\sigma(\mathcal{X}) : \mathcal{X} \times \mathcal{I} \longrightarrow \mathcal{X}$ be the first projection. Then $\mathbf{I} = ((\cdot) \times I, e_0, e_1, \sigma)$ is a cylinder on Gpd which satisfies DNE(2), so it satisfies DNE$(2, 1, 1)$ (see [**KP97**], III.1.8). Also note that it is suitable.

In the category of cofibrant objects associated with this model category the converse of Corollary 1.19 holds: every weak fibration will be shown to be a map in *fib*; hence in Gpd the notions of fibration and weak fibration coincide.

Proposition 3.5. *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a map of groupoids. If p is a weak fibration then p is a fibration.*

Proof. Let e be an object of $\mathcal{E}$ and $\beta : p(e) \longrightarrow b$ a map of $\mathcal{B}$. Let $*$ denote the category with one object $*$ and one morphism 1_*, a terminal object of Gpd. We define a commutative square

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d, "i_0"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{I} \arrow[r, "g"'] & \mathcal{B} \end{tikzcd}$$

by choosing $i_0(*) = 0$, $f(*) = e$ and $g(\iota) = \beta$. Clearly i_0 is a weak equivalence; we get a map of groupoids $h : \mathcal{I} \longrightarrow \mathcal{E}$ such that $p \circ h = g$ and a fibre homotopy $H : h \circ i_0 \simeq_p f : * \times \mathcal{I} \longrightarrow \mathcal{E}$. Now put $\epsilon = h(\iota) \circ H(1_*, \iota^{-1}) : e \longrightarrow h(1)$, then

$$p(\epsilon) = p(h(\iota)) \circ p(H(1_*, \iota^{-1})) = g(\iota) \circ p(f(\sigma(*)(1_*, \iota^{-1}))) = \beta.$$

This shows that p is a fibration of groupoids. $\square$

4. Comparing two notions of fibre homotopy in a model category

Throughout this section, we assume that we work in a model category

$$(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$$

and denote $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$ the associated category of cofibrant objects.

The following definition is inspired by the notion of relative homotopy in a fibration category [**Bau89**].

Definition 4.1. Let $p : E \longrightarrow B$ be a map in $\mathcal{M}$. Consider a kernel pair of p, i.e., the pair $\mathrm{pr}_0, \mathrm{pr}_1 : E \times_p E \longrightarrow E$ of projections in a pullback

$$\begin{tikzcd} E \times_p E \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

Note that both projections are fibrations if $p \in \mathit{fib}$. Any factorisation of the diagonal map (the unit of the pullback) $\Delta_p = (1_E, 1_E) : E \longrightarrow E \times_p E$ as a weak equivalence $\varsigma : E \longrightarrow E^p$ followed by a fibration $(\varepsilon_0, \varepsilon_1) : E^p \longrightarrow E \times_p E$ is called a *path object for p* and is denoted $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$.

Definition 4.2. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "p \circ f = p \circ g"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

in $\mathcal{M}$. We say that f is *right homotopic to g over p* if there is a path object $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$ for p and a map $H : X \longrightarrow E^p$ (a *right homotopy from f to g over p*) such that

$$\begin{cases} \varepsilon_0 \circ H = \mathrm{pr}_0 \circ (\varepsilon_0, \varepsilon_1) \circ H = f \\ \varepsilon_1 \circ H = \mathrm{pr}_1 \circ (\varepsilon_0, \varepsilon_1) \circ H = g. \end{cases}$$

Remark 4.3. If X is cofibrant and $H : X \longrightarrow E^p$ is a right homotopy from f to g over p for some path object $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$, then there is a right homotopy from f to g over p for any path object $((E^p)', \varepsilon_0', \varepsilon_1', \varsigma')$. Thus, in contrast to the fibre homotopy relation $\simeq_p$, which depends on the cylinder object choice, in this case, the relation of right homotopy over p does not depend on the chosen path object for p.

Theorem 4.4. *Let X, E and B be cofibrant objects of $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$. Suppose that on $(\mathcal{M}_c, \mathit{cof} \cap \mathcal{M}_c, \mathit{we} \cap \mathcal{M}_c)$, we have a cylinder object choice*

$$\mathfrak{I} = (X \times I, e_0(X), e_1(X), \sigma(X))_{X \in |\mathcal{M}_c|}.$$

Consider a commutative triangle as in the definition above. Then we have the following:

1. *If $f \simeq_p g$ with respect to $\Im$ then f is right homotopic to g over p.*
2. *Suppose that $p \in fib$. Then $f \simeq_p g$ with respect to $\Im$ if and only if f is right homotopic to g over p.*

Proof. Suppose that $L : X \times I \longrightarrow E$ is a fibre homotopy from f to g over p, and put $K = f \circ \sigma(X) : X \times I \longrightarrow E$. Then K is a fibre homotopy $K : f \simeq_p f$. Let $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$ be a path object for p. Note that $p \circ \sigma(X)$ is a map $X \times I \longrightarrow B$, and that the following diagram of unbroken arrows commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[rr, "\varsigma"] & & E^p \arrow[dd, "(\varepsilon_0,\varepsilon_1)"] \\
X \sqcup X \arrow[d, "e_0(X)+e_1(X)"'] & & & \\
X \times I \arrow[r, "\Delta_{p\circ\sigma(X)}"'] \arrow[urrr, dotted, "h"] & (X \times I) \times_{p\circ\sigma(X)} (X \times I) \arrow[rr, "K\times L"'] & & E \times_p E
\end{tikzcd}$$

Since $e_0 = (e_0(X) + e_1(X)) \circ \mathrm{in}_0$ is a trivial cofibration and $(\varepsilon_0, \varepsilon_1)$ is a fibration, there exists a map $h : X \times I \longrightarrow E^p$ such that the diagram above is commutative. Put $H = h \circ e_1(X) : X \longrightarrow E^p$. We now show that H is a right homotopy from f to g over p:

$$\varepsilon_0 \circ H = \varepsilon_0 \circ h \circ e_1(X) = K \circ e_1(X) = f$$

and

$$\varepsilon_1 \circ H = \varepsilon_1 \circ h \circ e_1(X) = L \circ e_1(X) = g.$$

Now let us assume that f is right homotopic to g over $p \in fib$. Then there exists a path object $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$ for p and a right homotopy $H : X \longrightarrow E^p$ from f to g over p. We note that the map $K = \varsigma \circ f : X \longrightarrow E^p$ is a right homotopy from f to f over p. This yields a commutative diagram such as the diagram of unbroken arrows below.

$$\begin{tikzcd}
X \sqcup X \arrow[r, "K\sqcup H"] \arrow[dd, "e_0(X)+e_1(X)"'] & E^p \sqcup E^p \arrow[r, "\nabla_{E^p}"] & E^p \arrow[d, "(\varepsilon_0,\varepsilon_1)"] \\
& & E \times_p E \arrow[d, "\mathrm{pr}_0"] \\
X \times I \arrow[r, "\sigma(X)"'] \arrow[urr, dotted, "l"] & X \arrow[r, "f"'] & E
\end{tikzcd}$$

Since $e_0(X) + e_1(X)$ is a cofibration and $\varepsilon_0 = \mathrm{pr}_0 \circ (\varepsilon_0, \varepsilon_1)$ is a fibration as well as a weak equivalence, there exists a map $l : X \times I \longrightarrow E^p$ such that the diagram above commutes. Put $L = \varepsilon_1 \circ l : X \times I \longrightarrow E$. Then the equalities

$$L \circ e_0(X) = \varepsilon_1 \circ l \circ e_0(X) = \varepsilon_1 \circ K = f$$

and

$$L \circ e_1(X) = \varepsilon_1 \circ l \circ e_1(X) = \varepsilon_1 \circ H = g$$

show that the map L constitutes a fibre homotopy $L : f \simeq_p g$. $\square$

Example 4.5. In this example, we show that if p does not belong to *fib*, then the two notions of homotopy over p considered in Theorem 4.4 need not coincide. Consider the Strøm model structure on Top with its choice of cylinder objects as in Example 3.1.

Lemma 4.6. *If $p : E \longrightarrow B$ is a map in Top, then a path object $(E^p, \varepsilon_0, \varepsilon_1, \varsigma)$ for p is defined by*

$$\begin{cases} E^p = \{\gamma \in E^I \,|\, p\gamma(t) = p\gamma(1-t) \quad \textit{for all } t \in I\}, \\ (\varepsilon_0, \varepsilon_1) : E^p \longrightarrow E \times_p E : \gamma \longmapsto (\gamma(0), \gamma(1)), \\ \varsigma : E \longrightarrow E^p : e \longmapsto (c_e : I \longrightarrow E : t \longmapsto e). \end{cases}$$

Proof. In Top we can take

$$E \times_p E = \{(e, e') \in E \times E \,|\, p(e) = p(e')\}.$$

Clearly ς and $(\varepsilon_0, \varepsilon_1)$ are continuous maps and $\Delta_p = (\varepsilon_0, \varepsilon_1) \circ \varsigma$. It is easy to see that ς is a homotopy equivalence, since $k \circ \varsigma = 1_E$ and $\varsigma \circ k \simeq 1_{E^p}$ where

$$k : E^p \longrightarrow E : \gamma \longmapsto \gamma(\tfrac{1}{2});$$

indeed if

$$\gamma_t(\tau) = \gamma((1-t)\tau + \tfrac{t}{2}),$$

then $H : E^p \times I \longrightarrow E^p : (\gamma, t) \longmapsto \gamma_t$ defines a homotopy from 1_{E^p} to $\varsigma \circ k$. Moreover the map ς is a weak cofibration, because it embeds E in E^p as a strong deformation retract (here we use [**DKP70**], Satz 2.29; cf. Theorem 6.3): the homotopy H is a homotopy relative $\varsigma(E)$.

Finally, $(\varepsilon_0, \varepsilon_1)$ is a Hurewicz fibration since

$$\Lambda(\gamma, \Gamma)(t)(\tau) = \begin{cases} \mathrm{pr}_0\Gamma(t - 3\tau) & \text{if } 0 \leqslant \tau \leqslant \frac{t}{3} \\ \gamma(\frac{3\tau - t}{3 - 2t}) & \text{if } \frac{t}{3} \leqslant \tau \leqslant 1 - \frac{t}{3} \\ \mathrm{pr}_1\Gamma(t + 3\tau - 3) & \text{if } 1 - \frac{t}{3} \leqslant \tau \leqslant 1 \end{cases}$$

defines a lifting function

$$\Lambda : \{(\gamma, \Gamma) \in E^p \times (E \times_p E)^I \,|\, \Gamma(0) = (\gamma(0), \gamma(1))\} \longrightarrow (E^p)^I$$

for $(\varepsilon_0, \varepsilon_1)$. $\square$

Take $E = I \times \{0\} \cup \{0\} \times I \cup I \times \{1\} \subset I \times I$, $B = I$, $p(x, y) = x$, $f : \{0\} \longrightarrow E : 0 \longmapsto (1, 1)$ and $g : \{0\} \longrightarrow E : 0 \longmapsto (1, 0)$. Then clearly f is right homotopic to g over p but not $f \simeq_p g$!

Remark 4.7. One could formulate Definition 4.1 of path object in a more restrictive manner by asking that the ς be a trivial cofibration (as, for example, in the context of model categories, one sometimes asks path objects to be *very good*; see [**DS95**], Definition 4.2). The relation of right homotopy over a map p in Definition 4.2 then becomes stronger than ours (since we have less path objects) and has the advantage of being independent of the chosen path object (cf. Remark 4.3). But yet, it would not be equivalent to the fibre homotopy relation $\simeq_p$: the topological counterexample in 4.5 still applies, since the map

$$\varsigma : E \longrightarrow E^p : e \longmapsto c_e$$

is a cofibration by Satz 3.26 of [**DKP70**]—$\varsigma(E)$ is the zeroset of the continuous map

$$E^p \longrightarrow \mathbb{R}^+ : \gamma \longmapsto \max_{t,t' \in I} \|\gamma(t) - \gamma(t')\|.$$

5. The dual situation: relative homotopy and weak cofibrations

All concepts introduced and theorems proved in the preceding sections can be dualized. We will give explicit definitions and formulations of theorems for the dual case. First we recall the axioms of a category of fibrant objects, as introduced by K. S. Brown in [**Bro73**].

Definition 5.1. Consider a triple $(\mathcal{F}, \mathit{fib}, \mathit{we})$, where $\mathcal{F}$ is a category with binary products and a terminal object e, and where fib and we are two classes of maps of $\mathcal{F}$. Maps in fib, we and $\mathit{fib} \cap \mathit{we}$ are respectively called *fibrations*, *weak equivalences* and *trivial fibrations*.

Let X be an object of $\mathcal{F}$ and let $\Delta_X = (1_X, 1_X) : X \longrightarrow X \times X$ denote the diagonal morphism. A *cocylinder object* $(X^I, \epsilon_0, \epsilon_1, s)$ on X consists of an object X^I of $\mathcal{F}$ and maps

$$\epsilon_0, \epsilon_1 : X^I \longrightarrow X, \qquad s : X \longrightarrow X^I$$

such that the map $(\epsilon_0, \epsilon_1) : X^I \longrightarrow X \times X$ is a fibration, s is a weak equivalence and $(\epsilon_0, \epsilon_1) \circ s = \Delta_X$.

The triple $(\mathcal{F}, \mathit{fib}, \mathit{we})$ is called a *category of fibrant objects* if the following axioms hold.

F1 Any isomorphism is a weak equivalence. For two morphisms f and g in $\mathcal{F}$ such that $g \circ f$ exists, two out of three morphisms f, g and $g \circ f$ being a weak equivalence implies that the third morphism is a weak equivalence.

F2 Any isomorphism is a fibration and the class fib is closed under composition.

F3 Given any pair of maps $i : A \longrightarrow X$, $u : B \longrightarrow X$ with $i \in \mathit{fib}$ the pullback

$$\begin{tikzcd} A \times_X B \arrow[r, "\overline{\imath}"] \arrow[d, "\overline{u}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "u"] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

exists and $\overline{\imath}$ is a fibration. If i is trivial, so is $\overline{\imath}$.

F4 For any object X of $\mathcal{F}$ there is a cocylinder object $(X^I, \epsilon_0, \epsilon_1, s)$.

F5 For any object X of $\mathcal{F}$ the unique map $X \longrightarrow e$ is a fibration.

Note that for any cocylinder object $(X^I, \epsilon_0, \epsilon_1, s)$ on an object X of $\mathcal{F}$, the maps ϵ_0 and ϵ_1 are trivial fibrations. We say that ϵ_0, ϵ_1 are *cocylinder fibrations* and that s is a *cocylinder section.*

Note that for any model category $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$, the full subcategory $\mathcal{M}_f$ of all fibrant objects, together with the classes $\mathit{fib} \cap \mathcal{M}_f$ and $\mathit{we} \cap \mathcal{M}_f$ of fibrations, resp. weak equivalences between fibrant objects, forms a category of fibrant objects $(\mathcal{M}_f, \mathit{fib} \cap \mathcal{M}_f, \mathit{we} \cap \mathcal{M}_f)$.

In order to define our notion of relative homotopy in a category of fibrant objects $(\mathcal{F}, \mathit{fib}, \mathit{we})$, we require that a cocylinder object is chosen for each object $X \in |\mathcal{F}|$:

Definition 5.2. If $(\mathcal{F}, \mathit{fib}, \mathit{we})$ is a category of fibrant objects, then a *cocylinder object choice* $\mathfrak{P}$ is a family

$$(X^I, \epsilon_0(X), \epsilon_1(X), s(X))_{X \in |\mathcal{F}|},$$

where for each object X of $\mathcal{F}$, $(X^I, \epsilon_0(X), \epsilon_1(X), s(X))$ is a cocylinder object on X.

Example 5.3. Let $\mathcal{F}$ be a category. A *cocylinder* or *cocylinder functor*

$$\mathbf{P} = ((\cdot)^I, \epsilon_0, \epsilon_1, s)$$

on $\mathcal{F}$ is a functor

$$(\cdot)^I : \mathcal{F} \longrightarrow \mathcal{F}$$

together with natural transformations

$$\epsilon_0, \epsilon_1 : (\cdot)^I \Longrightarrow 1_{\mathcal{F}}, \qquad s : 1_{\mathcal{F}} \Longrightarrow (\cdot)^I$$

such that $\epsilon_0 s = \epsilon_1 s = 1_{1_{\mathcal{F}}}$. Let $(\mathcal{F}, \mathit{fib}, \mathit{we})$ be a category of fibrant objects. A cocylinder $((\cdot)^I, \epsilon_0, \epsilon_1, s)$ on $\mathcal{F}$ is called *suitable* if $(X^I, \epsilon_0(X), \epsilon_1(X), s(X))$ is a cocylinder object on X for all $X \in |\mathcal{F}|$. Let $(\mathcal{M}, \mathit{fib}, \mathit{cof}, \mathit{we})$ be a model category. A cocylinder $(X^I, \epsilon_0(X), \epsilon_1(X), s(X))$ on $\mathcal{M}$ is called *suitable* if $(X^I, \epsilon_0(X), \epsilon_1(X), s(X))$ is a cocylinder object (see [**Qui67**] or [**Hov99**]) on X for all $X \in |\mathcal{M}|$.

If $\mathbf{P}$ is a suitable cocylinder on $(\mathcal{F}, \mathit{fib}, \mathit{we})$, then $\mathbf{P}$ evidently induces a cocylinder object choice on $(\mathcal{F}, \mathit{fib}, \mathit{we})$. Furthermore, note that if $(\mathcal{F}, \mathit{fib}, \mathit{we})$ is a category of fibrant objects generated by a cocylinder $\mathbf{P}$—see [**KP97**]—then $\mathbf{P}$ is automatically suitable.

Dualizing Definition 1.4 gives us the following notion of relative homotopy.

Definition 5.4. Let $(\mathcal{F}, \mathit{fib}, \mathit{we})$ be a category of cofibrant objects equipped with a cocylinder object choice $\mathfrak{P} = (X^I, \epsilon_0(X), \epsilon_1(X), s(X))_{X \in |\mathcal{F}|}$, and let $i : A \longrightarrow X$ be a map in $\mathcal{F}$. Suppose further that we have a commutative diagram as folows:

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "f\circ i = g \circ i"] & \\ X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y. \end{tikzcd}$$

Then we say that f is *homotopic to g under i*, and we write $f \simeq^i g$, if there is a map $H : X \longrightarrow Y^I$ such that

$$\begin{cases} \epsilon_0(X) \circ H = f \\ \epsilon_1(X) \circ H = g \\ H \circ i = s(Y) \circ f \circ i = s(Y) \circ g \circ i. \end{cases}$$

The map H is said to be a *relative homotopy (under i) from f to g*.

If $f : X \longrightarrow Y$ and $i : A \longrightarrow X$ are maps in $\mathcal{F}$, then being relatively homotopic under i is a relation on the set $[f]^i$ of all maps $\overline{f} : X \longrightarrow Y$ such that $\overline{f} \circ i = f \circ i$.

The following dualizes Proposition 1.7.

Proposition 5.5. *Let $(\mathcal{F}, \mathit{fib}, \mathit{we})$ be a category of fibrant objects equipped with a cocylinder object choice $\mathfrak{P}$ and let $i : A \longrightarrow X$ be a map in $\mathcal{F}$. Then the following properties hold:*

1. *For each map $f : X \longrightarrow Y$ in $\mathcal{F}$, we have that $f \simeq^i f$.*
2. *Suppose that we have the following commutative diagram.*

$$\begin{tikzcd} & A \arrow[dl, "i'"'] \arrow[d, "i"] \arrow[dr] & \\ X' \arrow[r, "h"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

If $f \simeq^i g$ then $f \circ h \simeq^{i'} g \circ h$.

Definition 5.6. Let $(\mathcal{F}, \mathit{fib}, \mathit{we})$ be a category of fibrant objects equipped with a cocylinder object choice $\mathfrak{P} = (X^I, \epsilon_0(X), \epsilon_1(X), s(X))_{X \in |\mathcal{F}|}$. Suppose that $i : A \longrightarrow X$ and $p : E \longrightarrow B$ are maps in $\mathcal{F}$. We say that i has the *weak left lifting property* (WLLP) with respect to p if whenever we have a commutative square as below,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd} \qquad \textbf{(E)}$$

then there exists a map $h : X \longrightarrow E$ such that $p \circ h \simeq^i g$ and $h \circ i = f$. A map $i : A \longrightarrow X$ in $\mathcal{F}$ is said to be a *weak cofibration* if it has the WLLP with respect to all weak equivalences $p : E \longrightarrow B$.

Weak cofibrations can be characterised using the *mapping path space factorisation.*

Definition 5.7. Let $(\mathcal{F}, \mathit{fib}, \mathit{we})$ be a category of fibrant objects equipped with a cocylinder object choice $\mathfrak{P}$. Let $f : X \longrightarrow Y$ be a map in $\mathcal{F}$. A *mapping path space* of f is a triple (P_f, π_f, j_f) (sometimes denoted shortly P_f) with $P_f \in |\mathcal{F}|$, and

$\pi_f : P_f \longrightarrow Y^I$ and $j_f : P_f \longrightarrow X$ maps in $\mathcal{F}$, such that the diagram

$$\begin{tikzcd} P_f \arrow[r, "\pi_f"] \arrow[d, "j_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "\epsilon_0(Y)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback in $\mathcal{F}$.

If $f : X \longrightarrow Y$ is a map in $\mathcal{F}$, then a mapping path space for f always exists by **F3**. Being a mapping path space depends on the cocylinder object choice $\mathfrak{P}$. The map j_f is a trivial fibration since $\epsilon_0(Y)$ is. We shall refer to the map $k_f = \epsilon_1(Y) \circ \pi_f : P_f \longrightarrow Y$ as the *mapping path space fibration.* If $f \in we$, then k_f is a trivial fibration.

Let $f : X \longrightarrow Y$ be a map in $\mathcal{F}$ and (P_f, π_f, j_f) a mapping path space of f. Due to pullback properties there is a unique map $q_f : X \longrightarrow P_f$ such that $j_f \circ q_f = 1_X$ and $\pi_f \circ q_f = s(Y) \circ f$. Thus we obtain a factorisation $f = k_f \circ q_f$ of f as a weak equivalence followed by a fibration.

Proposition 5.8. *Let $(\mathcal{F}, fib, we)$ be a category of fibrant objects equipped with a cocylinder object choice $\mathfrak{P}$. Let $p : E \longrightarrow B$ and $i : A \longrightarrow X$ be any maps in $\mathcal{F}$, then the following conditions are equivalent:*

1. *the map i has the* WLLP *with respect to p,*
2. *given the diagram of solid arrows below, then for any mapping path space (P_p, π_p, j_p) for p there exists a map $H : X \longrightarrow P_p$ such that the the diagram* **F** *below commutes,*
3. *given the diagram of solid arrows below, there exists a mapping path space (P_p, π_p, j_p) for p and a map $H : X \longrightarrow P_p$ such that the the diagram* **F** *below commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "i"'] & & E \arrow[dd, "p"] \arrow[dl, dotted, "q_p"'] \\ & P_p \arrow[dr, dotted, "k_p"] & \\ X \arrow[rr, "g"'] \arrow[ur, dotted, "H"] & & B. \end{tikzcd} \qquad (\mathbf{F})$$

Using this characterisation, one proves that in a category of fibrant objects equipped with a cocylinder object choice, the class of weak cofibrations is closed under pushout.

Now we restrict the class of cocylinder object choices in the following way:

Definition 5.9. Let $(\mathcal{F}, fib, we)$ be a category of fibrant objects. Then a cocylinder object choice $\mathfrak{P}$ is called *nice* if, for each map $i : A \longrightarrow X$ in $\mathcal{F}$, we have the following properties.

1. For each map $f : X \longrightarrow Y$ in $\mathcal{F}$, the relation $\simeq^i$ is an equivalence relation on $[f]^i$.

2. Suppose that we have the following commutative diagram.

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[d] \arrow[dr] & \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"'] & Y' \end{tikzcd}$$

If $f \simeq^i g$ then $h \circ f \simeq^i h \circ g$.

One can prove that in a category of fibrant objects equipped with a nice cocylinder object choice, the class of weak cofibrations is closed under composition. The following proposition gives an interesting situation in which a category of fibrant objects can be equipped with a nice cocylinder object choice.

Proposition 5.10. *Let $(\mathcal{F}, fib, we)$ be a category of fibrant objects equipped with a suitable cocylinder $\mathbf{P} = ((\cdot)^I, \epsilon_0, \epsilon_1, s)$ which satisfies the Kan filler condition* DNE(2, 1, 1). *Then the cocylinder object choice induced by $\mathbf{P}$ is nice.*

For the remaining part of this section, we suppose that the category of fibrant objects $(\mathcal{F}, fib, we)$ is equipped with a nice cocylinder object choice.

Proposition 5.11. *Let $i : A \longrightarrow X$ be a map in $\mathcal{F}$, then the following conditions are equivalent:*

1. *i has the* WLLP *with respect to all $p \in we$,*
2. *i has the* WLLP *with respect to all $p \in fib \cap we$.*

Definition 5.12. Let A be an object of $\mathcal{F}$ and $f : i \longrightarrow i'$ a map in the category ${}^{A}/\mathcal{F}$ of objects under A.

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X \arrow[rr, "f"'] & & X' \end{tikzcd} \qquad (\mathbf{G})$$

f is called a *relative homotopy equivalence* if there exists a *relative homotopy inverse* for f, i.e., a map $g : i' \longrightarrow i$ in ${}^{A}/\mathcal{F}$ such that $g \circ f \simeq^i 1_X$ and $f \circ g \simeq^{i'} 1_{X'}$.

We get the following version of Dold's theorem.

Theorem 5.13. *Suppose that in the commutative triangle of diagram $\mathbf{G}$, i and i' are weak cofibrations. If $f : X \longrightarrow X'$ is a weak equivalence then $f : i \longrightarrow i'$ is a relative homotopy equivalence.*

6. Examples of weak cofibrations

Example 6.1. On the category Top we consider the structure of category of fibrant objects induced by the Strøm model structure: weak equivalences are homotopy equivalences and fibrations are *Hurewicz-fibrations*, maps which have the homotopy lifting property with respect to all topological spaces. The cocylinder-functor

$$(\cdot)^I : \mathsf{Top} \longrightarrow \mathsf{Top} : X \longmapsto X^{[0,1]},$$

which maps a space X to the set of functions $[0,1] \longrightarrow X$ equipped with the compact-open topology, together with the obvious natural transformations is a suitable cocylinder which satisfies DNE(n) for all n.

Recall from [**DKP70**] the following definition: a map $i : A \longrightarrow X$ is said to have the *weak homotopy extension property* or *Homotopieerweiterungseigenschaft bis auf Homotopie* with respect to a topological space B if for any continuous map $f : X \longrightarrow B$ and homotopy $H : A \times I \longrightarrow B$ such that $H \circ e_0(A) = f \circ i$, there exists a homotopy $h : X \times I \longrightarrow B$ such that $h \circ (i \times I) = H$ and $h \circ e_0(X) \simeq^i f$. A map i is called a *(classical) weak cofibration* or *h-Cofaserung* if it has has the weak homotopy extension property with respect to all topological spaces $B \in |\mathsf{Top}|$.

In the category of fibrant objects Top, two maps are relatively homotopic if and only if they are relatively homotopic in the usual topological sense. Moreover, the categorical notion of weak cofibration coincides with the topological notion of classical weak cofibration, as proves the following theorem.

Theorem 6.2. *(cf. [**Kie87**], Theorem 1) Let $i : A \longrightarrow X$ be a map in* Top. *Then the following conditions are equivalent:*

1. *i has the weak homotopy extension property with respect to all $B \in |\mathsf{Top}|$,*
2. *i has the* WLLP *with respect to all $p \in fib \cap we$,*
3. *i has the* WLLP *with respect to all $p \in we$,*
4. *i has the* WLLP *with respect to all $\epsilon_0(B) : B^I \longrightarrow B$.*

Proof. We will use the Strøm model structure on Top and use the names fibration, cofibration and weak equivalence for maps in the respective classes. Suppose that condition (1) holds and that we have a commutative square as $\mathbf{E}$ above, where p is a trivial fibration. We can factor i as a cofibration $\bar{\imath} : A \longrightarrow M$ followed by a trivial fibration $\bar{p} : M \longrightarrow X$. We get a weak lifting $h : M \longrightarrow E$ in the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\bar{\imath}"'] & E \arrow[d, "p"] \\ M \arrow[r, "g \circ \bar{p}"'] & B. \end{tikzcd}$$

Now $\bar{p}$ is a map $\bar{\imath} \longrightarrow i$ in ${}^A/\mathsf{Top}$, p is a homotopy equivalence, $\bar{\imath}$ is a cofibration and i is a classical weak cofibration.

$$\begin{tikzcd} & A \arrow[dl, "\bar{\imath}"'] \arrow[dr, "i"] & \\ M \arrow[rr, "\bar{p}"] & & X \arrow[ll, dotted, bend left, "\tilde{p}"] \end{tikzcd}$$

Thus Dold's theorem [**DKP70**, 2.18] implies that $\overline{p}$ has a relative homotopy inverse $\widetilde{p} : i \longrightarrow \overline{\imath}$. In particular, we have that $\widetilde{p} \circ \overline{p} \simeq^{\overline{\imath}} 1_M$ and $\overline{p} \circ \widetilde{p} \simeq^{i} 1_X$. Put $H = h \circ \widetilde{p} : X \longrightarrow E$. Then $H \circ i = h \circ \widetilde{p} \circ i = h \circ \overline{\imath} = f$ and $p \circ H = p \circ h \circ \widetilde{p} = g \circ \overline{p} \circ \widetilde{p} \simeq^{i} g \circ 1_X = g$. Condition (2) follows.

Condition (3) follows from (2) using Proposition 5.11 and (4) follows from (3) because $\epsilon_0(B) \in we$ for all topological spaces $B \in |\mathsf{Top}|$. Condition (4) implies condition (1) because the functor $(\cdot) \times I$ is left adjoint to $(\cdot)^I$: there is a 1-1 correspondence between diagrams such as on the left below and diagrams such as on the right below.

$$\begin{tikzcd}
& X \arrow[dr, "e_0(X)"'] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "e_0(A)"'] & & X \times I \arrow[r, dotted, "h"] & B \\
& A \times I \arrow[ur, "i \times I"] \arrow[urr, bend right, "H"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\overline{H}"] \arrow[d, "i"'] & B^I \arrow[d, "\epsilon_0(B)"] \\
X \arrow[r, "\overline{f}"'] \arrow[ur, dotted, "\overline{h}"] & B
\end{tikzcd}$$

Moreover if $\overline{h} : X \longrightarrow B^I$ is a weak lifting in the right diagram, then the associated map $h : X \times I \longrightarrow B$ is a weak homotopy extension for the left diagram and vice versa. □

As Strøm's model structure on Top is a closed one, dualizing Proposition 2.7 yields the following dual of [**Kie87**], Theorem 2. Note that it contains Satz 2.29 of [**DKP70**].

Theorem 6.3. *Let $i : A \longrightarrow X$ be a map in* Top. *Then the following conditions are equivalent:*

1. *i is a weak cofibration and a homotopy equivalence,*
2. *i is a strong deformation retraction,*
3. *i has the* WLLP *with respect to all maps $p : E \longrightarrow B$ in* Top.

Example 6.4. Contrary to the dual case (Example 3.2), in the category of fibrant objects $(\mathsf{Top}, \mathit{fib}, we)$ induced by the Quillen model structure on Top, the notion of weak cofibration does *not* coincide with the classical notion, but instead is strictly stronger.

Note that the cocylinder from Example 6.1 is suitable and recall from Example 3.2 that in this model category, every object is fibrant. Clearly, every weak cofibration of $(\mathsf{Top}, \mathit{fib}, we)$ is a classical weak cofibration, because a map that has the WLLP with respect to all elements of we has the WLLP with respect to all homotopy equivalences. The converse is not true. To see this, suppose that every classical weak cofibration is a weak cofibration of $(\mathsf{Top}, \mathit{fib}, we)$, and let $f : X \longrightarrow X'$ be a weak equivalence. Let $\varnothing$ denote the empty topological space, the initial object of

Top, and let i and i' in the diagram below be the unique maps.

$$\begin{tikzcd} & \varnothing \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X \arrow[rr, "f"'] & & X' \end{tikzcd}$$

The maps i and i' are cofibrations in the sense of Example 3.1; hence, they are classical weak cofibrations, thus, by assumption, weak cofibrations of $(\mathsf{Top}, \mathit{fib}, \mathit{we})$. But then Dold's Theorem 5.13 implies that f is a homotopy equivalence. As there exist examples of weak equivalences that are not homotopy equivalences (see, for instance, [**Mau70**], Example 7.5.5), this is a contradiction.

This example proves that our notion of weak cofibration (and, dually, weak fibration) is not determined by the choice of (co)cylinder alone, but also by the given structure of category of (co)fibrant objects. Hence our notion of weak (co)fibration does in general not coincide with Kamps's notion as defined in [**Kam72**]—cf. Proposition 2.9.

Now we can strengthen Theorem 3.3 to the following version of Whitehead's Theorem, and prove it as a result of Dold's Theorem.

Theorem 6.5. *Let X and Y be topological spaces that are homotopy equivalent to cofibrant objects, and let $f : X \longrightarrow Y$ be a continuous map. If f is a weak equivalence then f is a homotopy equivalence.*

Proof. The dual of Theorem 2.5 implies that X and Y are homotopy equivalent to cofibrant objects if and only if the maps $\varnothing \longrightarrow X$ and $\varnothing \longrightarrow Y$ are weak cofibrations. But then Dold's Theorem 5.13 implies that f is a homotopy equivalence. $\square$

Example 6.6. It is easily proved that the category Gpd from Example 3.4 is cartesian closed. In particular, this means that the cylinder functor $(\cdot) \times I : \mathsf{Gpd} \longrightarrow \mathsf{Gpd}$ has a right adjoint $(\cdot)^I : \mathsf{Gpd} \longrightarrow \mathsf{Gpd}$. One can choose $\mathcal{X}^I$ to be a functor category $\mathsf{Fun}(\mathcal{I}, \mathcal{X}) = \mathsf{Gpd}(\mathcal{I}, \mathcal{X})$. The obvious natural transformations ϵ_0, ϵ_1 and s such that $\mathbf{P} = ((\cdot)^I, \epsilon_0, \epsilon_1, s)$ is a suitable cocylinder on Gpd are such that $(\mathbf{I}, \mathbf{P})$ is an adjoint cylinder/cocylinder pair ([**KP97**], II.3.5). Using [**KP97**], Proposition II.3.7, we get that $\mathbf{P}$ satisfies DNE$(2, 1, 1)$.

The structure of a category of fibrant objects associated to the model structure on Gpd, equipped with the nice cocylinder object choice induced by $\mathbf{P}$, gives rise to a notion of weak cofibration; as in the dual case, the weak cofibrations are exactly the cofibrations.

Proposition 6.7. *Let $i : \mathcal{A} \longrightarrow \mathcal{X}$ be a map of groupoids. If i is a weak cofibration then i is cofibration.*

Proof. Let a and a' be two objects of $\mathcal{A}$ such that $i(a) = i(a')$ and suppose that

$a \neq a'$. We define a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "i"'] & \mathcal{I} \arrow[d] \\ \mathcal{X} \arrow[r] & * \end{tikzcd}$$

by choosing $f(a') = 1$ and $f(a'') = 0$ for all objects $a'' \neq a'$ of $\mathcal{A}$. The unique functor $\mathcal{I} \longrightarrow *$ is a weak equivalence and i is a weak cofibration, so there is a functor $h : \mathcal{X} \longrightarrow \mathcal{I}$ such that $h \circ i = f$. But then $f(a) = f(a')$, a contradiction. If follows that $a = a'$ and i is a cofibration. $\square$

7. The case of chain complexes in an abelian category

In this last section we give a characterisation of the weak fibrations and weak cofibrations of unbounded chain complexes in an abelian category $\mathcal{A}$ that one gets when applying our definition to the model structures defined in [**CH02**]. We start by fixing some notations and giving a short description of these model structures.

Let $\mathcal{A}$ be an abelian category, for instance the category ${}_R\mathsf{Mod}$ of left R-modules over a ring R and R-linear maps. A *homomorphism* $f : C. \longrightarrow D.$ *of degree* n between *graded objects* $C.$ and $D.$, i.e. collections of objects $C. = (C_n)_{n\in\mathbb{Z}}$ and $D. = (D_n)_{n\in\mathbb{Z}}$ of $\mathcal{A}$, is a collection of maps $f = (f_i : C_i \longrightarrow D_{i+n})_{i\in\mathbb{Z}}$. We denote the abelian group of homomorphisms of degree n as $\mathsf{Ch.}(\mathcal{A})(C., D.)_n$. A *chain complex* then is a graded object $C.$ together with a homomorphism $d : C. \longrightarrow C.$ of degree -1 (its *differential*) such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$, and a *morphism* between chain complexes $C.$ and $D.$ is a homomorphism $f : C. \longrightarrow D.$ of degree 0 such that $f_{n-1} \circ d_n = d_n \circ f_n$ for all $n \in \mathbb{Z}$. Chain complexes and morphisms between them—morphisms are composed degreewise—form a category we denote $\mathsf{Ch.}(\mathcal{A})$ and $\mathsf{Ch.}(R) = \mathsf{Ch.}({}_R\mathsf{Mod})$. Dually, the category of *cochain complexes* and *cochain morphisms* will be denoted $\mathsf{Ch}^{\cdot}(\mathcal{A})$ and $\mathsf{Ch}^{\cdot}(R) = \mathsf{Ch}^{\cdot}({}_R\mathsf{Mod})$. Given objects $C.$ and $D.$ of $\mathsf{Ch.}(\mathcal{A})$, the graded object $\mathsf{Ch.}(\mathcal{A})(C., D.). = (\mathsf{Ch.}(\mathcal{A})(C., D.)_n)_{n\in\mathbb{Z}}$ has a natural structure of chain complex of $\mathbb{Z}$-modules with differential

$$\begin{array}{rlcl} d_n : & \mathsf{Ch.}(\mathcal{A})(C., D.)_n & \longrightarrow & \mathsf{Ch.}(\mathcal{A})(C., D.)_{n-1} : \\ & f = (f_i : C_i \longrightarrow D_{i+n})_{i\in\mathbb{Z}} & \longmapsto & d_n(f) = (d_n(f)_i : C_i \longrightarrow D_{i+n-1})_{i\in\mathbb{Z}}, \end{array}$$

where $d_n(f)_i = d_{i+n} \circ f_i + (-1)^{n-1} f_{i-1} \circ d_i$. For a given chain complex $C.$ we denote the object of *n-cycles* as $Z_nC. = \ker d_n$, the object of *n-boundaries* as $B_nC. = \operatorname{im} d_{n+1}$ and the *n-th homology* as $H_nC. = Z_nC./B_nC.$. If $\mathcal{A}$ is ${}_R\mathsf{Mod}$, the homology class of an n-cycle $z \in Z_nC.$ is denoted $[z]$. Note that for chain complexes $C.$ and $D.$, the 0-cycles of $\mathsf{Ch.}(\mathcal{A})(C., D.).$ are exactly the morphisms $C. \longrightarrow D.$. Homology induces exact functors $H_n : \mathsf{Ch.}(\mathcal{A}) \longrightarrow \mathcal{A}$. A *homology isomorphism* or *quasi-isomorphism* is a map $f : C. \longrightarrow D.$ of $\mathsf{Ch.}(\mathcal{A})$ such that $H_nf : H_nC. \longrightarrow H_nD.$ is an isomorphism for all $n \in \mathbb{Z}$. Two chain maps $f, g : C. \longrightarrow D.$ are called *chain homotopic*, notation $f \simeq g$, if there exists a homomorphism $H : C. \longrightarrow D.$ of degree 1 such that $f_n - g_n = d_1(H)_n = d_{n+1} \circ H_n + H_{n-1} \circ d_n$. A map is said to be *nullhomotopic* if it is chain homotopic to 0. A map $f : C. \longrightarrow D.$ is a *chain homotopy* *equivalence* if there is a $g : D. \longrightarrow C.$ with $g \circ f \simeq 1_{C.}$ and $f \circ g \simeq 1_{D.}$. A complex $C.$ is called *contractible* if the unique map $C. \longrightarrow 0.$ is a chain homotopy equivalence. Any chain homotopy equivalence is a quasi-isomorphism. Finally, for a chain complex $C.$, let $\Sigma^n C.$ denote its *n-fold suspension*, the chain complex given by $(\Sigma^n C.)_i = C_{i-n}$ and $d_i^{\Sigma^n C.} = (-1)^n d_{i-n}$, for $i \in \mathbb{Z}$; recall that the *mapping cone* $\operatorname{cone} f$ of a chain map $f : C. \longrightarrow D.$ is the chain complex given by $(\operatorname{cone} f)_n = C_{n-1} \oplus D_n$ and

$$d_n = \begin{pmatrix} -d_{n-1}^{C.} & 0 \\ -f_{n-1} & d_n^{D.} \end{pmatrix} : (\operatorname{cone} f)_n \longrightarrow (\operatorname{cone} f)_{n-1},$$

for $n \in \mathbb{Z}$.

In the article [**CH02**], model structures are defined on $\mathsf{Ch.}(\mathcal{A})$ with respect to a given *projective class*; this consists of a class of $\mathcal{A}$-objects one thinks of as the class of projective objects, together with a class of $\mathcal{A}$-maps one thinks of as the class of epimorphisms. This notion was originally introduced by Maranda in [**Mar64**].

Definition 7.1. [**CH02**, Definition 1.1], [**Mar64**] Let $\mathcal{C}$ be a category. Let P be an object of $\mathcal{C}$. Then a map $f : A \longrightarrow B$ is called *P-epic* if the induced map

$$\mathcal{C}(P, f) = f \circ (\cdot) : \mathcal{C}(P, A) \longrightarrow \mathcal{C}(P, B)$$

is a surjection. Given a class $\mathcal{P}$ of objects of $\mathcal{C}$, f is said to be *$\mathcal{P}$-epic* if it is P-epic for all P in $\mathcal{P}$.

A *projective class on* $\mathcal{C}$ is a class $\mathcal{P}$ of objects of $\mathcal{C}$ together with a class $\mathcal{E}$ of maps of $\mathcal{C}$ such that

1. $\mathcal{E}$ is the collection of all $\mathcal{P}$-epic maps,
2. $\mathcal{P}$ is the collection of all objects P such that each map in $\mathcal{E}$ is $\mathcal{P}$-epic,
3. for each object B there is a map $P \longrightarrow B$ in $\mathcal{E}$ with P in $\mathcal{P}$.

An object of $\mathcal{P}$ is called *$\mathcal{P}$-projective*.

Since the class $\mathcal{P}$ determines $\mathcal{E}$ we will sometimes speak of *the projective class* $\mathcal{P}$. If $\mathcal{A}$ is an abelian category with enough projectives then $(\mathcal{P}, \mathcal{E})$, where $\mathcal{P}$ is the class of projectives and $\mathcal{E}$ is the class of epimorphisms, forms a projective class on $\mathcal{A}$, called the *categorical* projective class. Dualizing the definition of projective class gives rise to a notion of *injective class* on a category $\mathcal{C}$. Note that a map $f : A \longrightarrow B$ is *I-monic* if the induced map $\mathcal{C}(f, I) = (\cdot) \circ f$ is a surjection. If $\mathcal{A}$ is an abelian category with enough injectives, then the class of injectives together with the class of monomorphisms forms an injective class on $\mathcal{A}$, the *categorical* injective class.

Given a projective class $\mathcal{P}$ or an injective class $\mathcal{I}$ on an abelian category $\mathcal{A}$, Christensen and Hovey construct a model structure on the category $\mathsf{Ch.}(\mathcal{A})$ of unbounded chain complexes in $\mathcal{A}$ as follows.

Definition 7.2. [**CH02**, Definition 2.1] Let $\mathcal{A}$ be an abelian category. For any object A of $\mathcal{A}$, there are hom-functors

$$\mathcal{A}(A, \cdot). : \mathsf{Ch.}(\mathcal{A}) \longrightarrow \mathsf{Ch.}(\mathbb{Z}) \qquad \text{and} \qquad \mathcal{A}(\cdot, A)^{\cdot} : \mathsf{Ch.}(\mathcal{A}) \longrightarrow \mathsf{Ch}^{\cdot}(\mathbb{Z}).$$

Let $\mathcal{P}$ be a projective class on $\mathcal{A}$. A map $f : X. \longrightarrow Y.$ in $\mathsf{Ch.}(\mathcal{A})$ is a *$\mathcal{P}$-equivalence* if the chain map $\mathcal{A}(P, f).$ is a homology isomorphism (of abelian groups) for each P

in $\mathcal{P}$. The map f is a *$\mathcal{P}$-fibration* if $\mathcal{A}(P, f).$ is an epimorphism of $\mathsf{Ch}.(\mathbb{Z})$ for each P in $\mathcal{P}$. f is a *$\mathcal{P}$-cofibration* if it has the left lifting property with respect to all maps that are both $\mathcal{P}$-fibrations and $\mathcal{P}$-equivalences (the *$\mathcal{P}$-trivial fibrations*). A complex $C.$ is called *$\mathcal{P}$-cofibrant* if the unique map $0. \longrightarrow C.$ is a $\mathcal{P}$-cofibration. A complex $K.$ is called *weakly $\mathcal{P}$-contractible* if the map $K. \longrightarrow 0.$ is a $\mathcal{P}$-equivalence.

Dually, let $\mathcal{I}$ be an injective class on $\mathcal{A}$. A map $f : X. \longrightarrow Y.$ in $\mathsf{Ch}.(\mathcal{A})$ is an *$\mathcal{I}$-equivalence* if the cochain map $\mathcal{A}(f, I)^{\cdot}$ is a cohomology isomorphism (of abelian groups) for each I in $\mathcal{I}$. The map f is an *$\mathcal{I}$-cofibration* if $\mathcal{A}(f, I)^{\cdot}$ is an epimorphism of $\mathsf{Ch}^{\cdot}(\mathbb{Z})$ for each I in $\mathcal{I}$. f is an *$\mathcal{I}$-fibration* if it has the right lifting property with respect to all maps that are both $\mathcal{I}$-cofibrations and $\mathcal{I}$-equivalences (the *$\mathcal{I}$-trivial cofibrations*). A complex $C.$ is called *$\mathcal{I}$-fibrant* if the unique map $C. \longrightarrow 0.$ is an $\mathcal{I}$-fibration. A complex $K.$ is called *weakly $\mathcal{I}$-contractible* if the map $0. \longrightarrow K.$ is an $\mathcal{I}$-equivalence.

Theorem 2.2 of [**CH02**] gives hypotheses for the classes of $\mathcal{P}$-fibrations, $\mathcal{P}$-cofibrations and $\mathcal{P}$-equivalences to form a model structure

$$(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{cof}(\mathcal{P}), \mathit{we}(\mathcal{P}))$$

on $\mathsf{Ch}.(\mathcal{A})$, the *$\mathcal{P}$-model structure*. Clearly, all of its objects are fibrant, and any chain homotopy equivalence is a $\mathcal{P}$-equivalence. Proposition 2.5 of [**CH02**] states that a map is a $\mathcal{P}$-cofibration exactly when it is a degreewise split monomorphism with a $\mathcal{P}$-cofibrant cokernel, and Lemma 2.4 that a chain complex $C.$ is $\mathcal{P}$-cofibrant if and only if each C_n is $\mathcal{P}$-projective and every map from $C.$ to a weakly $\mathcal{P}$-contractible object $K.$ is nullhomotopic. A detailed and direct proof that $\mathsf{Ch}.(R)$ with the categorical projective class on ${}_R\mathsf{Mod}$—the *projective model structure* on ${}_R\mathsf{Mod}$—forms a model category can be found in [**Hov99**, Section 2.3]. Its cofibrant objects are exactly the *DG-projective* chain complexes of [**AFL93**]; these are chain complexes $C.$ such that C_n is projective for all $n \in \mathbb{Z}$ and the functor $\mathsf{Ch}.(R)(C., \cdot). : \mathsf{Ch}.(R) \longrightarrow \mathsf{Ch}.(\mathbb{Z})$ preserves homology isomorphisms.

The dual model structure on $\mathsf{Ch}.(\mathcal{A})$, this time obtained from an injective class $\mathcal{I}$, is called the *$\mathcal{I}$-model structure on* $\mathsf{Ch}.(\mathcal{A})$, and is denoted

$$(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{I}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I})).$$

All of its objects are cofibrant and a map is an $\mathcal{I}$-fibration if and only if it is a degreewise split epimorphism with an $\mathcal{I}$-fibrant kernel; any chain homotopy equivalence is an $\mathcal{I}$-equivalence. Of course, $\mathcal{A}$ and the injective class must satisfy some hypotheses for this model structure to exist, for instance the dual of the hypotheses of [**CH02**, Theorem 2.2]. But also the *injective model structure* on $\mathsf{Ch}.(\mathcal{A})$, for $\mathcal{A}$ a Grothendieck category, constructed in [**Hov01**] is of this form: it is the model structure obtained from the categorical injective class. Its cofibrations are monomorphisms and its weak equivalences are homology isomorphisms. If $\mathcal{A}$ is ${}_R\mathsf{Mod}$ and $\mathcal{I}$ is the categorical injective class, then the $\mathcal{I}$-fibrant objects are exactly the *DG-injective* chain complexes of [**AFL93**]; these are chain complexes $C.$ such that C_n is injective for all $n \in \mathbb{Z}$ and the functor $\mathsf{Ch}.(R)(\cdot, C.). : \mathsf{Ch}.(R) \longrightarrow \mathsf{Ch}.(\mathbb{Z})$ preserves homology isomorphisms.

From now on we suppose that we work in $\mathsf{Ch}.(\mathcal{A})$ for $\mathcal{A}$ an abelian category, equipped with a model structure such as in Definition 7.2 above. We will define suitable cylinder and cocylinder functors for the associated categories of (co)fibrant objects, and characterise the resulting weak (co)fibrations. The standard cylinder $((\cdot)\times I, e_0, e_1, \sigma)$ on $\mathsf{Ch}.(\mathcal{A})$ is such that two chain maps are homotopic with respect to it (in the sense of [**KP97**]) if and only if they are chain homotopic. According to [**KP97**, Section III.3], this cylinder satisfies DNE$(2,1,1)$. Here the functor $(\cdot)\times I : \mathsf{Ch}.(\mathcal{A}) \longrightarrow \mathsf{Ch}.(\mathcal{A})$ is given by the equalities $(C.\times I)_n = C_n \oplus C_n \oplus C_{n-1}$ and

$$d_n = \begin{pmatrix} d_n & 0 & -1 \\ 0 & d_n & 1 \\ 0 & 0 & -d_{n-1} \end{pmatrix} : (C.\times I)_n \longrightarrow (C.\times I)_{n-1},$$

and the natural transformations e_0, e_1 and σ by

$$e_0(C.)_n = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : C_n \longrightarrow (C.\times I)_n, \quad e_1(C.)_n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : C_n \longrightarrow (C.\times I)_n$$

and $\sigma(C.)_n = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} : (C.\times I)_n \longrightarrow C_n$, for $C.$ in $\mathsf{Ch}.(\mathcal{A})$ and $n \in \mathbb{Z}$.

Let $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{cof}(\mathcal{P}), \mathit{we}(\mathcal{P}))$ be a model category obtained from a projective class $\mathcal{P}$ on $\mathcal{A}$. In [**CH02**, Lemma 2.13], it is shown that $e_0(C.) + e_1(C.) : C. \sqcup C. \longrightarrow C.\times I$ is a $\mathcal{P}$-cofibration as soon as $C.$ is $\mathcal{P}$-cofibrant, and $\sigma(C.)$, being a chain homotopy equivalence, is a $\mathcal{P}$-equivalence for all objects $C.$. It follows that $((\cdot)\times I, e_0, e_1, \sigma)$ suits the category of cofibrant objects $(\mathsf{Ch}.(\mathcal{A})_c, \mathit{cof}(\mathcal{P}) \cap \mathsf{Ch}.(\mathcal{A})_c, \mathit{we}(\mathcal{P}) \cap \mathsf{Ch}.(\mathcal{A})_c)$, because restricting the functor $(\cdot)\times I$ to $\mathsf{Ch}.(\mathcal{A})_c$ also corestricts it to $\mathsf{Ch}.(\mathcal{A})_c$. Dually, let $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{I}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$ be a model category obtained from an injective class $\mathcal{I}$ on $\mathcal{A}$, and let $C.$ be a chain complex in $\mathcal{A}$. Since $e_0(C.) + e_1(C.) : C. \sqcup C. \longrightarrow C.\times I$ is a degreewise split monomorphism, it is an $\mathcal{I}$-cofibration; the chain homotopy equivalence $\sigma(C.)$ is an $\mathcal{I}$-equivalence. It follows immediately that $((\cdot)\times I, e_0, e_1, \sigma)$ suits the category of cofibrant objects $(\mathsf{Ch}.(\mathcal{A}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$.

In the category of cofibrant objects $(\mathsf{Ch}.(\mathcal{A})_c, \mathit{cof}(\mathcal{P})\cap\mathsf{Ch}.(\mathcal{A})_c, \mathit{we}(\mathcal{P})\cap\mathsf{Ch}.(\mathcal{A})_c)$ equipped with the suitable cylinder defined above, the notion of weak fibration coincides with the notion of $\mathcal{P}$-fibration (between $\mathcal{P}$-cofibrant chain complexes). To prove this, consider a weak fibration $p : E. \longrightarrow B.$, and let P be a $\mathcal{P}$-projective object. We must show that for all $n \in \mathbb{Z}$, $p_n \circ (\cdot) : \mathcal{A}(P, E_n) \longrightarrow \mathcal{A}(P, B_n)$ is a surjection. Take $f : P \longrightarrow B_n$ in $\mathcal{A}(P, B_n)$. Let $D^n P$ denote the chain complex that is P in degrees n and $n-1$ and 0 elsewhere, and has differential $d_n = 1_P$. Then $D^n P$ is cofibrant, being a bounded below complex of $\mathcal{P}$-projectives (see [**CH02**, Lemma 2.7]). Moreover $0. \longrightarrow D^n P$ is a homotopy equivalence, thus also an $\mathcal{I}$-equivalence. Hence the commutative square

$$\begin{tikzcd} 0. \arrow[r] \arrow[d] & E. \arrow[d, "p"] \\ D^n P \arrow[r, "g"'] & B., \end{tikzcd}$$

in which the map $g : D^n P \longrightarrow B.$ is given by $g_n = f$ and $g_{n-1} = d_n \circ f$, has a lifting $h : D^n P \longrightarrow E.$, and $p_n \circ h_n = f$.

Note that in a similar way, one shows that the weak fibrations or $\mathcal{P}$-fibrations between $\mathcal{P}$-cofibrant, hence degreewise $\mathcal{P}$-projective, objects are exactly the degreewise split epimorphisms.

The weak fibrations of $(\mathsf{Ch}.(\mathcal{A}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$ equipped with the suitable cylinder defined above, are characterised by the following statements, of which the dual will be proved below (Proposition 7.6, Corollary 7.7 and Theorem 7.8).

Proposition 7.3. *Let $C.$ be a chain complex in $(\mathsf{Ch}.(\mathcal{A}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$. Then the following are equivalent:*

1. *$C.$ is* weakly $\mathcal{I}$-fibrant, *i.e., $C. \longrightarrow 0.$ is a weak fibration,*
2. *the functor $\mathsf{Ch}.(\mathcal{A})(\cdot, C).: \mathsf{Ch}.(\mathcal{A}) \longrightarrow \mathsf{Ch}.(\mathbb{Z})$ maps $\mathcal{I}$-equivalences to homology isomorphisms.*

This means that our notion of weakly $\mathcal{I}$-fibrant object is a generalisation of the *K-injective* objects of Spalenstein [**Spa88**].

Proposition 7.4. *Let $C.$ be a chain complex in $(\mathsf{Ch}.(\mathcal{A}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$. Then the following are equivalent:*

1. *$C.$ is $\mathcal{I}$-fibrant,*
2. *$C.$ is degreewise $\mathcal{I}$-injective and weakly $\mathcal{I}$-fibrant.*

Theorem 7.5. *Let $p : E. \longrightarrow B.$ be a chain map in $(\mathsf{Ch}.(\mathcal{A}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$. Then the following are equivalent:*

1. *$p : E. \longrightarrow B.$ is a weak fibration,*
2. *$p : E. \longrightarrow B.$ is a degreewise split epimorphism with weakly $\mathcal{I}$-fibrant kernel.*

It follows that a map is an $\mathcal{I}$-fibration exactly when it is a weak fibration with a degreewise $\mathcal{I}$-injective kernel. As an immediate consequence of Dold's Theorem 2.3, we get that any $\mathcal{I}$-equivalence between weakly $\mathcal{I}$-fibrant objects is a homotopy equivalence, as well as any weak fibration that is also an $\mathcal{I}$-equivalence. As shown in the dual case (Proposition 7.9), the classes of weak fibrations and $\mathcal{I}$-fibrations coincide if and only if $\mathcal{I}$ is the *trivial* injective class, i.e., $\mathcal{I} = |\mathcal{A}|$ and the $\mathcal{I}$-monos are the split monomorphisms.

Now we consider the dual case. The model structures described above determine structures of category of fibrant objects on $\mathsf{Ch}.(\mathcal{A})$. But in order to speak of weak cofibrations we also need to have a cocylinder on $\mathsf{Ch}.(\mathcal{A})$. Again, the standard cocylinder $((\cdot)^I, \epsilon_0, \epsilon_1, s)$ from [**KP97**] has the property that two maps are homotopic with respect to it if and only if they are chain homotopic, and it satisfies DNE$(2,1,1)$. Here the functor $(\cdot)^I : \mathsf{Ch}.(\mathcal{A}) \longrightarrow \mathsf{Ch}.(\mathcal{A})$ is given by the equalities $(C.^I)_n = C_n \oplus C_n \oplus C_{n+1}$ and

$$d_n = \begin{pmatrix} d_n & 0 & 0 \\ 0 & d_n & 0 \\ 1 & -1 & -d_{n+1} \end{pmatrix} : C^I_n \longrightarrow C^I_{n-1},$$

and the natural transformations ϵ_0, ϵ_1 and s by

$$s(C.)_n = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : C_n \longrightarrow C_n^I$$

and $\epsilon_0(C.)_n = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\epsilon_1(C.)_n = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} : C_n^I \longrightarrow C_n$, for $n \in \mathbb{Z}$ and $C.$ in $\mathsf{Ch}.(\mathcal{A})$.

Let $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{I}), \mathit{cof}(\mathcal{I}), \mathit{we}(\mathcal{I}))$ be a model category obtained from an injective class $\mathcal{I}$ on $\mathcal{A}$. For any chain complex $C.$, the map $s(C.)$ is a chain homotopy equivalence, hence an $\mathcal{I}$-equivalence; the map $(\epsilon_0(C.), \epsilon_1(C.)) : C_.^I \longrightarrow C. \times C.$ is an $\mathcal{I}$-fibration as soon as $C.$ is $\mathcal{I}$-fibrant. Now let $(\mathsf{Ch}.(\mathcal{A})_f, \mathit{fib}(\mathcal{I}) \cap \mathsf{Ch}.(\mathcal{A})_f, \mathit{we}(\mathcal{I}) \cap \mathsf{Ch}.(\mathcal{A})_f)$ denote the category of fibrant objects associated with this model structure. The restriction of $(\cdot)^I$ to $\mathsf{Ch}.(\mathcal{A})_f$ corestricts it to $\mathsf{Ch}.(\mathcal{A})_f$ because $0 = 0 \circ (\epsilon_0(C.), \epsilon_1(C.)) : C_.^I \longrightarrow 0.$, and for an $\mathcal{I}$-fibrant chain complex $C.$, $C. \times C.$ is $\mathcal{I}$-fibrant. As proved dually above, in this category of fibrant objects, the notion of weak cofibration coincides with the notion of $\mathcal{I}$-cofibration (between $\mathcal{I}$-fibrant chain complexes). Moreover, a map between $\mathcal{I}$-fibrant chain complexes is an $\mathcal{I}$-cofibration exactly when it is a degreewise split monomorphism.

Now reconsider the model structure $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{cof}(\mathcal{P}), \mathit{we}(\mathcal{P}))$ generated by a projective class $\mathcal{P}$. The cocylinder functor $(\cdot)^I$ defined above suits the category of fibrant objects $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{we}(\mathcal{P}))$ associated with this model category. Its weak cofibrations can be characterized by the following statements.

Proposition 7.6. *Let $C.$ be a chain complex in $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{we}(\mathcal{P}))$. Then the following are equivalent:*

1. *$C.$ is* weakly $\mathcal{P}$-cofibrant, *i.e., the map $0. \longrightarrow C.$ is a weak cofibration,*
2. *any map from $C.$ to a weakly $\mathcal{P}$-contractible object $K.$ is nullhomotopic,*
3. *for every weakly $\mathcal{P}$-contractible object $K.$, the chain complex $\mathsf{Ch}.(\mathcal{A})(C., K.).$ is acyclic,*
4. *the functor $\mathsf{Ch}.(\mathcal{A})(C., \cdot). : \mathsf{Ch}.(\mathcal{A}) \longrightarrow \mathsf{Ch}.(\mathbb{Z})$ maps $\mathcal{P}$-equivalences to homology isomorphisms,*
5. *given chain maps*

$$\begin{tikzcd} & E. \arrow[d, "p"] \\ C. \arrow[r, "f"'] & B. \end{tikzcd}$$

such that p is a $\mathcal{P}$-equivalence, there exists a map $h : C. \longrightarrow E.$ such that $p \circ h \simeq f$; moreover, this map h is unique up to chain homotopy.

Proof. This is just a reformulation and slight generalization of [**Spa88**, Proposition 1.4]. To show that (1) implies (2), let $f : C. \longrightarrow K.$ be a map from $C.$ to a weakly $\mathcal{P}$-contractible object $K.$. Then the map $0 : 0. \longrightarrow K.$ in the diagram below is a

$\mathcal{P}$-equivalence by axiom **F1**.

$$\begin{tikzcd} 0. \arrow[r, equal] \arrow[d] & 0. \arrow[d, "0"] \\ C. \arrow[r, "f"'] & K. \end{tikzcd}$$

It follows that there is a map $h : C. \longrightarrow 0.$ such that $f \simeq 0 \circ h = 0$.

Now suppose that (2) holds and let $K.$ be any chain complex. Note that, for $n \in \mathbb{Z}$, the abelian group $H_n\mathsf{Ch}.(\mathcal{A})(C., K.).$ consists of the chain homotopy classes of morphisms $C. \longrightarrow \Sigma^{-n}K..$ Let $K.$ be weakly $\mathcal{P}$-contractible. But then $\Sigma^{-n}K.$ is weakly $\mathcal{P}$-contractible for any $n \in \mathbb{Z}$; hence, condition (2) implies that $\mathsf{Ch}.(\mathcal{A})(C., K.).$ is acyclic, and condition (3) holds.

Next, suppose that (3) holds, and let $p : E. \longrightarrow B.$ be a $\mathcal{P}$-equivalence. Then the left exactness of $\mathsf{Ch}.(\mathcal{A})(C., \cdot).$ already implies that the group homomorphism

$$H_n\mathsf{Ch}.(\mathcal{A})(C., p). : H_n\mathsf{Ch}.(\mathcal{A})(C., E.). \longrightarrow H_n\mathsf{Ch}.(\mathcal{A})(C., B.).$$

is a monomorphism for any $n \in \mathbb{Z}$. To see that it is an epimorphism as well, take $[f] \in H_n\mathsf{Ch}.(\mathcal{A})(C., B.)..$ This $[f]$ is the chain homotopy class of a morphism $f : C. \longrightarrow \Sigma^{-n}B..$ Note that, because p is a $\mathcal{P}$-equivalence, the mapping cone $\operatorname{cone} p$ of p is a weakly $\mathcal{P}$-contractible chain complex; thus, (3) implies that the complex $\mathsf{Ch}.(\mathcal{A})(C., \operatorname{cone} p).$ is acyclic. Hence, the extension of f to a morphism $\overline{f} = (0, f) : C. \longrightarrow \Sigma^{-n}\operatorname{cone} p$ is nullhomotopic. Let H denote the chain homotopy $\overline{f} \simeq 0$, then the homomorphism $h = \mathrm{pr}_{\Sigma^{-n}E.} \circ H : C. \longrightarrow \Sigma^{-n}E.$ of degree 0 is in fact a chain map, and moreover $\Sigma^{-n}p \circ h \simeq f$. It follows that $[f] = [\Sigma^{-n}p \circ h] = H_n\mathsf{Ch}.(\mathcal{A})(C., p).[h]$ and $H_n\mathsf{Ch}.(\mathcal{A})(C., p).$ is an epimorphism of abelian groups. This proves that (4) holds.

Condition (5) follows easily from condition (4), and (5) obviously implies (1). ☐

It is clear that our notion of weakly $\mathcal{P}$-cofibrant object is a generalisation of the *K-projective* objects of Spaltenstein [**Spa88**].

Corollary 7.7. *Let $C.$ be a chain complex in $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{we}(\mathcal{P}))$. Then the following are equivalent:*

1. *$C.$ is $\mathcal{P}$-cofibrant,*
2. *$C.$ is degreewise $\mathcal{P}$-projective and weakly $\mathcal{P}$-cofibrant.*

Proof. This follows immediately from Proposition 7.6, the dual of Corollary 1.19 and [**CH02**, Lemma 2.4]. ☐

Theorem 7.8. *Let $i : A. \longrightarrow X.$ be a chain map in $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{we}(\mathcal{P}))$. Then the following are equivalent:*

1. *$i : A. \longrightarrow X.$ is a weak cofibration,*
2. *$i : A. \longrightarrow X.$ is a degreewise split monomorphism with weakly $\mathcal{P}$-cofibrant cokernel.*

Proof. Suppose that $i : A. \longrightarrow X.$ is a weak cofibration and consider the pushout square

$$\begin{tikzcd} A. \arrow[r] \arrow[d, "i"'] & 0. \arrow[d] \\ B. \arrow[r] & \operatorname{coker} i. \end{tikzcd}$$

As the class of weak cofibrations in a category of fibrant objects is closed under pushout, this already shows that $\operatorname{coker} i$ is weakly $\mathcal{P}$-cofibrant. Now let $D^{n+1}A_n$ denote the chain complex that is A_n in degrees $n+1$ and n and 0 elsewhere, and has differential $d_{n+1} = 1_{A_n}$. Then $D^{n+1}A_n \longrightarrow 0.$ is a $\mathcal{P}$-equivalence, being a homotopy equivalence, and the commutative square

$$\begin{tikzcd} A. \arrow[r, "f"] \arrow[d, "i"'] & D^{n+1}A_n \arrow[d] \\ X. \arrow[r] \arrow[ur, dotted, "r"'] & 0., \end{tikzcd}$$

in which the map $f : A. \longrightarrow D^{n+1}A_n$ is given by $f_n = 1_{A_n}$ and $f_{n+1} = d_{n+1}$, has a lifting $r : X. \longrightarrow D^{n+1}A_n$. We get that $r_n : X_n \longrightarrow A_n$ is a left inverse of $i_n : A_n \longrightarrow X_n$, and i_n is a split monomorphism.

Now suppose that (2) holds. In the commutative square

$$\begin{tikzcd} A. \arrow[r, "f"] \arrow[d, "i"'] & E. \arrow[d, "p"] \\ X. \arrow[r, "g"'] & B., \end{tikzcd}$$

let i be a degreewise split monomorphism with $\mathcal{P}$-cofibrant cokernel $q : X. \longrightarrow C. = \operatorname{coker} i$, and let p be a weak equivalence. We must show that i has the WLLP with respect to p. Now for all $n \in \mathbb{Z}$,

$$0 \longrightarrow A_n \xrightarrow{i_n} X_n \xrightarrow{q_n} C_n \longrightarrow 0$$

is a short exact sequence with i_n a split monomorphism. Thus, q_n is a split epimorphism and we have a commutative diagram

$$\begin{tikzcd} A_n \arrow[r, "i_n"] \arrow[d, equal] & X_n \arrow[r, "q_n"] \arrow[dl, "r_n"] & C_n \arrow[d, equal] \\ A_n & & C_n. \arrow[ul, "s_n"'] \end{tikzcd}$$

Moreover, $X_n \cong A_n \oplus C_n$, where $i_n + s_n : A_n \oplus C_n \longrightarrow X_n$ is an isomorphism with inverse $(r_n, q_n) : X_n \longrightarrow A_n \oplus C_n$. Now the differentials on $X.$ induce a structure of chain complex on $(A_n \oplus C_n)_{n\in\mathbb{Z}}$ such that they are isomorphic; in particular, $\forall n \in \mathbb{Z}$,

$$d_n = (d_n + r_{n-1} \circ d_n \circ s_n, d_n) : A_n \oplus C_n \longrightarrow A_{n-1} \oplus C_{n-1}.$$

The collection $\tau = (\tau_n = r_{n-1} \circ d_n \circ s_n)_{n\in\mathbb{Z}}$ is a homomorphism $C. \longrightarrow A.$ of degree -1 such that for all $n \in \mathbb{Z}$, $d_{n-1}\circ\tau_n+\tau_{n-1}\circ d_n = 0$, because $\tau_n\circ q_n = r_{n-1}\circ d_n - d_n\circ r_n$ by definition of s_n, and consequently $(d_{n-1}\circ\tau_n+\tau_{n-1}\circ d_n)\circ q_n = 0\circ q_n$, and because q_n is epi. This means that $d_{-1}(\tau) = 0$, i.e., $\tau \in \mathsf{Ch}.(\mathcal{A})(C., A.)_{-1}$ is a (-1)-cycle of $\mathsf{Ch}.(\mathcal{A})(C., A.)$.. Also, $\tau \circ q = d_0(-r) \in B_{-1}\mathsf{Ch}.(\mathcal{A})(X., A.)$. is a (-1)-boundary of $\mathsf{Ch}.(\mathcal{A})(X., A.)$.. Moreover $q : X. \longrightarrow C.$ is epi, hence

$$\mathsf{Ch}.(\mathcal{A})(q, A.). : \mathsf{Ch}.(\mathcal{A})(C., A.). \longrightarrow \mathsf{Ch}.(\mathcal{A})(X., A.).$$

is mono, and in particular

$$H_{-1}\mathsf{Ch}.(\mathcal{A})(q, A.). : H_{-1}\mathsf{Ch}.(\mathcal{A})(C., A.). \longrightarrow H_{-1}\mathsf{Ch}.(\mathcal{A})(X., A.).$$

is mono. But $H_{-1}\mathsf{Ch}.(\mathcal{A})(q, A.).[\tau] = [\tau \circ q] = 0$, hence $[\tau] = 0$ and $\tau \in \operatorname{im} d_0$. This implies that there exists a homomorphism $H \in \mathsf{Ch}.(\mathcal{A})(C., A.)_0$ of degree 0 such that for all $n \in \mathbb{Z}$, $\tau_n = d_n \circ H_n - H_{n-1} \circ d_n$.

Because $g : X. \longrightarrow B.$ is a chain morphism, we can write $g_n \circ (i_n + s_n) = (p_n \circ f_n + \sigma_n)$, where $\sigma = (\sigma_n)_{n\in\mathbb{Z}} = (g_n \circ s_n)_{n\in\mathbb{Z}} : C. \longrightarrow B.$ is a homomorphism of degree 0 such that

$$d_n \circ \sigma_n = \sigma_{n-1} \circ d_n + p_{n-1} \circ f_{n-1} \circ \tau_n, \quad \forall n \in \mathbb{Z}.$$

We get that the homomorphism $\sigma - p \circ f \circ H : C. \longrightarrow B.$ of degree 0 is a chain map. Now $C.$ is weakly $\mathcal{P}$-cofibrant; thus, there exists a map $h : C. \longrightarrow E.$ such that $p\circ h \simeq \sigma - p\circ f\circ H$. It follows that there is a homomorphism $K : C. \longrightarrow B.$ of degree 1, yielding for all $n \in \mathbb{Z}$ the equality $p_n\circ h_n - \sigma_n + p_n\circ f_n\circ H_n = d_{n+1}\circ K_n + K_{n-1}\circ d_n$. Now the homomorphism $f \circ r + h \circ q + f \circ H \circ q : X \longrightarrow E$ of degree 0 is in fact a chain map and

$$\begin{aligned} & p_n \circ (f_n \circ r_n + h_n \circ q_n + f_n \circ H_n \circ q_n) - g_n \\ = \quad & d_{n+1} \circ (K_n \circ q_n) + (K_{n-1} \circ q_{n-1}) \circ d_n. \end{aligned}$$

The collection $L = (p_n \circ (f_n \circ r_n + h_n \circ q_n + f_n \circ H_n \circ q_n), g_n, K_n \circ q_n)_{n\in\mathbb{Z}}$ is a relative homotopy $L : p \circ (f \circ r + h \circ q + f \circ H \circ q) \simeq^i g$, and $(f \circ r + h \circ q + f \circ H \circ q) \circ i = f$. This shows that i has the WLLP with respect to p. $\square$

It follows that a map is a $\mathcal{P}$-cofibration exactly when it is a weak cofibration with a degreewise $\mathcal{P}$-projective cokernel. As an immediate consequence of Dold's Theorem 5.13, we get that any $\mathcal{P}$-equivalence between weakly $\mathcal{P}$-cofibrant objects is a chain homotopy equivalence, as well as any weak cofibration that is also a $\mathcal{P}$-equivalence.

To end this section, we now show that the class of weak cofibrations is almost never trivial, in the following sense:

Proposition 7.9. *The class of weak cofibrations of* $(\mathsf{Ch}.(\mathcal{A}), fib(\mathcal{P}), we(\mathcal{P}))$ *coincides with the class of* $\mathcal{P}$*-cofibrations if and only if* $\mathcal{P}$ *is the* trivial *projective class, i.e.,* $\mathcal{P} = |\mathcal{A}|$ *and the* $\mathcal{P}$*-epis are the split epimorphisms.*

Proof. If $\mathcal{P}$ is the trivial projective class, then any chain complex in $\mathcal{A}$ is degreewise $\mathcal{P}$-projective. By Corollary 7.7 then, any weakly $\mathcal{P}$-cofibrant object is $\mathcal{P}$-cofibrant.

But then any weak cofibration is a degreewise split monomorphism with $\mathcal{P}$-cofibrant cokernel, hence a $\mathcal{P}$-cofibration.

Reciprocally, suppose that $A \in |\mathcal{A}| \setminus \mathcal{P}$ is a non-$\mathcal{P}$-projective object. Then the chain complex $C.$ given by $C_n = A$ for any $n \in \mathbb{Z}$, $d_n = 0$ for n even and $d_n = 1_A$ for n odd, is contractible. This implies that $C.$ is weakly $\mathcal{P}$-cofibrant but not $\mathcal{P}$-cofibrant. Consequently, the class of weak cofibrations of $(\mathsf{Ch}.(\mathcal{A}), \mathit{fib}(\mathcal{P}), \mathit{we}(\mathcal{P}))$ strictly contains the class of $\mathcal{P}$-cofibrations. $\square$

References

[AFL93] L. L. Avramov, H.-B. Foxby, and J. Lescot, *Bass series of local ring homomorphisms of finite flat dimension*, Trans. Amer. Math. Soc. **335** (1993), 497–523.

[Bau89] H. J. Baues, *Algebraic homotopy*, Cambridge studies in advanced mathematics, vol. 15, Cambridge University Press, Cambridge, 1989.

[Boo93] I. Booth, *Local to global properties in the theory of fibrations*, Cah. Top. Géom. Diff. Catég. **XXXIV-2** (1993), 127–151.

[Bro73] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458.

[CH02] J. D. Christensen and M. Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Camb. Phil. Soc. **133** (2002), 261–293.

[DKP70] T. tom Dieck, K. H. Kamps, and D. Puppe, *Homotopietheorie*, Lecture notes in mathematics, vol. 157, Springer, 1970.

[Dol63] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. **78** (1963), 223–255.

[DS95] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (I. M. James, ed.), Elsevier Science, 1995.

[HKK96] K. A. Hardie, K. H. Kamps, and R. W. Kieboom, *Dold type theorems in cubical homotopy theory*, Categorical Topology (l'Aquila 1994) (E. Giuli, ed.), Kluwer, Dordrecht, 1996, pp. 139–154.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, 1999.

[Hov01] M. Hovey, *Model category structures on chain complexes of sheaves*, Trans. Amer. Math. Soc. **353** (2001), 3181–3191.

[Jam84] I. M. James, *General topology and homotopy theory*, Springer, 1984.

[Kam72] K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopietheorie*, Math. Z. **124** (1972), 215–236.

[Kie87] R. W. Kieboom, *New lifting properties of weak fibrations*, Analele ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi **XXXIII** (1987), 63–66.

[KP97] K. H. Kamps and T. Porter, *Abstract homotopy and simple homotopy theory*, World Scientific Publishing, Singapore, 1997.

[Mar64] J. M. Maranda, *Injective structures*, Trans. Amer. Math. Soc. **110** (1964), 98–135.

[Mau70] C. R. F. Maunder, *Algebraic topology*, The New University Mathematics Series, Van Nostrand Reinhold, London, 1970.

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture notes in mathematics, vol. 43, Springer, 1967.

[Spa88] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.

[Str72] A. Strøm, *The homotopy category is a homotopy category*, Arch. Math. **23** (1972), 435–441.

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