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EXTENSIONS OF HOMOGENEOUS COORDINATE RINGS TO A_∞-ALGEBRAS

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(*communicated by James Stasheff*)

Abstract

We study A_∞-structures extending the natural algebra structure on the cohomology of $\oplus_{n\in\mathbb{Z}} L^n$, where L is a very ample line bundle on a projective d-dimensional variety X such that $H^i(X, L^n) = 0$ for $0 < i < d$ and all $n \in \mathbb{Z}$. We prove that there exists a unique such *nontrivial* A_∞-structure up to a strict A_∞-isomorphism (i.e., an A_∞-isomorphism with the identity as the first structure map) and rescaling. In the case when X is a curve we also compute the group of strict A_∞-automorphisms of this A_∞-structure.

1. Introduction

Let X be a projective variety over a field k, L be a very ample line bundle on X. Recall that the graded k-algebra

$$R_L = \oplus_{n\geqslant 0} H^0(X, L^n)$$

is called the *homogeneous coordinate ring* corresponding to L. More generally, one can consider the bigraded k-algebra

$$A_L = \oplus_{p,q\in\mathbb{Z}} H^q(X, L^p).$$

We call A_L the *extended homogeneous coordinate ring* corresponding to L.

Since A_L can be represented naturally as the cohomology algebra of some dg-algebra (say, using injective resolutions or Čech cohomology with respect to an affine covering), it is equipped with a family of higher operations called Massey products. A better way of recording this additional structure uses the notion of A_∞-algebra due to Stasheff. Namely, by the theorem of Kadeishvili the product on A_L extends to a canonical (up to A_∞-isomorphism) A_∞-algebra structure with $m_1 = 0$ (see [**4**] 3.3 and references therein). More precisely, this structure is unique up to a *strict A_∞-isomorphism*, i.e., an A_∞-isomorphism with the identity map as the first structure map (see section 2.1 for details). Note that the axioms of A_∞-algebra use the *cohomological* grading on A_L (where $H^q(X, L^p)$ has cohomological degree q), and all the operations (m_n) have degree zero with respect to the *internal*

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grading (where $H^q(X, L^p)$ has *internal degree* p). The natural question is whether it is possible to characterize intrinsically this canonical class of A_∞-structures on A_L. This question is partly motivated by the homological mirror symmetry. Namely, in the case when X is a Calabi-Yau manifold, the A_∞-structure on A_L is supposed to be A_∞-equivalent to an appropriate A_∞-algebra arising on a mirror dual symplectic side. An intrinsic characterization of the A_∞-isomorphism class of our A_∞-structure could be helpful in reducing the problem of constructing such an A_∞-equivalence to constructing an isomorphism of the usual associative algebras. More generally, it is conceivable that the algebra A_L can appear as cohomology algebra of some other dg-algebras (for example, if there is an equivalence of the derived category of coherent sheaves on X with some other such category), so one might be interested in comparing corresponding A_∞-structures on A_L.

Thus, we want to study all A_∞-structures (m_n) on A_L (with respect to the cohomological grading), such that $m_1 = 0$, m_2 is the standard double product and all m_n have degree 0 with respect to the internal grading. Let us call such an A_∞-structure on A_L *admissible*. As we have already mentioned before, there is a *canonical strict A_∞-isomorphism class* of such structures coming from the realization of A_L as cohomology of the dg-algebra $\oplus_n \mathcal{C}^\bullet(L^n)$ where $\mathcal{C}^\bullet(?)$ denotes the Čech complex with respect to some open affine covering of X. By definition, an A_∞-structure belongs to the canonical class if there exists an A_∞-morphism from A_L equipped with this A_∞-structure to the above dg-algebra inducing identity on the cohomology. The simplest picture one could imagine would be that all admissible A_∞-structures are strictly A_∞-isomorphic, i.e., that A_L is intrinsically formal. It turns out that this is not the case. However, our main theorem below shows that if the cohomology of the structure sheaf on X is concentrated in degrees 0 and $\dim X$ then for sufficiently ample L the situation is not too much worse.

We will recall the notion of a homotopy between A_∞-morphisms in section 2.1 below. [1] Let us say that an A_∞-structure is *nontrivial* if it is not A_∞-isomorphic to an A_∞-structure with $m_n = 0$ for $n \neq 2$. By rescaling of an A_∞-structure we mean the change of the products (m_n) to $(\lambda^{n-2} m_n)$ for some constant $\lambda \in k^*$. Our main result gives a classification of admissible A_∞-structures on A_L up to a strict A_∞-isomorphism and rescaling (under certain assumptions).

Theorem 1.1. *Let L be a very ample line bundle on a d-dimensional projective variety X such that $H^q(X, L^p) = 0$ for $q \neq 0, d$ and all $p \in \mathbb{Z}$. Then*
(i) up to a strict A_∞-isomorphism and rescaling there exists a unique non-trivial admissible A_∞-structure on A_L; moreover, A_∞-structures on A_L from the canonical strict A_∞-isomorphism class are nontrivial;
(ii) for every pair of strict A_∞-isomorphisms $f, f' : (m_i) \to (m'_i)$ between admissible A_∞-structures on A_L there exists a homotopy from f to f'.

Remarks. 1. One can unify strict A_∞-isomorphisms with rescalings by considering A_∞-isomorphisms (f_n) with the morphism f_1 of the form $f_1(a) = \lambda^{\deg(a)}$ for

[1] All A_∞-morphisms and homotopies between them are assumed to respect the internal grading on A_L.

some $\lambda \in k^*$ (where a is a homogeneous element of A_L). In particular, part (i) of the theorem implies that all non-trivial admissible A_∞-structures on A_L are A_∞-isomorphic.
2. As we will show in section 2.1, part (ii) of the theorem is equivalent to its particular case when $f' = f$. In this case the statement is that every strict A_∞-automorphism of f is homotopic to the identity.
3. If one wants to see more explicitly how a canonical A_∞-structure on A_L looks like, one has to choose one of the natural dg-algebras with cohomology A_L (an obvious algebraic choice is the Čech complex; in the case $k = \mathbb{C}$ one can also use the Dolbeault complex), choose a projector π from the dg-algebra to some space of representatives for the cohomology such that $\pi = 1 - dQ - Qd$ for some operator Q, and then apply formulas of [**5**] for the operations m_n (they are given by certain sums over trees).

The above theorem is applicable to every line bundle of sufficiently large degree on a curve. In higher dimensions it can be used for every sufficiently ample line bundle on a d-dimensional projective variety X such that there exists a dualizing sheaf on X and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$. For example, this condition is satisfied for complete intersections in projective spaces. At present we do not know how to extend this theorem to the case when $\mathcal{O}_X$ has some nontrivial middle cohomology. Note that for a smooth projective variety X over $\mathbb{C}$ the natural (up to a strict A_∞-isomorphism) A_∞-structure on $H^*(X, \mathcal{O}_X)$ is trivial as follows from the formality theorem of [**1**]. This suggests that for sufficiently ample line bundle L one could try to characterize the canonical A_∞-structure on A_L (up to a strict A_∞-isomorphism and rescaling) as an admissible A_∞-structure whose restriction to $H^*(X, \mathcal{O}_X)$ is trivial.

In the case when X is a curve we can also compute the group of strict A_∞-automorphisms of an A_∞-structure on A_L. As we will explain below 2.1, strict A_∞-isomorphisms on A_L form a group HG, which is a subgroup of automorphisms of the free coalgebra $\mathrm{Bar}(A_L)$ (preserving both gradings). The dual to the degree zero component of $\mathrm{Bar}(A_L)$ (with respect to both gradings) can be identified with the completed tensor algebra $\hat{T}(H^1(X, \mathcal{O}_X)^*) = \prod_{n \geqslant 0} T^n(H^1(X, \mathcal{O}_X)^*)$. Therefore, we obtain a natural homomorphism from HG to to the group G of continuous automorphisms of $\hat{T}(H^1(X, \mathcal{O}_X)^*)$.

Theorem 1.2. *Let L be a very ample line bundle on a projective curve X such that $H^1(X, L) = 0$. Let also $HG(m) \subset HG$ be the group of strict A_∞-automorphisms of an admissible A_∞-structure m on A_L. Then the above homomorphism $HG \to G$ restricts to an isomorphism of $HG(m)$ with the subgroup $G_0 \subset G$ consisting of inner automorphisms of $\hat{T}(H^1(X, \mathcal{O}_X)^*)$ by elements in $1 + \prod_{n>0} T^n(H^1(X, \mathcal{O}_X)^*)$.*

Assume that X is a smooth projective curve. Then there is a canonical non-commutative thickening $\widetilde{J}$ of the Jacobian J of X (see [**3**]). As was shown in [**10**], a choice of an A_∞-structure in the canonical strict A_∞-isomorphism class gives rise to a formal system of coordinates on $\widetilde{J}$ at zero. More precisely, by this we mean an isomorphism of the formal completion of the local ring of $\widetilde{J}$ at zero with $\hat{T}(H^1(X, \mathcal{O}_X)^*)$ inducing the identity map on the tangent spaces. Formal coordinates associated with two strictly isomorphic A_∞-structures are related by the coordinate change given by the image of the corresponding A_∞-isomorphism under the homomorphism $HG \to G$. Now Theorem 1.2 implies that two A_∞-structures in the canonical class that induce the same formal coordinate on $\widetilde{J}$ can be connected by a unique strict A_∞-isomorphism. Indeed, two such isomorphisms would differ by a strict A_∞-automorphism inducing the trivial automorphism of $\hat{T}(H^1(X, \mathcal{O}_X)^*)$, but such an A_∞-automorphism is trivial by Theorem 1.2.

Convention. Throughout the paper we work over a fixed ground field k. The symbol $\otimes$ without additional subscripts always denotes the tensor product over k.

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2. Preliminaries

2.1. Strict A_∞-isomorphisms and homotopies

We refer to [**4**] for an introduction to A_∞-structures. We restrict ourselves to several remarks about A_∞-isomorphisms and homotopies between them.

A *strict A_∞-isomorphism* between two A_∞-structures (m) and (m') on the same graded space A is an A_∞-morphism $f = (f_n)$ from (A, m) to (A, m') such that $f_1 = \mathrm{id}$. The equations connecting f, m and m' can be interpreted as follows. Recall that m and m' correspond to coderivations d_m and $d_{m'}$ of the bar-construction $\mathrm{Bar}(A) = \oplus_{n\geqslant 1} T^n(A[1])$ such that $d_m^2 = d_{m'}^2 = 0$. Now every collection $f = (f_n)_{n\geqslant 1}$, where $f_n : A^{\otimes n} \to A$ has degree $1-n$, $f_1 = \mathrm{id}$, defines a coalgebra automorphism $\alpha_f : \mathrm{Bar}(A) \to \mathrm{Bar}(A)$, with the component $\mathrm{Bar}(A) \to A[1]$ given by (f_n). The condition that f is an A_∞-morphism is equivalent to the equation $\alpha_f \circ d_m = d_{m'} \circ \alpha_f$. In other words, strict A_∞-isomorphisms between A_∞-structures precisely correspond to the action of the group of automorphisms of $\mathrm{Bar}(A)$ as a coalgebra on the space of coderivations d such that $d^2 = 0$. More precisely, we consider only automorphisms of $\mathrm{Bar}(A)$ of degree 0 inducing the identity map $A \to A$. Let us denote by $HG = HG(A)$ the group of such automorphisms which we will also call *the group of strict A_∞-isomorphisms* on A. We will denote by $m \to g * m$, where $g \in HG$, the natural action of this group on the set of all A_∞-structures on A.

One can define a decreasing filtration (HG_n) of HG by normal subgroups by setting

$$HG_n = \{f = (f_i) \mid f_i = 0, 1 < i \leqslant n\}.$$

Note that $f \in HG_n$ if and only if the restriction of α_f to the sub-coalgebra $\mathrm{Bar}(A)_{\leqslant n} = \oplus_{i\leqslant n}(A[1])^{\otimes i}$ is the identity homomorphism. Furthermore, it is also clear that $HG \simeq \operatorname{proj\,lim}_n HG/HG_n$. In particular, an infinite product of strict A_∞-isomorphisms $\ldots * f(3) * f(2) * f(1)$ is well-defined as long as $f(n) \in HG_{i(n)}$, where $i(n) \to \infty$ as $n \to \infty$.

The notion of a homotopy between A_∞-morphisms is best understood in a more general context of A_∞-categories. Namely, for every pair of A_∞-categories $\mathcal{C}$, $\mathcal{D}$ one can define the A_∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ having A_∞-functors $F : \mathcal{C} \to \mathcal{D}$ as objects (see [**6**], [**8**]). In particular, there is a natural notion of closed morphisms between two A_∞-functors $F, F' : \mathcal{C} \to \mathcal{D}$. Specializing to the case when $\mathcal{C}$ and $\mathcal{D}$ are A_∞-categories with one object corresponding to A_∞-algebras A and B we obtain a notion of a closed morphism between a pair of A_∞-morphisms $f, f' : A \to B$. Following [**4**] we call such a closed morphism a *homotopy between A_∞-morphisms f and f'*. More explicitly, a homotopy h is given by a collection of maps $h_n : A^{\otimes n} \to B$ of degree $-n$, where $n \geqslant 1$, satisfying some equations. These equations are written as follows: there exists a unique linear map $H : \mathrm{Bar}(A) \to \mathrm{Bar}(B)$ of degree -1 with the component $\mathrm{Bar}(A) \to B$ given by (h_n), such that

$$\Delta \circ H = (\alpha_f \otimes H + H \otimes \alpha_{f'}) \circ \Delta, \tag{2.1.1}$$

where $\alpha_f, \alpha_{f'} : \mathrm{Bar}(A) \to \mathrm{Bar}(B)$ are coalgebra homomorphisms corresponding to f and f', Δ denotes the comultiplication. Then the equation connecting h, f and f' is

$$\alpha_f - \alpha_{f'} = d_A \circ H + H \circ d_B, \tag{2.1.2}$$

where d_A (resp., d_B) is the coderivation of $\mathrm{Bar}(A)$ (resp., $\mathrm{Bar}(B)$) corresponding to the A_∞-structure on A (resp., B). It is not difficult to check that for a given A_∞-morphism f from A to B the equations (2.1.1) and (2.1.2) imply that $\alpha_{f'}$ is a homomorphism of dg-coalgebras, so it defines an A_∞-morphism f' from A to B. Moreover, similarly to the case of strict A_∞-isomorphisms we have the following result.

Lemma 2.1. *Let A and B be A_∞-algebras and $f = (f_n)$ be an A_∞-morphism from A to B. For every collection $(h_n)_{n\geqslant 1}$, where $h_n : A^{\otimes n} \to B$ has degree $-n$, there exists a unique A_∞-morphism f' from A to B such that h is a homotopy from f to f'.*

Proof. It is easy to see that equation (2.1.1) is equivalent to the following formula

$$\begin{aligned} &H[a_1|\ldots|a_n] = \textstyle\sum_{i_1<\ldots<i_k<m<j_1<\ldots<j_l=n} \\ &\pm[f_{i_1}(a_1,\ldots,a_{i_1})|\ldots|f_{i_k-i_{k-1}}(a_{i_{k-1}+1},\ldots,a_{i_k})| \\ &h_{m-i_k}(a_{i_k+1},\ldots a_m)|f'_{j_1-m}(a_{m+1},\ldots,a_{j_1})|\ldots|f'_{j_l-j_{l-1}}(a_{j_{l-1}+1},\ldots,a_{j_l})], \end{aligned} \tag{2.1.3}$$

where $a_1, \ldots, a_n \in A$, $n \geqslant 1$. We are going to construct the maps $H|_{\mathrm{Bar}(A)_{\leqslant n}}$ and $\alpha_{f'}|_{\mathrm{Bar}(A)_{\leqslant n}}$ recursively, so that at each step the equations (2.1.2) and (2.1.3) are satisfied when restricted to $\mathrm{Bar}(A)_{\leqslant n}$. Of course, we also want H to have (h_n) as components. Then such a construction will be unique. Note that $H|_{A[1]}$ is given by h_1 and $\alpha_{f'}|_{A[1]}$ is given by $f'_1 = f_1 - m_1 \circ h_1 - h_1 \circ m_1$. Now assume that the restrictions of H and $\alpha_{f'}$ to $\mathrm{Bar}(A)_{\leqslant n-1}$ are already constructed, so that the maps $f'_i : A^{\otimes i} \to B$ are defined for $i \leqslant n-1$. Then the formula (2.1.3) defines uniquely the extension of H to $\mathrm{Bar}(A)_{\leqslant n}$ (note that in the RHS of this formula only f'_i with $i \leqslant n-1$ appear). It remains to apply formula (2.1.2) to define $\alpha_{f'}|_{\mathrm{Bar}(A)_{\leqslant n}}$. □

Let HG be the group of strict A_∞-isomorphisms on a given graded space A. In other words, HG is the group of degree 0 coalgebra automorphisms of $\mathrm{Bar}(A)$ with the componenent $A \to A$ equal to the identity map. This group acts on the set of all A_∞-structures on A. The stabilizer subgroup of some A_∞-structure m is the group of strict A_∞-automorphisms $HG(m)$. We can consider the set of all strict A_∞-automorphisms $f_h \in HG(m)$ such that there exists a homotopy h from the trivial A_∞-automorphism f^{tr} to f_h (where $f_i^{tr} = 0$ for $i > 1$). It is easy to see that A_∞-automorphisms of the form f_h constitute a normal subgroup in $HG(m)$ that we will denote by $HG(m)^0$. Furthermore, for every $g \in HG$ we have $HG_{g*m} = gHG(m)^0g^{-1}$. Also, for a pair of elements $g_1, g_2 \in HG$ such that $m' = g_1 * m = g_2 * m$, there exists a homotopy between g_1 and g_2 (where g_i are considered as A_∞-morphisms from (A, m) to (A, m')) if and only if $g_1^{-1}g_2 \in HG(m)^0$.

2.2. Obstructions

Below we use Hochschild cohomology $HH(A) := HH(A, A)$ for a graded associative algebra A (see [**7**] for the corresponding sign convention). When considering $A = A_L$ as a graded algebra we equip it with the cohomological grading, so in the situation of Theorem 1.1 this grading has only 0-th and d-th non-trivial graded components.

The following lemma is well known and its proof is straightforward.

Lemma 2.2. *Let m and m' be two admissible A_∞-structures on A. Assume that $m_i = m_i'$ for $i < n$, where $n \geqslant 3$.*
(i) Set $c(a_1, \dots, a_n) = (m_n' - m_n)(a_1, \dots, a_n)$. Then c is a Hochschild n-cocycle, i.e.,

$$\delta c(a_1, \dots, a_{n+1}) = \sum_{j=1}^{n} (-1)^j c(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) +$$
$$(-1)^{n \deg(a_1)} a_1 c(a_2, \dots, a_{n+1}) + (-1)^{n+1} c(a_1, \dots, a_n) a_{n+1} = 0.$$

(ii) If $m' = f * m$ for a strict A_∞-isomorphism f such that $f_i = 0$ for $1 < i < n-1$, then setting $b(a_1, \dots, a_{n-1}) = (-1)^{n-1} f_{n-1}(a_1, \dots, a_{n-1})$ we get

$$c(a_1, \dots, a_n) = \delta b(a_1, \dots, a_n),$$

where c is the n-cocycle defined in (i). Hence, c is a Hochschild coboundary.

Thus, the study of admissible A_∞-structures on A is closely related to the study of certain components of Hochschild cohomology of A. More precisely, let us denote $C^n_{p,q}(A)$ (resp. $HH^n_{p,q}(A)$) the space of reduced Hochschild n-cochains (resp. of n-th Hochschild cohomology classes) of internal grading $-p$ and of cohomological grading $-q$. In other words, $C^n_{p,q}(A)$ consists of cochains $c : A^{\otimes n} \to A$ such that $\operatorname{intdeg} c(a_1, \dots, a_n) = \operatorname{intdeg} a_1 + \dots + \operatorname{intdeg} a_n - p$, $\deg c(a_1, \dots, a_n) = \deg a_1 + \dots + \deg a_n - q$. Since, all the operations m_n respect the internal grading and have (cohomological) degree $2 - n$, we see that the cocycle c defined in Lemma 2.2 lives in $C^n_{0,n-2}(A)$.

There is an analogue of Lemma 2.2 for strict A_∞-isomorphisms.

Lemma 2.3. *Let m and m' be admissible A_∞-structures on A, f, f' be a pair of strict A_∞-isomorphisms from m to m'. Assume that $f_i = f_i'$ for $i < n$, where $n \geqslant 2$.*
(i) Set $c(a_1, \dots, a_n) = (f_n' - f_n)(a_1, \dots, a_n)$. Then c is a Hochschild n-cocycle in $C^n_{0,n-1}(A)$.
(ii) If $\phi : f \to f'$ is a homotopy such that $\phi_i = 0$ for $i < n-1$, then for $b(a_1, \dots, a_{n-1}) = \pm\phi_{n-1}(a_1, \dots, a_{n-1})$ one has $c = \delta b$.

3. Calculations

3.1. Hochschild cohomology

In this subsection we calculate the components of the Hochschild cohomology of $A = A_L$ that are relevant for the proof of Theorem 1.1.

Let us set $R = R_L$ and let $R_+ = \oplus_{n \geqslant 1} R_n$ be the augmentation ideal in R, so that $R/R_+ = k$. Recall that the bar-construction provides a free resolution of k as R-module of the form

$$\ldots \to R_+ \otimes R_+ \otimes R \to R_+ \otimes R \to R \to k. \tag{3.1.1}$$

For graded R-bimodules $M_1, \ldots, M_n$ we consider the bar-complex

$$B^\bullet(M_1, \ldots, M_n) = M_1 \otimes T(R_+) \otimes M_2 \otimes \ldots T(R_+) \otimes M_n,$$

where $T(R_+)$ is the tensor algebra of R_+. This is just the tensor product over $T(R_+)$ of the bar-complexes of $M_1, \ldots, M_n$ (where M_1 is considered as a right R-module, $M_2, \ldots, M_{n-1}$ as R-bimodules, and M_n as a left R-module). The grading in this complex is induced by the *cohomological grading* of the tensor algebra $T(R_+)$ defined by $\deg T^i(R_+) = -i$, so that $B^\bullet(M_1, \ldots, M_n)$ is concentrated in nonnegative degrees and the differential has degree 1. For example, $B^\bullet(k, R)$ is the bar-resolution (3.1.1) of k.

Proposition 3.1. *Under the assumptions of Theorem 1.1 let us consider the graded R-module $M = \oplus_{i \in \mathbb{Z}} H^d(X, L^i)$. Let $M_1, \ldots, M_n$ be graded R-bimodules such that each of them is isomorphic to M as a (graded) right R-module and as a left R-module.*
(i) The complex $B^\bullet(k, M) = T(R_+) \otimes M$ has one-dimensional cohomology, which is concentrated in degree $-d-1$ and internal degree 0.
(ii) $H^i(B^\bullet(M_1, M_2)) = 0$ for $i \neq -d-1$ and $H^{-d-1}(B^\bullet(M_1, M_2))$ is isomorphic to M as a (graded) right R-module and as a left R-module.
(iii) $H^i(B^\bullet(M_1, \ldots, M_n)) = 0$ for $i > -(n-1)(d+1)$.
(iv) $H^i(B^\bullet(k, M_1, \ldots, M_n, k)) = 0$ for $i > -n(d+1)$. In addition, the space $H^{-d-1}(B^\bullet(k, M_1, k))$ is one-dimensional and has internal degree 0.

Proof. (i) Localizing the exact sequence (3.1.1) on X and tensoring with L^m, where $m \in \mathbb{Z}$, we obtain the following exact sequence of vector bundles on X:

$$\ldots \oplus_{n_1, n_2 > 0} R_{n_1} \otimes R_{n_2} \otimes L^{m-n_1-n_2} \to \oplus_{n>0} R_n \otimes L^{m-n} \to L^m \to 0. \tag{3.1.2}$$

Each term in this sequence is a direct sum of a number of copies of line bundles L^n: for a finite-dimensional vector space V we denote by $V \otimes L^n$ the direct sum of $\dim V$ copies of L^n. Now let us consider the spectral sequence with E_1-term given by the cohomology of all sheaves in this complex and abutting to zero (this sequence converges since we can compute cohomology using Čech resolutions with respect to a finite open affine covering of X). The E_1-term consists of two rows: one obtained by applying the functor $H^0(X, \cdot)$ to (3.1.2), another obtained by applying $H^d(X, \cdot)$. The row of H^0's has form

$$\ldots \oplus_{n_1, n_2 > 0} R_{n_1} \otimes R_{n_2} \otimes R_{m-n_1-n_2} \to \oplus_{n>0} R_n \otimes R_{m-n} \to R_m \to 0$$

which is just the m-th homogeneous component of the bar-resolution. Hence, this complex is exact for $m \neq 0$. Since the sequence abuts to zero the row of H^d's should also be exact for $m \neq 0$. For $m = 0$ the row of H^0's reduces to the single term $H^0(X, \mathcal{O}_X) = k$, hence, the row of H^d's in this case has one-dimensional cohomology at $-(d+1)$th term and is exact elsewhere.
(ii) Consider the filtration on $B^\bullet(M_1, M_2)$ associated with the $\mathbb{Z}$-grading on M_2. By part (i) the corresponding spectral sequence has the term $E_1 \simeq H^{-d-1}(M_1 \otimes T(R_+)) \otimes M_2 \simeq M_2$. Hence, it degenerates in this term and

$$H^*(K^\bullet) \simeq H^{-d-1}(K^\bullet) \simeq M$$

as a right R-module. Similarly, the spectral sequence associated with the filtration on $K^\bullet$ induced by the $\mathbb{Z}$-grading on M_2 gives an isomorphism of left R-modules $H^{-d-1}(K^\bullet) \simeq M$.
(iii) For $n = 2$ this follows from (ii). Now let $n > 2$ and assume that the assertion holds for $n' < n$. We can consider $B^\bullet(M_1, \ldots, M_n)$ as the total complex associated with a bicomplex, where the bidegree $(\deg_0, \deg_1)$ on $M_1 \otimes T(R_+) \otimes \ldots \otimes T(R_+) \otimes M_n$ is given by

$$\deg_0(x_1 \otimes t_1 \otimes \ldots \otimes t_{n-1} \otimes x_n) = \sum_{i \equiv 0(2)} \deg(t_i),$$

$$\deg_1(x_1 \otimes t_1 \otimes \ldots \otimes t_{n-1} \otimes x_n) = \sum_{i \equiv 1(2)} \deg(t_i),$$

where $t_i \in T(R_+)$, $x_i \in M_i$, deg denotes the cohomological degree on $T(R_+)$. Therefore, there is a spectral sequence abbuting to cohomology of $B^\bullet(M_1, \ldots, M_n)$ with the E_1-term

$$H^*(M_1 \otimes T(R_+) \otimes M_2) \otimes T(R_+) \otimes H^*(M_3 \otimes T(R_+) \otimes M_4) \otimes \ldots,$$

where the last factor of the tensor product is either M_n or $H^*(M_{n-1} \otimes T(R_+) \otimes M_n)$. Using part (ii) we see that E_1 is isomorphic to the complex of the form

$$B^\bullet(M'_1, \ldots, M'_{n'})[(n - n')(d + 1)]$$

with $n' < n$. It remains to apply the induction assumption.
(iv) Consider first the case $n = 1$. The complex $B^\bullet(k, M_1, k) = T(R_+) \otimes M_1 \otimes T(R_+)$ is the total complex of the bicomplex $(\partial_1 \otimes \mathrm{id}, \mathrm{id} \otimes \partial_2)$, where ∂_1 and ∂_2 are bar-differentials on $T(R_+) \otimes M_1$ and $M_1 \otimes T(R_+)$. Our assertion follows immediately from (i) by considering the spectral sequence associated with this bicomplex.

Now assume that for some $n > 1$ the assertion holds for all $n' < n$. As before we view $B^\bullet(k, M_1, \ldots, M_n, k)$ as the total complex of a bicomplex by defining the bidegree on $T(R_+) \otimes M_1 \otimes \ldots \otimes M_n \otimes T(R_+)$ as follows:

$$\deg_0(t_0 \otimes x_1 \otimes t_1 \ldots \otimes x_n \otimes t_n) = \sum_{i \equiv 0(2)} \deg(t_i),$$

$$\deg_1(t_0 \otimes x_1 \otimes t_1 \ldots \otimes x_n \otimes t_n) = \sum_{i \equiv 1(2)} \deg(t_i).$$

Assume first that n is even. Then there is a spectral sequence associated with the above bicomplex abutting to the cohomology of $B^\bullet(k, M_1, \dots, M_n, k)$ and with the E_1-term

$$T(R_+) \otimes H^*(M_1 \otimes T(R_+) \otimes M_2) \otimes T(R_+) \otimes \dots \otimes H^*(M_{n-1} \otimes T(R_+) \otimes M_n) \otimes T(R_+).$$

Using (ii) we see that E_1 is isomorphic to the complex of the form

$$B^\bullet(k, M'_1, \dots, M'_{n/2}, k)[n(d+1)/2],$$

so we can apply the induction assumption. If n is odd then we consider another spectral sequence associated with the above bicomplex, so that

$$\begin{aligned} E_1 = &H^*(T(R_+) \otimes M_1) \otimes T(R_+) \otimes \\ &H^*(M_2 \otimes T(R_+) \otimes M_3) \otimes T(R_+) \otimes \dots \otimes H^*(M_{n-1} \otimes T(R_+) \otimes M_n) \otimes T(R_+). \end{aligned}$$

By (i) and (ii) this complex is isomorphic to $B^\bullet(k, M'_1, \dots, M'_{(n-1)/2}, k)[(n+1)(d+1)/2]$. Again we can finish the proof by applying the induction assumption. $\square$

We will also need the following simple lemma.

Lemma 3.2. *Let $C^\bullet$ be a complex in an abelian category equipped with a decreasing filtration $C^\bullet = F^0C^\bullet \supset F^1C^\bullet \supset F^2C^\bullet \supset \dots$ such that $C^n = \text{proj.}\lim_i C^n/F^iC^n$ for all n. Let $\operatorname{gr}_i C^\bullet = F^iC^\bullet/F^{i+1}C^\bullet$, $i = 0, 1, \dots$ be the associated graded factors. Assume that $H^n \operatorname{gr}_i C^\bullet = 0$ for all $i > 0$ and for some fixed n. Then the natural map $H^nC^\bullet \to H^n \operatorname{gr}_0 C^\bullet$ is injective.*

Proof. Considering an exact sequence of complexes

$$0 \to F^1C^\bullet \to C^\bullet \to \operatorname{gr}_0 C^\bullet \to 0$$

one can easily reduce the proof to the case $H^n \operatorname{gr}_i C^\bullet = 0$ for all $i \geqslant 0$. In this case we have to show that $H^nC^\bullet = 0$. Let $c \in C^n$ be a cocycle and let c_i be its image in C^n/F^iC^n. It suffices to construct a sequence of elements $x_i \in C^{n-1}/F^iC^{n-1}$, $i = 1, 2, \dots$, such that $x_{i+1} \equiv x_i \mod F^iC^{n-1}$ and $c_i = d(i)x_i$, where $d(i)$ is the differential on $C^\bullet/F^iC^\bullet$. Since n-th cohomology of $C^\bullet/F^1C^\bullet = \operatorname{gr}_0 C^\bullet$ is trivial we can find x_1 such that $c_1 = d(1)x_1$. Then we proceed by induction: once $x_1, \dots, x_i$ are chosen an easy diagram chase using the exact triple of complexes

$$0 \to \operatorname{gr}_i C^\bullet \to C^\bullet/F^{i+1}C^\bullet \to C^\bullet/F^iC^\bullet \to 0$$

and the vanishing of $H^n(\operatorname{gr}_i C^\bullet)$ show that x_{i+1} exists. $\square$

Theorem 3.3. *Under the assumptions of Theorem 1.1 one has $HH^i_{0,md}(A) = 0$ for $i < m(d+2)$ and $\dim HH^{d+2}_{0,d}(A) \leqslant 1$, where $A = A_L$.*

Proof. Set $C^i = C^i_{0,md}(A)$ (see 2.2). Note that Hochschild differential maps C^i into C^{i+1} (since m_2 preserves both gradings on A). Recall that the decomposition of A into graded pieces with respect to the cohomological degree has form $A = R \oplus M$, where R has degree 0 and $M = \oplus_{i \in \mathbb{Z}} H^d(X, L^i)$ has degree d. The natural augmentation of A is given by the ideal $A_+ = R_+ \oplus M$. Each of the spaces C^i

decomposes into a direct sum $C^i = C^i(0) \oplus C^i(d)$, where $C^i(0) \subset \mathrm{Hom}(A_+^{\otimes i}, R)$, $C^i(d) \subset \mathrm{Hom}(A_+^{\otimes i}, M)$. More precisely, the space $C^i(0)$ consists of linear maps

$$[T(R_+) \otimes M \otimes T(R_+) \otimes \ldots \otimes M \otimes T(R_+)]_i \to R \tag{3.1.3}$$

preserving the internal grading, where there are m factors of M in the tensor product and the index i refers to the total number of factors $H^*(L^*)$ (so that the LHS can be considered as a subspace of $A_+^{\otimes i}$). Similarly, the space $C^i(d)$ consists of linear maps

$$[T(R^+) \otimes M \otimes T(R_+) \otimes \ldots \otimes M \otimes T(R_+)]_i \to M$$

preserving the internal grading, where there is $m+1$ factors of M in the tensor product. Clearly, $C^\bullet(d)$ is a subcomplex in $C^\bullet$, so we have an exact sequence of complexes

$$0 \to C^\bullet(d) \to C^\bullet \to C^\bullet(0) \to 0.$$

Therefore, it suffices to prove that $H^i(C^\bullet(0)) = H^i(C^\bullet(d)) = 0$ for $i < m(d+2)$, and that in the case $m = 1$ one has in addition $H^{d+2}(C^\bullet(d)) = 0$ and $\dim H^{d+2}(C^\bullet(0)) \leqslant 1$.

To compute the cohomology of these two complexes we can use spectral sequences associated with some natural filtrations to reduce the problem to simpler complexes. First, let us consider the decomposition

$$C^\bullet(0) = \prod_{j \geqslant 0} C^\bullet(0)_j,$$

where $C^i(0)_j \subset C^i(0)$ is the space of maps (3.1.3) with the image contained in $H^0(L^j) \subset R$. The differential on $C^\bullet(0)$ has form

$$\delta(x_j)_{j \geqslant 0} = (\sum_{j' \leqslant j} \delta_{j',j} x_{j'})_{j \geqslant 0}$$

for some maps $\delta_{j',j} : C^\bullet(0)_{j'} \to C^\bullet(0)_j$, where $j' \leqslant j$. By Lemma 3.2 it suffices to prove that one has $H^i(C^\bullet(0)_j, \delta_{j,j}) = 0$ for $i < m(d+2)$ and all j, while for $m = 1$ one has in addition $H^{d+2}(C^\bullet(0)_j, \delta_{j,j}) = 0$ for $j > 0$ and $\dim H^{d+2}(C^\bullet(0)_0, \delta_{0,0}) \leqslant 1$. But

$$(C^\bullet(0)_j, \delta_{j,j}) = \mathrm{Hom}(K^\bullet_{m,j}, R_j)[-m],$$

where $K^\bullet_m = B^\bullet(k, M, \ldots, M, k)$ (m copies of M) and $K^\bullet_{m,j}$ is its j-th graded component with respect to the internal grading. Here we use the following convention for the grading on the dual complex: $\mathrm{Hom}(K^\bullet, R)^i = \mathrm{Hom}(K^{-i}, R)$. Therefore, Proposition 3.1(iv) implies that cohomology of $C^\bullet(0)_j$ is concentrated in degrees $\geqslant m(d+1) + m = m(d+2)$. Moreover, for $m = 1$ the $(d+2)$-th cohomology space is non-zero only for $j = 0$, in which case it is one-dimensional.

For the complex $C^\bullet(d)$ we have to use a different filtration (since M is not bounded below with respect to the internal grading). Consider the decreasing filtration on $C^\bullet(d)$ induced by the following grading on $T(R^+) \otimes M \otimes T(R_+) \otimes \ldots \otimes M \otimes T(R_+)$:

$$\deg(t_1 \otimes x_1 \otimes t_2 \otimes \ldots \otimes x_{m+1} \otimes t_{m+2}) = \deg(t_1) + \deg(t_{m+2}),$$

where $t_i \in T(R_+)$, $x_i \in M$, the degree of R_+ is taken to be -1. The associated graded complex is

$$\mathrm{Hom}_{\mathrm{gr}}(T(R^+) \otimes B^\bullet(M, \ldots, M) \otimes T(R^+), M)[-m-1],$$

where there are $m+1$ factors of M in the bar-construction. It remains to apply Proposition 3.1(iii). □

3.2. Some Massey products

In this subsection we will show the nontriviality of the canonical class of A_∞-structures on A_L and combine it with our computations of the Hochschild cohomology to prove the main theorem.

Note that the canonical class of A_∞-structures can be defined in a more general context. Namely, if $\mathcal{C}$ is an abelian category with enough injectives then we can define the canonical class of A_∞-structures on the derived category $D^+(\mathcal{C})$ of bounded below complexes over $\mathcal{C}$. Indeed, one can use the equivalence of $D^+(\mathcal{C})$ with the homotopy category of complexes with injective terms and apply Kadeishvili's construction to the dg-category of such complexes (see [**10**],1.2 for more details). In the case when $\mathcal{C}$ is the category of coherent sheaves the resulting A_∞-structure is strictly A_∞-isomorphic to the structure obtained using Čech resolutions (since the relevant dg-categories are equivalent). In this context we have the following construction of nontrivial Massey products.

Lemma 3.4. *Let $\mathcal{C}$ be an abelian category with enough injectives,*

$$0 \to \mathcal{F}_0 \stackrel{\alpha_1}{\to} \mathcal{F}_1 \stackrel{\alpha_2}{\to} \mathcal{F}_2 \to \ldots \stackrel{\alpha_n}{\to} \mathcal{F}_n \to 0$$

be an exact sequence in $\mathcal{C}$, where $n \geqslant 2$, and let $\beta : \mathcal{F}_n \to \mathcal{F}_0[n-1]$ be a morphism in the derived category $D^b(\mathcal{C})$ corresponding to the Yoneda extension class in $\mathrm{Ext}^{n-1}(\mathcal{F}_n, \mathcal{F}_0)$ represented by the above sequence. Assume that $\mathrm{Ext}^{j-i-1}(\mathcal{F}_j, \mathcal{F}_i) = 0$ when $0 \leqslant i < j \leqslant n-1$. Then

$$m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm\, \mathrm{id}_{\mathcal{F}_0}$$

for any A_∞-structure (m_i) on $D^b(\mathcal{C})$ from the canonical class.

Proof. Assume first that $n = 2$. Then $m_3(\alpha_1, \alpha_2, \beta)$ is the unique value of the well-defined triple Massey product in $D^b(\mathcal{C})$ (see [**9**], 1.1). Using the standard recipe for its calculation (see [**2**], IV.2) we immediately get that $m_3(\alpha_1, \alpha_2, \beta) = \mathrm{id}$.

For general n we can proceed by induction. Assume that the statement is true for $n-1$. Set $\mathcal{F}'_{n-1} = \ker(\alpha_n)$. Then we have exact sequences

$$0 \to \mathcal{F}_0 \stackrel{\alpha_1}{\to} \mathcal{F}_1 \to \ldots \to \mathcal{F}_{n-2} \stackrel{\alpha'_{n-1}}{\to} \mathcal{F}'_{n-1} \to 0, \tag{3.2.1}$$

$$0 \to \mathcal{F}'_{n-1} \stackrel{i}{\to} \mathcal{F}_{n-1} \stackrel{\alpha_n}{\to} \mathcal{F}_n \to 0, \tag{3.2.2}$$

where $m_2(\alpha'_{n-1}, i) = i \circ \alpha'_{n-1} = \alpha_{n-1}$. By the definition, we have $\beta = m_2(\gamma, \beta')$, where $\beta' \in \mathrm{Ext}^{n-2}(\mathcal{F}'_{n-1}, \mathcal{F}_0)$ and $\gamma \in \mathrm{Ext}^1(\mathcal{F}_n, \mathcal{F}'_{n-1})$ are the extension classes corresponding to these exact sequences. Applying the A_∞-axiom to the elements

$(\alpha_1, \ldots, \alpha_n, \gamma, \beta')$ and using the vanishing of $m_{n-i+2}(\alpha_{i+1}, \ldots, a_n, \gamma, \beta') \in \mathrm{Ext}^{i-1}(\mathcal{F}_i, \mathcal{F}_0)$ for $0 < i < n$, we get

$$m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm m_n(\alpha_1, \ldots, \alpha_{n-1}, m_3(\alpha_{n-1}, \alpha_n, \gamma), \beta'). \tag{3.2.3}$$

Furthermore, applying the A_∞-axiom to the elements $(\alpha'_{n-1}, i, \alpha_n, \gamma)$ we get

$$m_3(\alpha_{n-1}, \alpha_n, \gamma) = \pm m_2(\alpha'_{n-1}, m_3(i, \alpha_n, \gamma)). \tag{3.2.4}$$

Next, we claim that sequences (3.2.1) and (3.2.2) satisfy the assumptions of the lemma. Indeed, for (3.2.1) this is clear, so we just have to check that $\mathrm{Hom}(\mathcal{F}_{n-1}, \mathcal{F}'_{n-1}) = 0$. The exact sequence (3.2.1) gives a resolution $\mathcal{F}_0 \to \ldots \to \mathcal{F}_{n-2}$ of $\mathcal{F}'_{n-1}$ and we can compute $\mathrm{Hom}(\mathcal{F}_{n-1}, \mathcal{F}'_{n-1})$ using this resolution. Now the required vanishing follows from our assumption that $\mathrm{Ext}^{n-i-2}(\mathcal{F}_{n-1}, \mathcal{F}_i) = 0$ for $0 \leqslant i \leqslant n-2$. Therefore, we have

$$m_3(i, \alpha_n, \gamma) = \mathrm{id}\,.$$

Together with (3.2.4) this implies that

$$m_3(\alpha_{n-1}, \alpha_n, \gamma) = \pm\alpha'_{n-1}.$$

Substituting this into (3.2.3) we get

$$m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm m_n(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_{n-1}, \beta').$$

It remains to apply the induction assumption to the sequence (3.2.1). □

Proof of Theorem 1.1. (i) Since the algebra $A = A_L$ is concentrated in degrees 0 and d, the first potentially nontrivial higher product of an admissible A_∞-structure (m_i) on A is m_{d+2}. Therefore, by Lemma 2.2 for every such A_∞-structure (m_i) on A the map m_{d+2} induces a cohomology class $[m_{d+2}] \in HH^{d+2}_{0,d}(A)$. We claim that if (m'_i) is another admissible A_∞-structure on A then (m_i) is strictly A_∞-isomorphic to (m'_i) if and only if $[m_{d+2}] = [m'_{d+2}]$. Indeed, this follows from Lemma 2.2 and from the vanishing of higher obstructions due to Theorem 3.3 (these obstructions lie in $HH^{md+2}_{0,md}(A)$ where $m > 1$, and the vanishing follows since $md+2 < m(d+2)$). In particular, an admissible A_∞-structure (m_i) is nontrivial if and only if $[m_{d+2}] \neq 0$. Since by Theorem 3.3 the space $HH^{d+2}_{0,d}(A)$ is at most one-dimensional, it remains to prove the nontriviality of an admissible A_∞-structure from the canonical class. Replacing L by its sufficiently high power if necessary we can assume that there exists $d+1$ sections $s_1, \ldots, s_{d+1} \in H^0(L)$ without common zeroes. The corresponding Koszul complex gives an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}^{\oplus(d+1)} \otimes_{\mathcal{O}} L \to \mathcal{O}^{\oplus\binom{d+1}{2}} \otimes_{\mathcal{O}} L^2 \to \ldots \to \mathcal{O}^{\oplus(d+1)} \otimes_{\mathcal{O}} L^d \to L^{d+1} \to 0.$$

By our assumptions this sequence satisfies the conditions required in Lemma 3.4, hence we get a nontrivial $(d+2)$-ple Massey product for our A_∞-structure.

(ii) Applying Lemma 2.3 we see that obstructions for connecting two strict A_∞-isomorphisms by a homotopy lie in $\oplus_{m \geqslant 1} HH^{md+1}_{0,md}(A)$. But this space is zero by Theorem 3.3. □

Corollary 3.5. *Under assumptions of Theorem 1.1 the space $HH^{d+2}_{0,d}(A_L)$ is one-dimensional.*

Proof. Indeed, from Theorem 3.3 we know that $\dim HH^{d+2}_{0,d}(A_L) \leqslant 1$. If this space were zero then the above argument would show that all admissible A_∞-structures on A_L are trivial. But we know that A_∞-structures on A_L from the canonical class are nontrivial. □

Remark. One can ask whether there exists an A_∞-structure on A_L from the canonical class such that $m_n = 0$ for $n > d+2$ or at least $m_n = 0$ for all sufficiently large n. However, even in the case of smooth curves of genus $\geqslant 1$ the answer is "no". The proof can be obtained using the construction of a universal deformation of a coherent sheaf (when it exists) using the canonical A_∞-structure, outlined in [**10**]. For example, it is shown there that the products

$$m_{n+2} : H^1(\mathcal{O}_X)^{\otimes n} \otimes H^0(L^{n_1}) \otimes H^0(L^{n_2}) \to H^0(L^{n_1+n_2})$$

appear as coefficients in the universal deformation of the structure sheaf. The base of this family is $\operatorname{Spec} R$, where $R \simeq k[[t_1, \dots, t_g]]$ is the completed symmetric algebra of $H^1(\mathcal{O}_X)^\vee$. If all sufficiently large products were zero, this family would be induced by the base change from some family over an open neighborhood U of zero in the affine space $\mathbb{A}^g$. But this would imply that the embedding of $\operatorname{Spec} R$ into the Jacobian (corresponding to the isomorphism of R with the completion of the local ring of the Jacobian at zero) factors through U, which is false.

3.3. Proof of Theorem 1.2

Theorem 1.1(i) easily implies that every admissible A_∞-structure on $A = A_L$ is (strictly) A_∞-isomorphic to some strictly unital A_∞-structure. Therefore, it is enough to prove our statement for strictly unital structures. Recall that the group of strict A_∞-isomorphisms HG is the group of coalgebra automorphisms of $\operatorname{Bar}(A_L)$ inducing the identity map $A_L \to A_L$ and preserving two grading on $\operatorname{Bar}(A_L)$ induced by the two gradings of A_L. Thus, we can identify HG with a subgroup of algebra automorphisms of the completed cobar-construction $\operatorname{Cobar}(A_L) = \prod_{n \geqslant 0} T^n(A_L^*[-1])$ (our convention is that passing to dual vector space changes the grading to the opposite one).

By Theorem 1.1(ii) for every strict A_∞-automorphism f of an A_∞-structure m there exists a homotopy from f to the trivial A_∞-automorphism f^{tr}. Let $\alpha = \alpha_f^*$ be the automorphism of $\operatorname{Cobar}(A_L)$ corresponding to f and $h = H^* : \operatorname{Cobar}(A_L) \to \operatorname{Cobar}(A_L)[-1]$ be the map giving the homotopy from f to f^{tr}. The equations dual to (2.1.1) and (2.1.2) in our case have form

$$h(xy) = h(x)y \pm \alpha(x)h(y),$$

$$\alpha = \operatorname{id} + d \circ h + h \circ d,$$

where d is the differential on $\operatorname{Cobar}(A_L)$ associated with m. Recall that $A_L = H^0 \oplus H^1$, where $H^0 = \oplus_{n \geqslant 0} H^0(X, L^n)$, $H^1 = \oplus_{n \leqslant 0} H^1(X, L^n)$. Since h has degree

-1 we have $h((H^1)^*[-1]) = 0$ and $h((H^0)^*[-1]) \subset \hat{T}((H^1)^*[-1])$. Furthermore, since h preserves the internal degree, we have $h(H^0(X, L^n)^*[-1]) = 0$ for all $n > 0$. Let $\epsilon \in (H^0)^*[-1] \subset \mathrm{Cobar}(A_L)$ be an element corresponding to the natural projection $H^0 \to H^0(X, \mathcal{O}_X) \simeq k$. Then we have $A_L^*[-1] = k\epsilon \oplus V$, where $V = (H^1)^*[-1] \oplus (\oplus_{n>0} H^0(X, L^n))^*[-1]$, and $h(V) = 0$. Let $\langle V \rangle \subset \mathrm{Cobar}(A_L)$ be the subalgebra topologically generated by V. Then h vanishes on $\langle V \rangle$. Also, for every $x \in V$ we have

$$dx = \epsilon x + x\epsilon \mod \langle V \rangle$$

since our A_∞-structure is strictly unital. Hence, for $x \in V$ we have

$$\alpha(x) = x + h(dx) = x + h(\epsilon)x - \alpha(x)h(\epsilon),$$

which implies that

$$\alpha(x) = (1 + h(\epsilon))x(1 + h(\epsilon))^{-1}.$$

In particular, the restriction of α to the subalgebra $\hat{T}(H^1(X, \mathcal{O}_X)^*[-1])$ is the inner automorphism associated with the invertible element $1+h(\epsilon) \in \hat{T}(H^1(X, \mathcal{O}_X)^*[-1])$. On the other hand, we have

$$d\epsilon = \epsilon^2 \mod \langle V \rangle.$$

Hence,

$$\alpha(\epsilon) = \epsilon + dh(\epsilon) + h(\epsilon)\epsilon - \alpha(\epsilon)h(\epsilon),$$

so that

$$\alpha(\epsilon) = (1 + h(\epsilon))\epsilon(1 + h(\epsilon))^{-1} + dh(\epsilon) \cdot (1 + h(\epsilon))^{-1}.$$

Thus, α is uniquely determined by $h(\epsilon)$. Also, by Lemma 2.1 $h(\epsilon)$ can be an arbitrary element of $\prod_{n \geqslant 1} T^n(H^1(X, \mathcal{O}_X)^*[-1])$. □

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