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HOCHSCHILD COHOMOLOGY AND MODULI SPACES OF STRONGLY HOMOTOPY ASSOCIATIVE ALGEBRAS

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Abstract

Motivated by ideas from stable homotopy theory we study the space of strongly homotopy associative multiplications on a two-cell chain complex. In the simplest case this moduli space is isomorphic to the set of orbits of a group of invertible power series acting on a certain space. The Hochschild cohomology rings of resulting A_∞-algebras have an interpretation as totally ramified extensions of discrete valuation rings. All A_∞-algebras are supposed to be unital and we give a detailed analysis of unital structures which is of independent interest.

1. Introduction

The notion of a strongly homotopy associative algebra or of an A_∞-algebra was introduced in [**26**] by Stasheff and was recently much studied in connection with deformation quantization and the Deligne conjecture, cf. [**15**], [**22**]. From the point of view of homotopy theory, an A_∞-algebra is the same as a differential graded algebra (dga). However, for the purposes of explicit computations, it is often more convenient to work with A_∞-algebras rather than with dga's.

The purpose of this paper is to study 'homotopy invariant' or 'derived' moduli spaces for A_∞-algebras. It should be noted that other authors also considered the problem of constructing derived moduli spaces. Here we mention the works of M.Schlessinger and J.Stasheff, cf. [**27**] and of V.Hinich, [**10**],[**11**]. Another approach making heavy use of simplicial methods and homotopical algebra is developed in [**3**]. The case of A_∞-algebras considered here, exhibits, on the one hand, most of the representative features of derived moduli space theory and, on the other hand, allows one to perform concrete computations without the need of too much apparatus.

Though our examples are purely algebraic, they are motivated by the study of complex-oriented cohomology theories. There is a parallel notion of an S-algebra, or an A_∞-ring spectrum in stable homotopy theory, cf. [**4**]. Many important spectra of algebraic topology, especially those related to complex cobordisms admit structures of A_∞-ring spectra. The existence of such structures was proved in [**17**], [**24**] and in [**8**] by methods of obstruction theory, but up until now there was no attempt

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to classify such structures. Of particular interest is the space of A_∞ structures on $K(n)$'s, the higher Morava K-theories. A closely related problem is computation of $THH^*(K(n), K(n))$, the topological Hochschild cohomology of $K(n)$. It should be noted that in the topological situation one is forced to work in the abstract setting of Quillen closed model categories and this makes the classification problem much more difficult then the corresponding algebraic one. Therefore it seems natural to consider the algebraic problem first and this is what we do in the present paper.

We now outline the problem under consideration and our approach. Consider the field of p elements $\mathbb{F}_p$ as an algebra over $\mathbb{Z}$. Then we could form the derived Hochschild cohomology of $\mathbb{F}_p$ with coefficients in itself as a $\mathbb{Z}$-algebra (sometimes called Shukla cohomology, cf. [**29**]):

$$HH_{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) := RHom_{\mathbb{F}_p \otimes^L \mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p).$$

Here $\otimes^L$ denotes the derived tensor product over $\mathbb{Z}$ and $RHom$ denotes the derived module of homomorphisms. An easy computation then shows that $HH^*_{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[[z]]$ with z having cohomological degree 2.

This result is valid because $\mathbb{F}_p$ has a unique structure of a $\mathbb{Z}$-algebra which happens to be commutative. If R is an evenly graded commutative ring and $x \in R$ is a homogeneous element that is not a zero divisor we obtain similarly $HH^*_R(R/x, R/x) = R/x[[z]]$ with z having cohomological degree $|x| + 2$. Notice, however, that we implicitly resolved the R-algebra R/x by the differential graded algebra $\Lambda_R(y)$, with one generator y in degree $|x|+1$ whose square is 0 and $dy = x$. In other words we assumed that R/x is given the usual structure of an R-algebra, in particular that it is commutative. In general there are many different structures of an A_∞-algebra on the complex $R \xrightarrow{x} R$ (which is a model for R/x in the derived category of R-modules). Take for instance $R = \mathbb{Z}[v, v^{-1}]$ where the element v has degree 2 and x is a prime number $p \neq 2$. Then the differential graded algebra $R[y]/(y^2 - v)$ with differential $dy = p$ is a model for R/x (that is, its homology ring is R/x), but it is not commutative even up to homotopy. It turns out that its Hochschild cohomology is $\hat{\mathbb{Z}}_p[v, v^{-1}]$ where $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers. In general for any structure of an A_∞-R-algebra on R/p the ring $HH^*_R(R/p, R/p)$ is filtered and complete with associated graded isomorphic to the formal power series algebra over $\mathbb{F}_p[v, v^{-1}]$. It follows that the ring $HH^*_R(R/p, R/p)$ is either an $\mathbb{F}_p$-algebra or it has no p-torsion. The torsion-free case corresponds to totally ramified extensions of the field of the p-adic numbers (cf.[**25**]) and we show that by varying A_∞ structures one can obtain extensions of arbitrary ramification index that is coprime to p, the so-called tamely ramified extensions.

So the natural problem is now to classify all possible A_∞ structures of R/x (or, equivalently all differential graded R-algebras whose homology ring is R/x). We consider this as part of a more general problem, namely the classification of all A_∞ structures on a two cell complex $\{\Sigma^d R \xrightarrow{\partial} R\}$ with no restrictions on d or the differential ∂. An A_∞-algebra of this sort is called a Moore algebra as being analogous to the Moore spectrum of stable homotopy theory. The resulting theory bears striking resemblance with the theory of one-dimensional formal groups, cf. [**9**] although we could not establish a direct link between the two. There are essentially two different cases: d odd and d even. We concentrate mainly on the even case since it is most relevant to the parallel topological problem. However, the odd case is also quite interesting and apparently related to the homology of moduli spaces of algebraic curves, cf. [**13**].

We note that there is a close connection between the results of this paper and algebraic deformation theory, cf. [**5**], [**23**]. For example Theorem 4.4 implies that any deformation of a unital A_∞-algebra is equivalent to a unital one. This and other implications of our results in deformation theory will be explained elsewhere.

We need to make a comment about grading. Throughout the paper we work with $\mathbb{Z}$-graded complexes of modules over a $\mathbb{Z}$-graded even commutative ring R. Some other authors, e.g. [**14**] work in the slightly less general $\mathbb{Z}/2\mathbb{Z}$-graded context. These two approaches are closely related. If we have a $\mathbb{Z}$-graded object we could always forget down to a $\mathbb{Z}/2\mathbb{Z}$-graded object. Conversely, tensoring everything in sight with the ring $\mathbb{Z}[v, v^{-1}]$ with $|v| = 2$ we obtain a $\mathbb{Z}$-graded object from a $\mathbb{Z}/2\mathbb{Z}$-graded one. This procedure is routinely employed in topology when studying complex oriented cohomology theories, cf. for example [**1**].

This paper is organized as follows. In section 2 we recall the definition of an A_∞-algebra and A_∞-morphism and collect various formulae which will be needed later on. The material presented here is fairly standard, except that we define A_∞-algebras over commutative graded rings rather than over fields as normally done.

In section 3 we study unital structures and prove a formula for the action of a unital automorphism on a given A_∞ structure. It turns out that this and other formulae in the theory of A_∞-algebras are best handled using the language of the dual cobar-construction (or, perhaps, of the 'cobar-construction of the dual') which could be thought of as a formal noncommutative (super)manifold in the sense of [**13**] . The seemingly trivial passage from bar to cobar construction is the main technical invention of this paper. We hope that it will have further applications.

In section 4 we define the Hochschild complex for A_∞-algebras and prove that it is homotopically equivalent to a normalized complex. This theorem is well-known for (strictly) associative algebras.

Section 5 introduces Moore algebras which are our main object of study. Moore algebras are in several respects similar to one-dimensional formal group laws and we prove the analogue of Lazard's theorem stating that the functor associating to a ring the set of Moore algebras over it is representable by a certain polynomial algebra on infinitely many generators.

In section 6 we consider the problem of classification of even Moore algebras over a field or a complete discrete valuation ring. This problem is equivalent to the classification of orbits of a certain action of the group of formal power series without the constant term. We obtain complete classification in characteristic zero and some partial results in characteristic p.

In section 7 we compute Hochschild cohomology of even Moore algebras and discuss its relation with totally ramified extensions of discrete valuation rings.

Notations and conventions. In sections $2-4$ we work over a fixed evenly graded commutative ground ring R. The symbols Hom and $\otimes$ always mean Hom_R and $\otimes_R$. In Sections $5-7$ the emphasis is shifted somewhat in that the ground ring is varied. It is still denoted by R sometimes with subscripts, e.g. R_e and R_o, and we use unadorned Hom and $\otimes$ where it does not cause confusion.

A graded ring whose homogeneous nonzero elements are invertible is referred to as a graded field. A graded discrete valuation ring is a Noetherean graded ring having a unique homogeneous ideal generated by a nonnilpotent element.

The set of invertible elements in a ring R is denoted by $R^{\times}$.

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2. Preliminaries

In this section we collect the necessary definitions and facts about A_∞-algebras. The details may be found in the definitive monograph by Markl, Shnider and Stasheff [**21**] or in the papers by Getzler-Jones [**7**], and Keller [**12**].

Let R be an evenly graded commutative ring. Since we need to work in the derived category of unbounded complexes of R-modules we will recall some basic facts following [**16**]. An n-sphere R-module is a free R-module on one generator in degree n. A cell R-module is the union of an expanding sequence of R-submodules M_n such that $M_0 = 0$ and M_{n+1} is the mapping cone of a map $\phi_n : F_n \to M_n$ where F_n is a direct sum of sphere modules (perhaps of different degrees). Thus a cell R-module is necessarily a complex of free R-modules. Conversely, it is easy to see that a bounded below complex of free R-modules is a cell R-module. In our applications we will be concerned with only such cell complexes.

Further, any R-module M admits a cell approximation, that is there is a cell R-module ΓM and a a quasi-isomorphism of complexes $\Gamma M \to M$. The functors $? \otimes M$ and $Hom(M, ?)$ where M is a cell R-module preserve quasi-isomorphisms and exact sequences in the variable ?. This allows one to define the derived functors of $\otimes$ and Hom by setting $M \otimes^L N := \Gamma M \otimes N$ and $RHom(M, N) := Hom(\Gamma M, N)$.

Let A be a cell R-module and TA the tensor algebra of A:

$$TA = R \oplus A \oplus A^{\otimes 2} \oplus \dots.$$

Then TA is a coalgebra via the comultiplication $\Delta : TA \to TA \otimes TA$ where

$$\Delta(a_1, \dots, a_n) = \sum (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n).$$

It is standard that a coderivation $\xi : TA \to TA$ of the coalgebra TA is determined by the composition $TA \xrightarrow{\xi} TA \to A$ where the second map is the canonical projection. Denoting the components of this composite map by ξ_i we see that a coderivation $\xi \in Coder(TA)$ is specified by a collection of maps $\xi_i : A^{\otimes i} \to A$. Let us introduce a filtration on the coalgebra TA by setting $F_pTA = \oplus_{i=0}^{p} T^iA$. Then the space of coderivations of TA also acquires a filtration. Namely, we say that a coderivation ξ has weight $\geqslant p$ if $\xi(F_iTA) \subset F_{i-p}TA$. Then clearly the elements of weight $\geqslant -1$ form the whole space $Coder(TA)$ and filtration by weight is an exhaustive Hausdorff filtration on the space $Coder(TA)$. We will denote the set of coderivations of weight $\geqslant p$ by $O(p)$.

For a graded R-module A we will denote by ΣA its suspension: $(\Sigma A)_i = A_{i-1}$.

Definition 2.1. The structure of an A_∞-algebra on a cell R-module A is a coderivation $m : T\Sigma A \to T\Sigma A$ of degree -1 such that $m^2 = 0$, $m(T^0\Sigma A) = 0$ and the first component m_1 of m is (the suspension of) the original differential on A. Thus an A_∞-algebra is a pair (A, m). We will frequently omit mentioning m and simply refer to the A_∞-algebra A.

Remark 2.2. The condition that $m(T^0\Sigma A) = 0$ means that $m \in O(0)$, or that the zeroth component m_0 of m vanishes. Some authors consider A_∞-algebras with nonvanishing m_0, cf. [**6**].

This definition is slightly more general then the usual one in that R is not assumed to be a field. We emphasize here that the grading on the ground ring R will be essential for our constructions. For an A_∞-algebra A we will call the coalgebra $T\Sigma A$ with the differential m the bar-construction of A and use the symbol BA to denote it. Following the usual tradition we will denote the element $\Sigma a_1 \otimes \Sigma a_2 \ldots \Sigma a_n \in (\Sigma A)^{\otimes n}$ by $[a_1|a_2|\ldots|a_n]$. The following formula shows how to recover the coderivation $m : T\Sigma A \to T\Sigma A$ from its components:

$$m[a_1|\ldots|a_n]$$

$$= \sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{|a_1|+\ldots+|a_i|+i}[a_1|\ldots|a_i|m_k[a_{i+1}|\ldots|a_{i+k}]|a_{i+k+1}|\ldots|a_n] \qquad (2.1)$$

The components $m_i : \Sigma A^{\otimes i} \to \Sigma A$ of the coderivation m correspond to maps $\tilde{m}_i : A^{\otimes i} \to A$ of degree $i-2$. The map $\tilde{m}_1$ is the original differential in A, the map $\tilde{m}_2$ is a multiplication up to homotopy and $\tilde{m}_i : A^{\otimes i} \to A$ are higher multiplications on A.

The space $Coder(T\Sigma A) \cong Hom(T\Sigma A, \Sigma A)$ is a differential graded Lie algebra with respect to the (graded) commutator. Let $m, n \in Hom(T\Sigma A, \Sigma A)$ so that $m = (m_1, m_2, \ldots)$ and $n = (n_1, n_2, \ldots)$ where $m_i, n_i \in Hom((\Sigma A^{\otimes i}), \Sigma A)$. Then the commutator of m and n is clearly determined by commutators of their components: $[m_i, n_j] = m_i \circ n_j - (-1)^{|m_i||n_i|} n_i \circ m_i$. Furthermore we have the following formula for the composition $m_i \circ n_j \in Hom((\Sigma A)^{\otimes i+j-1}, \Sigma A)$:

$$(m_i \circ n_j)[a_1|\ldots|a_{i+j-1}] = \sum_{k=0}^{i-1}(-1)^{|n_j|(|a_1|+\ldots+|a_k|+k)} m_i[a_1|\ldots$$
$$|a_k|n_j[a_{k+1}|\ldots,|a_{k+j}]|a_{k+j+1}|\ldots|a_{i+j-1}] \qquad (2.2)$$

Definition 2.3. For two A_∞-algebras A and C an A_∞-morphism (or A_∞-map) $A \to C$ is a map of differential graded coalgebras $f : T\Sigma A \to T\Sigma C$ for which $f(T^0\Sigma A) = T^0\Sigma C = R$.

It is clear that any A_∞-map $T\Sigma A \to T\Sigma C$ between two A_∞-algebras A and C is a map of filtered coalgebras. Furthermore a coalgebra map $f : T\Sigma A \to \Sigma C$ is determined by the composition $T\Sigma A \xrightarrow{f} T\Sigma C \to \Sigma C$ where the second arrow is the canonical projection. Denoting the components of the composite map by f_i we see that f is determined by the collection $(f_1, f_2, \ldots)$ where $f_i : (\Sigma A)^{\otimes i} \to \Sigma C$. The map f could be recovered from the collection $\{f_i\}$ as follows:

$$f[a_1|\ldots|a_n] = \sum [f_{i_1}[a_1|\ldots|a_{i_1}]|\ldots|f_{i_k}[a_{i_{k-1}+1}|\ldots|a_n]] \tag{2.3}$$

where the summation is over all partitions $(i_1, \ldots, i_k)$ of n.

The components f_i of the A_∞-map f correspond to the maps $\tilde{f}_i : A^{\otimes i} \to C$ of degrees $i-1$. The map $\tilde{f}_1 : A \to C$ is a map of complexes which is multiplicative up to higher homotopies provided by $\tilde{f}_2, \tilde{f}_3, \ldots$.

We say that an A_∞-map f is a weak equivalence if $f_1 : \Sigma A \to \Sigma C$ is a quasi-isomorphism of complexes. Further we say that two A_∞-algebras A and C are weakly equivalent if there is a chain of weak equivalences $A \to A_1 \leftarrow A_2 \to \ldots \leftarrow A_n = C$.

Remark 2.4. In fact it is possible to prove that for a weak equivalence $A \to C$ there is always a weak equivalence $C \to A$. This could be proved by constructing a closed model category of cocomplete differential graded R-coalgebras and identifying bar-constructions of A_∞-algebras as fibrant-cofibrant objects in this category. The discussion of such matters would take us too far afield and we refer the reader to [**20**] where this construction is carried out.

The weak equivalences $f = (f_1, f_2, \ldots)$ that we consider later on in the paper will always have the property that the morphism f_1 is invertible (this is always the case for so-called *minimal* A_∞-algebras, i.e. such that $m_1 = 0$). The following proposition is a version of the formal implicit function theorem.

Proposition 2.5. *Let $f = (f_1, f_2, \ldots) : TA \to TC$ be a map of (filtered) coalgebras. Then f is invertible if and only if $f_1 : A \to C$ is invertible.*

Proof. If f is invertible with the inverse $g = (g_1, g_2, \ldots)$ then clearly g_1 is the inverse to f_1. Conversely suppose that f_1 is invertible. We will construct a sequence of maps $g^n : TC \to TA$ such that $f \circ g^n = id \mod O(n)$ as follows. Set $g^1 = (f_1^{-1}, 0, 0, \ldots)$. Clearly $f \circ g^1 = id \mod O(1)$. Now assume by induction that the maps $g^n = (g_1^n, g_2^n, \ldots)$ have been constructed for $n \leqslant k$. Then up to the terms of filtration $\geqslant k+1$ we have $f \circ g^k = id + f_1 \circ g_{k+1}^k + X$ where X is some map $TA \to TA$ having filtration k. Set $g^{k+1} = g^k - g_{k+1}^k - f_1^{-1} \circ X$. Then g^{k+1} agrees with g^k up to the terms of filtration $\geqslant k-1$ and has the property that $f \circ g^{k+1} = id \mod O(k+1)$. The sequence $\{g^n\}$ clearly converges in the sense of the filtration on $Hom(TC, TA)$and setting $g = lim_{n\to\infty} g^n$ we obtain $f \circ g = id$. Similarly there exists a right inverse to f so f is invertible. ☐

3. Unital structures

Definition 3.1. An A_∞ structure $m = (m_1, m_2, \ldots)$ on a cell complex A is called unital if there exists an element $1 = 1_A$ of degree zero (called the unit of A) such that $m_2[1_A|a] = a = (-1)^{|a|} m_2[a, 1_A]$ for all $a \in A$ and such that $m_i(a_1, \ldots, a_i) = 0$ for all $i \neq 2$ if one of a_i equals 1_A. An A_∞-morphism $f = (f_1, f_2, \ldots)$ between two unital A_∞-algebras A and B is called unital if $f_1[1_A] = [1_B]$ and $f_i[a_1|\ldots|a_i] = 0$ for all $i \geqslant 2$ if one of a_i equals 1_A.

Remark 3.2. Notice that $m_2[1_A|a] = a = (-1)^{|a|}m_2[a|1_A]$ is equivalent to the more customary $\tilde{m}_2(1_A, a) = \tilde{m}_2(a, 1_A) = a$.

From now on we will use the term A_∞-algebra for a unital A_∞-algebra and an A_∞-morphism for a unital A_∞-morphism (unless indicated otherwise).

One important consequence of unitality (which will not be used in this paper however) is that the complex BA with the differential determined by the collection $m = (m_1, m_2, \ldots)$ is exact for a unital A_∞-algebra A. We leave it to the interested reader to check that the map $s : [a_1|a_2|\ldots|a_n] \to [1|a_1|a_2|\ldots|a_n]$ is a contracting homotopy for BA.

The classification problem of A_∞-algebras naturally leads one to consider the group $Aut(T\Sigma A)$ of automorphisms of the coalgebra $T\Sigma A$ where A is a graded R-module. In the unital case the relevant group is the group of *normalized* automorphisms $\overline{Aut}(T\Sigma A)$ which we will now define.

Definition 3.3. Let A be a free graded R-module with a distinguished element $[1] \in \Sigma A$ of degree 1. We call an automorphism $f = (f_1, f_2, \ldots), f_i : (\Sigma A)^{\otimes i} \to \Sigma A$ of the coalgebra $T\Sigma A$ normalized if $f_1[1] = [1]$ while $f_i[a_1|\ldots|a_n] = 0$ for $i > 1$ if one of a_i's is equal to 1. The set of normalized automorphisms will be denoted by $\overline{Aut}(T\Sigma A)$.

The set $\overline{Aut}(T\Sigma A)$ is in fact a group. Indeed using the formula (2.3) one sees immediately that the composition of two normalized automorphisms is normalized. Therefore $\overline{Aut}(T\Sigma A)$ is a subgroup in $Aut(T\Sigma A)$.

The concomitant notion to a normalized automorphism is that of a normalized coderivation.

Definition 3.4. A coderivation $\xi = (\xi_0, \xi_1, \ldots) \in Coder(T\Sigma A)$ will be called normalized if $\xi_i[a_1|\ldots|a_i] = 0$ each time one of $a_k = 1$ for $i = 1, 2, \ldots$. The set of all normalized derivations is denoted by $\overline{Coder}(T\Sigma A)$.

Remark 3.5. Clearly the set $\overline{Coder}(T\Sigma A)$ forms a (graded) Lie subalgebra in the Lie algebra $Coder(T\Sigma A)$. It is natural to consider $\overline{Aut}(T\Sigma A)$ as the associated Lie group.

It is often extremely convenient to work in the dual setting. Suppose that the element $1 \in A$ can be completed to a basis $\{1, y_i, i \in I\}$ of the R-module A. The indexing set I will be finite in our examples but need not be in general.

Remark 3.6. If our ground ring R is local, than the above assumption is always satisfied. Indeed let $\{e_i\}$ be a basis of A over R. Then $1 = r_1e_1 + \ldots + r_ne_n$. Clearly the element 1 remains nonzero after reducing modulo the maximal ideal in R. Therefore one of the coefficients $r_1, \ldots, r_n$, say r_1 must be invertible in R. Then $1, e_2, e_3, \ldots$ form a basis in A. Thus the assumption that 1 can be completed to a basis in A is not really a restriction since we can always argue 'one prime at a time'.

Then the R-module dual to the coalgebra $T\Sigma A$ (usually referred to as the cobar-construction) is the algebra of noncommutative power series in variables $\{\tau, \mathbf{t}\} = \{\tau, t_1, t_2, \ldots\}$. Here the elements τ, t_i form the basis in ΣA^* dual to $[1], [y_i] \in \Sigma A$:

$$(T\Sigma A)^* = k\langle\langle \tau, \mathbf{t} \rangle\rangle$$

Notice that τ has degree -1 whereas $|t_i| = -|y_i| - 1$.

The algebra $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ has a linear topology where the fundamental system of neighborhoods of 0 is formed by those series whose constant term is 0 and which annihilate a finite dimensional submodule in $T\Sigma A$. It is clear that $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ is Hausdorff and complete with respect to this topology.

Clearly the coalgebra endomorphisms of $T\Sigma A$ are in one-to-one correspondence with continuous endomorphisms of the algebra $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ while coderivations of $T\Sigma A$ are in one-to-one correspondence with continuous derivations of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$. A continuous endomorphism f of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ is specified by its values on τ, which is a series $G(\tau, \mathbf{t})$ of degree -1 and on t_i's which are series $F_i(\tau, \mathbf{t})$ whose degree equals that of t_i. So f corresponds to a collection of power series of the form $(G(\tau, \mathbf{t}), F_1(\tau, \mathbf{t}), F_2(\tau, \mathbf{t})\ldots)$. (Observe that if the indexing set I is infinite then continuity imposes certain restrictions on the collection $G(\mathbf{t}), F_1(\mathbf{t}), F_2(\mathbf{t}), \ldots)$). The composition of endomorphisms corresponds to substitution of power series. Similarly any continuous derivation ξ could be uniquely represented in the form $\xi = A(\tau, \mathbf{t})\partial_\tau + \sum_i B_i(\tau, \mathbf{t})\partial_{t_i}$. Here ∂_τ and ∂_{t_i} are standard derivations corresponding to the coordinates τ, t_i.

Definition 3.7. A continuous derivation of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ is called normalized if the corresponding coderivation of $T\Sigma A$ is normalized. We will denote the set of normalized derivations of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ by $\overline{Der}(R\langle\langle\tau, \mathbf{t}\rangle\rangle)$. Similarly we call a continuous automorphism of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ normalized if such is the corresponding automorphism of $T\Sigma A$. The set of normalized automorphisms of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ will be denoted by $\overline{Aut}(R\langle\langle\tau, \mathbf{t}\rangle\rangle)$.

Recall that the space $Coder(T\Sigma A)$ has a filtration $O(-1) \supset O(0) \supset \ldots$ where $O(n)$ consists of those coderivations $\xi = (\xi_0, \xi_1, \ldots)$ for which $\xi_1 = \xi_2 = \ldots = \xi_n = 0$. Then the space of (continuous) derivations of $(T\Sigma A)^*$ acquires filtration so that the derivation $A(\tau, \mathbf{t})\partial_\tau + \sum_{i\in I} B_i(\tau, \mathbf{t})\partial_{t_i}$ has weight $\geqslant n$ if and only if the expressions $A(\tau, \mathbf{t}), B_1(\tau, \mathbf{t}), B_2(\tau, \mathbf{t})\ldots)$ do not contain terms of degree $\leqslant n$. We will still denote the collection of elements of weight $\geqslant n$ by $O(n)$.

Proposition 3.8. *(i) Any normalized derivation ξ of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ has the form*

$$\xi = A(\mathbf{t})\partial_\tau + \sum_{i\in I} B_i(\mathbf{t})\partial_{t_i}.$$

(ii) Any unital A_∞ structure m on A corresponds to a derivation m^* of $(T\Sigma A)^*$ of the form

$$m^* = (A(\mathbf{t}) + \tau^2)\partial_\tau + \sum_{i\in I}([\tau, t_i] + B_i(\mathbf{t}))\partial_{t_i}.$$

where the series $A(\mathbf{t}), B(\mathbf{t})$ have vanishing constant terms.

Proof. Denote by $\langle, \rangle$ the R-linear pairing between $(T\Sigma A)$ and $(T\Sigma A)^*$. Associated to a homogeneous endomorphism T of the R-module $(T\Sigma A)$ is the endomorphism T^* of $(T\Sigma A)^*$ for which

$$\langle T(a), b\rangle = (-1)^{|a||T|}\langle a, T^*(b)\rangle \tag{3.1}$$

The rest is just a routine exercise in dualization using (3.1) which we can safely leave to the reader. Note that the quadratic term $\sum_{i\in I}[\tau, t_i]\partial_{t_i} + \tau^2\partial_\tau$ corresponds to the identities $m_2[1|y_i] = (-1)^{|y_i|}m_2[y_i|1] = [y_i]$ and $m[1|1] = [1]$. The condition that $A(\mathbf{t}), B(\mathbf{t})$ have vanishing constant terms means that $m^* \in O(0)$. $\square$

Remark 3.9. Of course not every derivation ξ of $(T\Sigma A)^*$ of the form $\xi = (A(\mathbf{t}) + \tau^2)\partial_\tau + \sum_{i\in I}([\tau, t_i] + B_i(\mathbf{t}))\partial_{t_i}$ is an A_∞ structure. The condition which specifies an A_∞ structure is $\xi \circ \xi = 0$ (or, equivalently, $[\xi, \xi] = 0$ if R has no 2-torsion). Also the condition that m^* has degree -1 puts further restrictions on $A(\mathbf{t})$ and $B_i(\mathbf{t})$. For example if the variables t_i have even degrees then all B_i's necessarily vanish.

Remark 3.10. It is easy to check that the derivation $\sum_{i\in I}([\tau, t_i])\partial_{t_i} + \tau^2\partial_\tau$ can be compactly written as $ad\tau - \tau^2\partial_\tau$ where $ad\tau(?) := [\tau, ?]$. Therefore the formula for m^* could be written as

$$m^* = A(\mathbf{t})\partial_\tau + \sum_{i\in I} B_i(\mathbf{t})\partial_{t_i} + ad\tau - \tau^2\partial_\tau.$$

Similarly we could translate the notion of a normalized automorphism to the dual setting. Consider the continuous endomorphism of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ corresponding to the collection $(G, \mathbf{F}) := (\tau + G(\mathbf{t}), F_1(\mathbf{t}), F_2(\mathbf{t}), \ldots)$ of power series without constant terms. Here we require that $\mathbf{F}(\mathbf{t}) = (F_1(\mathbf{t}), F_2(\mathbf{t}), \ldots) : R\langle\langle\mathbf{t}\rangle\rangle \to R\langle\langle\mathbf{t}\rangle\rangle$ be invertible with inverse $\mathbf{F}^{-1}(\mathbf{t})$. Then clearly $(G, \mathbf{F})$ is invertible and $(G, \mathbf{F})^{-1} = (-G(\mathbf{F}^{-1}), \mathbf{F}^{-1})$. Moreover such endomorphisms form a subgroup of all continuous automorphisms of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$. Then we have the following result whose proof is similar to part (i) of Proposition 3.8.

Proposition 3.11. *The group of continuous automorphisms of $R\langle\langle\tau, \mathbf{t}\rangle\rangle$ consisting of pairs $(G, \mathbf{F})$ as above is isomorphic to $\overline{Aut}(R\langle\langle\tau, \mathbf{t}\rangle\rangle)$.*

Remark 3.12. The condition that a multiplicative automorphism necessarily has degree zero puts certain restrictions on $\mathbf{F}$ and G. For example if all variables t_i have even degrees then $G(\mathbf{t}) = 0$.

Remark 3.13. It is illuminating to consider the unit map $R \to A$ from the point of view of the cobar construction. Observe that the canonical structure of an associative algebra on R corresponds to the derivation $\tau^2\partial_\tau$ of the power series ring $R[[\tau]]$. Then the unit map $R \to A$ considered as an A_∞-map is the map of cobar constructions

$$(T\Sigma A)^* = R\langle\langle\tau, \mathbf{t}\rangle\rangle \xrightarrow{i} (T\Sigma R)^* = R[[\tau]]$$

where $i(\tau) = \tau$ and $i(\mathbf{t}) = 0$. The unitality condition ensures that i is a map of dga's. Further the maps of dga's $R[[\tau]] \to R\langle\langle\tau, \mathbf{t}\rangle\rangle$ should be considered as 'A_∞-points' of A. The existence of A_∞-points is a subtle question in general and we hope to return to it in in the future. If the A_∞ structure m^* has the form $m^* = ad\tau - \tau^2\partial_\tau + \sum_{i\in I} B(\mathbf{t})\partial_{t_i}$ then the map $\epsilon : R[[\tau]] \to R\langle\langle\tau, \mathbf{t}\rangle\rangle : \epsilon(\tau) = \tau$ is a 'canonical' A_∞-point of A.

Next observe that the group $Aut(T\Sigma A)$ acts on the set of coderivations of ΣTA according to the formula $f : m \to m^f = f \circ m \circ f^{-1}$ for $m \in Coder(\Sigma TA)$ and

$f \in Aut(\Sigma TA)$. Obviously if $m \circ m = 0$ then $m^f \circ m^f = 0$ so $Aut(\Sigma TA))$ acts on the set of (nonunital) A_∞ structures on A. It turns out the the group $\overline{Aut}(T\Sigma A) = \overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ acts on the set of *unital* A_∞ structures.

Denote by $(A, \mathbf{B})$ the derivation of $R\langle\langle \tau, \mathbf{t}\rangle\rangle$ corresponding to a unital A_∞ structure:

$$(A, \mathbf{B}) = (A(\mathbf{t}) + \tau^2)\partial_\tau + \sum_{i\in I}([\tau, t_i] + B_i(\mathbf{t}))\partial_{t_i}.$$

Proposition 3.14. *The group $\overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ acts on the right on the set of unital A_∞ structures according to the formula*

$$(A, \mathbf{B}) * (G, \mathbf{F}) = (G, \mathbf{F}) \circ (A, \mathbf{B}) \circ (G, \mathbf{F})^{-1} = (A(\mathbf{F}(\mathbf{t})) - G(\mathbf{t})^2 + \sum_{j\in I}[B_j(\mathbf{F}(\mathbf{t}))\partial_{t_j} G(\mathbf{F^{-1}})](\mathbf{F}(\mathbf{t})), \sum_{i,j\in I}([G(\mathbf{t}), t_i] + (B_j(\mathbf{F}(\mathbf{t}))\partial_{t_j}\mathbf{F}^{-1})(\mathbf{F}(t_i)). \quad (3.2)$$

Proof. This is one of the examples where the use of the dual language leads to considerable simplifications; the relatively painless calculations below become exceedingly gruesome when performed in terms of coderivations of the coalgebra $T\Sigma A$. We compute:

$$\begin{aligned}((A, \mathbf{B}) * (G, \mathbf{F}))(t_i) &= (G, \mathbf{F}) \circ (A, \mathbf{B}) \circ (G, \mathbf{F})^{-1}(t_i)\\ &= (G, \mathbf{F}) \circ (A, \mathbf{B})(\mathbf{F}^{-1}(t_i))\\ &= (G, \mathbf{F})([\tau, \mathbf{F}^{-1}(t_i)] + \sum_{j\in I} B_j(\mathbf{t})\partial_{t_j}\mathbf{F}^{-1}(t_i))\\ &= [\tau, t_i] + [G(\mathbf{t}), t_i] + \sum_{j\in I}(B_j(\mathbf{F}(\mathbf{t}))\partial_{t_j}\mathbf{F}^{-1})(\mathbf{F}(t_i)) \quad (3.3)\end{aligned}$$

Further

$$\begin{aligned}((A, \mathbf{B}) * (G, \mathbf{F}))(\tau) &= (G, \mathbf{F}) \circ (A, \mathbf{B}) \circ (G, \mathbf{F})^{-1}(\tau)\\ &= (G, \mathbf{F}) \circ (A, \mathbf{B})(\tau - G(\mathbf{F}^{-1}(\mathbf{t})))\\ &= (G, \mathbf{F})(\tau^2 + A(\mathbf{t}) - [\tau, G(\mathbf{F}^{-1}(\mathbf{t}))] - \sum_{j\in I} B_j(\mathbf{t})\partial_{t_j} G(\mathbf{F}^{-1}(\mathbf{t})))\\ &= (\tau + G(\mathbf{t}))^2 + A(\mathbf{F}(\mathbf{t})) - [\tau + G(\mathbf{t}), G(\mathbf{t})] + \sum_{j\in I}[B_j(\mathbf{F}(\mathbf{t}))\partial_{t_j} G(\mathbf{F^{-1}})](\mathbf{F}(\mathbf{t})).\end{aligned}$$

Since $G(\mathbf{t})$ and τ have odd degrees we have the equalities $(\tau + G(\mathbf{t}))^2 = \tau^2 + G(\mathbf{t})^2 + [\tau, G(\mathbf{t})]$ and $[G(\mathbf{t}), G(\mathbf{t})] = 2G(\mathbf{t})^2$. It follows that

$$\begin{aligned}&((A, \mathbf{B}) * (G, \mathbf{F}))(\tau) \quad (3.4)\\ &= \tau^2 + A(\mathbf{F}(\mathbf{t})) - G(\mathbf{t})^2 + \sum_{j\in I}[B_j(\mathbf{F}(\mathbf{t}))\partial_{t_j} G(\mathbf{F^{-1}})](\mathbf{F}(\mathbf{t}))\end{aligned}$$

The formula (3.14) is a consequence of (3) and (3.4) and our proposition is proved. □

Remark 3.15. The above proposition has two parts: the statement that the group $\overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ acts on the set of unital A_∞ structures and an explicit formula for this action. While the formula clearly requires the assumption that 1 can be completed to an R-basis in A, the statement about the group action is valid without this assumption. The proof of this could be deduced from Remark 3.6 using standard localization techniques.

Remark 3.16. Proposition 3.14 admits the following infinitesimal analogue: if m is a unital A_∞ structure and ξ is a normalized coderivation of $T\Sigma A$ then the commutator $[\xi, m]$ is also normalized. This can be interpreted as saying that the normalized Hochschild cochains of a unital A_∞-algebra form a subcomplex with respect to the Hochschild differential, cf. next section of the present paper.

Remark 3.17. We have seen that if the variables t_i all have even degrees then $B_i = 0$ and $G = 0$. In other words the group of normalized automorphisms is just the group of formal power series $\mathbf{F}(\mathbf{t})$ under composition and a unital A_∞ structure corresponds to the derivation of the form $A(\mathbf{t})\partial_\tau$. The formula (3.14) in this case takes an especially simple form: $A * \mathbf{F} = A(\mathbf{F})$.

The next result we are going to discuss requires a certain knowledge of *operads*. We do not intend to discuss this subject in detail here and refer the interested reader to the nice exposition in [**30**]. An A_∞-algebra is in fact an algebra over a certain operad $\mathcal{A}_\infty$ in the category of differential graded R-modules, sometimes called the *Stasheff operad*. The operad $\mathcal{A}_\infty$ maps into another operad $\mathcal{A}ss$ whose algebras are strictly associative differential graded R-algebras and this map is a quasi-isomorphism. In particular any differential graded algebra is an A_∞-algebra.

Proposition 3.18. *There is a functor that assigns to each unital A_∞-algebra A a strictly associative differential graded algebra $\tilde{A}$ which is weakly equivalent to A.*

Proof. We will only give a sketch following [**16**], *V*.1.7. Associated to any operad is a monad having the same algebras. Denote the monad in the category of complexes of R-modules associated to $\mathcal{A}ss$ by C and the one associated to $\mathcal{A}_\infty$ by C_∞. Then there is a canonical map of monads $C \to C_\infty$. Consider the following maps of $\mathcal{A}_\infty$-algebras

$$A \leftarrow B(C_\infty, C_\infty, A) \to B(C, C_\infty, A) \tag{3.5}$$

Here $B(-,-,A)$ stands for a two-sided monadic bar construction. Both maps in (3.5) are homology isomorphisms and our proposition is proved. ☐

4. Hochschild Cohomology of A_∞-algebras.

Let A be an A_∞-algebra. Consider the graded Lie algebra $Coder(BA)$ of all coderivations of the coalgebra $BA = T\Sigma A$. There is a preferred coderivation $m : BA \to BA$ of degree -1 which is given by the A_∞ structure on A. We will define a differential ∂ on $Coder(BA)$ by the formula $\partial(f) = [f, m]$ where the right hand side is the (graded) commutator of two coderivations f and m. The condition $m \circ m = 0$ implies that $\partial \circ \partial = 0$.

Definition 4.1. The complex $C^*(A, A) := Coder(BA)$ with the differential ∂ is called the Hochschild complex of an A_∞-algebra A. Its cohomology $H^*(A, A)$ is called the Hochschild cohomology of A with coefficients in itself.

Remark 4.2. Since the coderivation m has weight $\geqslant 0$ the differential on $C^*(A, A)$ agrees with the filtration on BA in the sense that $d(O(n)) \subset O(n)$.

Recall that since BA is cofree in the category of cocomplete coalgebras there is a natural identification $C^*(A, A) \cong Hom(BA, \Sigma A)$ which we will use without explicitly mentioning. Using the the formula (2.1) one can recover the coderivation of $BA = T\Sigma A$ from its components $c_k \in Hom((T\Sigma A)^{\otimes k}, \Sigma A) \subset Hom(BA, \Sigma A)$.

We will now introduce the normalized Hochschild complex for A_∞-algebras which is smaller and easier to compute with.

Definition 4.3. Let A be an A_∞-algebra. Then a Hochschild cochain $c \in Hom(T\Sigma A, \Sigma A)$ is called normalized if c is normalized as a coderivation of BA.

It is easy to check using (2.2) that the normalized cochains form a subcomplex of the Hochschild complex. We will denote this subcomplex by $\bar{C}^*(A, A)$.

Theorem 4.4. *Let A be an A_∞-algebra. Then there is a chain deformation retraction of $C^*(A, A)$ onto the subcomplex $\bar{C}^*(A, A)$. In particular both complexes have the same cohomology.*

Proof. The proof is similar to that of the classical theorem of Eilenberg-MacLane on normalized simplicial modules. Note that this theorem cannot be applied directly since the Hochschild cohomology of an A_∞-algebra *is not* a cohomology of a simplicial object. The resulting calculations in the A_∞ context are considerably more involved.

Let us call a cochain $c \in C^*(A, A)$ i-normalized if c vanishes each time one if its first i arguments is equal to 1. Then c is normalized if and only if it is i-normalized for all i.

We define a sequence of cochain maps $h_i : C^*(A, A) \to C^*(A, A)$ as follows. Let $c \in Hom((\Sigma A)^{\otimes n}, \Sigma A)$ for some n and consider the cochain $s_i(c) \in Hom((\Sigma A)^{\otimes n-1}, \Sigma A)$ defined by the formula

$$s_i(c)[a_1| \ldots |a_{n-1}] = (-1)^{|a_1|+\ldots+|a_l|+i+1} c[a_1| \ldots |a_i|1|a_{i+1}| \ldots |a_{n-1}].$$

Extending by linearity we define s_i on the whole $C^*(A, A)$. Then set $h_i(c) := c - \partial(s_i(c)) - s_i(\partial c)$. We claim that h_i takes an i-normalized Hochschild cochain to an $i+1$-normalized cochain. Indeed, let c be an i-normalized cochain. We could assume without loss of generality that $c \in Hom((\Sigma A^{\otimes n}), \Sigma A)$ for some $n > i$. We want to show that

$$c(?) = [s_i(c), m_k](?) + s_i[c, m_k](?) \tag{4.1}$$

for any k as long as the $i+1$st argument in ? is 1.

Notice that the left hand side of (4.1) is only nonzero if the number of arguments in ? is n whereas the right hand side of (4.1) is nonzero if the number of arguments is $n - k + 2$. Therefore we need to consider the cases $k = 2$ and $k \neq 2$ separately. For $k = 2$ we have

$$\begin{aligned}(m_2 \circ s_i c)[a_1| \ldots |a_n] = (-1)^{|a_1|+\ldots+|a_i|+i+1} & m_2[c[a_1| \ldots |a_i|1|a_{i+1}| \ldots a_{n-1}]a_n] \\ & \pm m_2[a_1|c[a_2| \ldots |a_{i+1}|1|a_{i+2}| \ldots |a_n].\end{aligned}$$

Setting $a_{i+1} = 1$ and taking into account that c is i-normalized we obtain

$$(m_2 \circ s_i c)[a_1|\ldots|a_i|1|a_{i+2}|\ldots|a_n]$$
$$= (-1)^{|a_1|+\ldots+|a_i|+i+1} m_2[c[a_1|\ldots|a_i|1|1|a_{i+2}|\ldots a_{n-1}]a_n].$$

Similarly we obtain

$$s_i(m_2 \circ c)[a_1|\ldots|a_i|1|a_{i+2}|\ldots|a_n]$$
$$= (-1)^{|a_1|+\ldots+|a_i|+i+1} m_2[c[a_1|\ldots|a_i|1|1|a_{i+2}|\ldots a_{n-1}]a_n].$$

It follows that

$$(m_2 \circ s_i c - s_i(m_2 \circ c))[a_1|\ldots|a_i|1|a_{i+2}|\ldots|a_n] = 0 \tag{4.2}$$

Taking into account the identities $m_2[a_i|1] = (-1)^{|a_i|}[a_i]$ and $m_2[1|a_{i+1}] = [a_{i+1}]$ we have

$$(s_i c \circ m_2)[a_1|\ldots|a_n] = \sum_{l=0}^{i} \pm s_i c[a_1|\ldots|a_l|m_2[a_{l+1}|a_{l+2}]|a_{l+3}|\ldots|\ldots|a_n]$$
$$+ \sum_{l=i+1}^{n-2} (-1)^{|a_1|+\ldots+|a_l|+l} s_i c[a_1|\ldots|a_i|a_{i+1}|\ldots|a_l|m_2[a_{l+1}|a_{l+2}]|a_{l+3}|\ldots|a_n].$$

After substituting $a_{i+1} = 1$ the term having the sign $\pm$ in front of it vanishes and we get

$$(s_i c \circ m_2)[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n]$$
$$= \sum_{l=i+1}^{n-2} (-1)^{|a_1|+\ldots+|a_l|+l+|a_1|+\ldots+|a_i|+i+1} c[a_1|\ldots|a_i|1|1|\ldots$$
$$|a_l|m_2[a_{l+1}|a_{l+2}]|a_{l+3}|\ldots|a_n].$$

And similarly

$$s_i(c \circ m_2)[a_1|\ldots|a_i|1|\ldots|a_n] = c[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n]$$
$$+ \sum_{l=i+1}^{n-2} (-1)^{|a_1|+\ldots+|a_l|+l+1+|a_1|+\ldots+|a_i|+i} c[a_1|\ldots|a_i|1|1|a_{i+2}|\ldots$$
$$|a_l|m_2[a_{l+1}|a_{l+2}]|a_{l+3}|\ldots|a_n]$$

Therefore

$$(s_i c \circ m_2 + s_i(c \circ m_2))[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n] = c[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n] \tag{4.3}$$

Taking into account that $|s_i c| = |c| + 1$ we obtain from (4.3), and (4.2)

$$([s_i c, m_2] + s_i[c, m_2])[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n]$$
$$= (s_i c \circ m_2 - (-1)^{|c|+1} m_2 \circ s_i c + s_i(c \circ m_2)$$
$$-(-1)^{|c|} s_i(m_2 \circ c))[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n]$$
$$= c[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_n].$$

Now let $k \neq 2$. We have:

$$(m_k \circ s_i c)[a_1|\dots|a_i|1|a_{i+2}|a_{n+k-2}] = m_k[s_i c[a_1|\dots|a_{n-1}]a_n|\dots|a_{n+k-2}]$$
$$= (-1)^{|a_1|+\dots+|a_i|+i+1} m_k[c[a_1|\dots|a_i|1|1|a_{i+2}|\dots|a_{n-1}]|a_n|\dots|a_{n+k-2}]$$

(the remaining terms in the expansion for $(m_k \circ s_i c)[a_1|\dots|a_i|1|a_{i+2}|a_{n+k-2}]$ vanish because the Hochschild cochains $s_i c$ is i-normalized and m_k is normalized). Likewise

$$s_i(m_k \circ c)[a_1|\dots|a_i|1|a_{i+1}|\dots|a_{n+k-2}]$$
$$= (-1)^{|a_1|+\dots+|a_i|+i+1} m_k \circ c[a_1|\dots|a_i|1|1|a_{i+2}|\dots|a_{n+k-2}]$$
$$= (-1)^{|a_1|+\dots+|a_i|+i+1} m_k[c[a_1|\dots|a_i|1|1|a_{i+2}|\dots|a_{n-1}]|a_n|\dots|a_{n+k-2}]$$

It follows that

$$(m_k \circ s_i c - s_i(m_k \circ c))[a_1|\dots|a_i|1|a_{i+2}|\dots|a_n] = 0. \tag{4.4}$$

Further

$$(s_i c \circ m_k)[a_1|\dots|a_i|1|a_{i+1}|\dots|a_{n+k-2}]$$
$$= \sum_{l=i}^{n-2} (-1)^{|a_1|+\dots+|a_l|+l} s_i c[a_1|\dots|a_i|1|a_{i+1}|\dots$$
$$|a_l|m_k[a_{l+1}|\dots|a_{l+k}]|a_{l+k+1}|\dots|a_{k+n-2}]$$
$$= \sum_{l=i}^{n-2} \epsilon_l c[a_1|\dots|a_i|1|a_{i+1}|\dots|a_l|m_k[a_{l+1}|\dots|a_{l+k}]|a_{l+k+1}|\dots|a_{k+n-2}]$$

where $\epsilon_l = (-1)^{|a_1|+\dots+|a_i|+i+1+|a_1|+\dots+|a_l|+l}$ and similarly

$$s_i(c \circ m_k)[a_1|\dots|a_i|1|a_{i+1}|\dots|a_{n+k-2}]$$
$$= (-1)^{|a_1|+\dots+|a_i|+i+1} c \circ m_k[a_1|\dots|a_i|1|a_{i+1}|\dots|a_{k+n-2}]$$
$$= \sum_{l=i+1}^{n-2} (-\epsilon_l) c[a_1|\dots|a_i|1|1|a_{i+2}|\dots|a_l|m_k[a_{l+1}|\dots|a_{l+k}]|a_{l+k+1}|\dots|a_{k+n-2}].$$

Therefore

$$(s_i c \circ m_k + s_i(c \circ m_k))[a_1|\dots|a_i|1|a_{i+1}|\dots|a_{n+k-2}] = 0 \tag{4.5}$$

Finally (4.4) and (4.5) imply that $[s_i c, m_k] + s_i[c, m_k] = 0$.

This proves (4.1) and, therefore, our claim that h_i takes i-normalized cochains into $i+1$-normalized cochains. It follows that the composition $\dots \circ h_l \circ h_{l-1} \dots h_0$ takes an arbitrary cochain $c \in C^*(A, A)$ into a normalized cochain and exhibits the subcomplex $\bar{C}(A, A)$ of normalized cochains as a chain deformation retract of $C^*(A, A)$. ◻

Remark 4.5. Part of Theorem 4.4 could be interpreted as saying that if the cochain $c \in C^*(A, A)$ has the property that $[c, m]$ belongs to the Lie subalgebra of normalized cochains then there exists a normalized cochain c' for which $[c, m] = [c', m]$. This result has the following globalization proved in [**20**]: if two minimal A_∞ structures are equivalent through a nonunital A_∞-morphism, then they are equivalent also through a unital one. It would be interesting to deduce this result from Theorem 4.4 (the proof in the cited reference uses obstruction theory).

Remark 4.6. Let us call an A_∞-algebra A *homotopy unital* if there exists an element $1 \in A$ of degree 0 for which $m_1[1] = 0$, and $m_2[1|a] = (-1)^{|a|}m_2[a|1]$ for any $a \in A$. It is easy to see that (in contrast with strict unitality) weak equivalences preserve homotopy unitality for minimal A_∞-algebras. Then it is proved in [**20**] that any minimal homotopy unital A_∞-algebra is weakly equivalent to a (strictly) unital one. This result combined with the previous remark and Proposition 3.14 shows that the classification problem of (minimal) homotopy unital A_∞-algebras up to a nonunital weak equivalence is equivalent to classification of unital A_∞-algebras up to a unital weak equivalence.

For the next result we will need a slightly more general definition of Hochschild cohomology than the one already given. Let (A, m_A), (C, m_C) be two A_∞-algebras and $i : BA \to BC$ an A_∞-morphism between them. We say that a map $f : BA \to BC$ is a coderivation of the coalgebra BA with values in the coalgebra BC if the following diagram is commutative:

$$\begin{tikzcd} BA \arrow[r, "\Delta_{BA}"] \arrow[d, "f"] & BA \otimes BA \arrow[d, "f\otimes i + i\otimes f"] \\ BC \arrow[r, "\Delta_{BC}"] & BC \otimes BC \end{tikzcd}$$

Here Δ_{BA} and Δ_{BC} denote the diagonals in the coalgebras BA and BC. Then the space $Coder(BA, BC)$ becomes a complex with the differential $df = m_C \circ f - (-1)^{|f|} f \circ m_A$. We will denote this complex by $C^*(A, C)$.

Now let $c \in C^*(A, A)$ be a Hochschild cochain. Define the cochain $i_*(c) \in C^*(A, C)$ by the formula $i_*(c) = i \circ c : BA \to BC$. Likewise for a cochain $c' \in C^*(C, C)$ define the cochain $i^*(c') \in C^*(A, C)$ by the formula $i^*(c') = c' \circ i$. It is straightforward to check i_* and i^* give maps of cochain complexes:

$$i_* : C^*(A, A) \to C^*(A, C) \leftarrow C^*(C, C) : i^*.$$

Proposition 4.7. *For two weakly equivalent A_∞-algebras A and C their Hochschild complexes $C^*(A, A)$ and $C^*(C, C)$ are quasi-isomorphic as complexes of R-modules. In particular, $H^*(A, A) \cong H^*(C, C)$.*

Proof. Let $i : BA \to BC$ be an A_∞-morphism establishing a weak equivalence between A and C. Since the cochain map $i_* : C^*(A, A) \to C^*(A, C)$ is a filtered map it induces a map on associated spectral sequences. Since i induces a quasi-isomorphism $A \to C$ we see that i_* induces an isomorphism of the E_1-terms of the corresponding spectral sequences. Therefore i_* is itself a quasi-isomorphism. Similar considerations show that the cochain map $i^* : C^*(C, C) \to C^*(A, A)$ is a quasi-isomorphism and our proposition is proved. □

Remark 4.8. In general the complex $C^*(A, A)$ as well as its cohomology $H^*(A, A)$ is not functorial with respect to A. It is possible to define the Hochschild complex $C^*(A, M)$ of an A_∞-algebra with coefficients in a A_∞-bimodule M, cf. for example, [**6**]. Then $C^*(A, M)$ is contravariant in the variable A and covariant in the variable M. However we don't need such level of generality here and the discussion of A_∞-bimodules would take us too far afield.

Now let A be an A_∞-algebra. Propositions 4.7 and 3.18 shows that the complex $C^*(A, A)$ is quasi-isomorphic to the complex $C^*(\tilde{A}, \tilde{A})$ where $\tilde{A}$ is a differential graded (unital) algebra weakly equivalent to A. The complex $C^*(\tilde{A}, \tilde{A})$ is the usual Hochschild complex of the dga $\tilde{A}$ and it is well-known that it possesses itself a structure of a homotopy commutative dga; something that we did not see from the point of view of the A_∞-algebra A. The Hochschild complex of a dga admits a different (but of course equivalent) description. Namely, we can define the complex $C^*(\tilde{A}, \tilde{A})$ as an object in the derived category of $\tilde{A} \otimes \tilde{A}^{op}$-modules:

$$C^*(\tilde{A}, \tilde{A}) := RHom_{\tilde{A} \otimes^L \tilde{A}^{op}}(\tilde{A}, \tilde{A}).$$

Here $\tilde{A}^{op}$ is the differential graded algebra having the same underlying R-module and differential as $\tilde{A}$ but the opposite multiplication. Since $\tilde{A}$ has the same homology algebra as A we get the following result:

Proposition 4.9. *There exists a spectral sequence of R-modules*

$$Ext^{**}_{H_*(A_* \otimes^L A_*^{op})}(H_*(A), H_*(A)) \Longrightarrow H^*(A, A).$$

It is of standard cohomological type, lies in the right half plane and converges conditionally.

5. Moore algebras

In this section we introduce and study a class of A_∞-algebras which will be called Moore A_∞-algebras or just Moore algebras. The terminology comes from stable homotopy theory - a Moore algebra is analogous to the Moore spectrum which is a cofibre of the map $S \xrightarrow{p} S$ where S is the sphere spectrum. In some sense Moore algebras are the simplest nontrivial examples of A_∞-algebras which are not differential graded algebras.

Definition 5.1. An A_∞-algebra over a commutative evenly graded ring R is called a Moore algebra if its underlying complex is $A = \{\Sigma^d R \xrightarrow{\partial} R\}$ for some differential ∂ . The integer d is called the degree of A.

Obviously the generator in degree 0 is $1 \in R$. We will denote the generator in $\Sigma^d R$ by y, so $|y| = d+1$. The structure of an A_∞-algebra on A is clearly determined by the collection $m_i[y]^{\otimes i}, i = 1, 2, \ldots$. Notice that the map ∂ is necessarily given by a multiplication by some $x \in R$ so that $\partial(y) = x \cdot 1$. If d is odd, then $\partial = 0$. If d is even and x is not a zero divisor in R, then the (internal) homology of A is simply R/x. For an E_∞ ring spectrum R the structure of the (homotopy) associative algebra on R/x was investigated in [**4**] and [**28**]. This parallel topological theory was our original motivation for introducing the notion of a Moore algebra.

Let R' be another evenly graded commutative ring and $f : R \to R'$ be a ring map. Consider a Moore algebra A over R specified by the collection $\{m_i[y]^{\otimes i} \in R\}$.

Then the collection $\{f(m_i[y]^{\otimes i}) \in R'\}$ will determine a Moore algebra f_*A over R'. In other words the set $\mathcal{S}(d)$ which associates to any evenly graded commutative ring R the set of Moore algebras over R of degree d is a functor of R.

Theorem 5.2. (i). *Let d be even. Then the functor $\mathcal{S}(d)$ is representable by the polynomial algebra $R_e = \mathbb{Z}[u_1, u_2, \ldots]$ where $|u_i| = i(d+2)-2$. More precisely there exists a Moore algebra A_e over R_e of degree d such that for any R and any Moore algebra A over R of degree d there exists a unique ring map $R_o \to R$ for which $f_*A_e = A$. The universal Moore algebra A_e is specified by the formulae $m_i[y]^{\otimes i} = u_i[1], i = 1, 2, \ldots$.*

(ii). *Let d be odd. Then $\mathcal{S}(d)$ is represented by the polynomial algebra $R_o = \mathbb{Z}[v_1, v_2, \ldots] \otimes \mathbb{Z}[w_1, w_2, \ldots]$ where $|v_i| = 2i(d+2) - d - 3$ and $|w_i| = 2i(d+2) - 2$. More precisely there exists a Moore algebra A_o over R_o of degree d such that for any R and any Moore algebra A over R of degree d there exists a unique ring map $R_o \to R$ for which $f_*A_o = A$. The universal Moore algebra A_o is specified by the formulae $m_{2i-1}[y]^{\otimes 2i-1} = 0$ and $m_{2i}[y]^{\otimes 2i} = v_i[y] + w_i[1], i = 1, 2, \ldots$.*

Proof. In both cases (i) and (ii) the universality is obvious and we only need to prove that $m \circ m = 0$. Note that apriori the latter equation could impose nontrivial relations on the generators u_i, v_i and w_i; the theorem effectively states that no such relations except commutativity are in fact present.

The equality $m \circ m = 0$ could be checked directly using the composition formula (2.2). However this path is rather long-winded and unenlightening and we will choose the approach via the cobar-construction. So consider the algebra $(T\Sigma A)^* = R\langle\langle \tau, t\rangle\rangle$ where τ and t are dual to $[1]$ and $[y]$ respectively so $|\tau| = -1, |t| = -d-2$. Then the coderivation m of $T\Sigma A$ determines the continuous derivation m^* of $(T\Sigma A)^*$. Routine inspection shows that in the case (i)

$$m^* = \sum_{i=1}^{\infty} u_i t^i \partial_\tau + ad\tau - \tau^2 \partial_\tau$$

whereas in the case (ii) we have

$$m^* = \sum_{i=1}^{\infty} v_i t^{2i} \partial_t + \sum_{i=1}^{\infty} w_i t^{2i} \partial_\tau - \tau^2 \partial_\tau + ad\tau.$$

For (i) we compute

$$(m^* \circ m^*)(t) = m^*([\tau, t]) = [\sum_{i=1}^{\infty} u_i t^i + \tau^2, t] - [\tau, [\tau, t]].$$

The elements t and $\sum_{i=1}^{\infty} u_i t^i$ are both even and therefore $[\sum_{i=1}^{\infty} u_i t^i, t] = 0$. Clearly $[\tau^2, t] - [\tau, [\tau, t]] = 0$ This implies that $(m^* \circ m^*)(t) = 0$. Next,

$$(m^* \circ m^*)(\tau) = m^*(\sum_{i=1}^{\infty} u_i t^i + \tau^2)$$

$$= [\tau, \sum_{i=1}^{\infty} u_i t^i] + m^*(\tau^2) = [\tau, \sum_{i=1}^{\infty} u_i t^i] + \sum_{i=1}^{\infty} u_i t^i \partial_\tau(\tau^2) = 0$$

It proves that $m^*\circ m^*=0$. Similarly for (ii) we have

$$(m^*\circ m^*)(t)=m^*(\sum_{i=1}^{\infty}v_it^{2i}+[\tau,t])$$

$$=(\sum_{i=1}^{\infty}v_it^{2i}\partial_t)(\sum_{i=1}^{\infty}v_it^{2i})-[\tau,\sum_{i=1}^{\infty}v_it^{2i}]+[\tau,\sum_{i=1}^{\infty}v_it^{2i}]+[\tau^2,t]-[\tau,[\tau,t]].$$

Since now the element t is odd the derivation $\sum_{i=1}^{\infty}v_it^{2i}\partial_t$ is also odd while $\sum_{i=1}^{\infty}v_it^{2i}$ is even and it follows that $(\sum_{i=1}^{\infty}v_it^{2i}\partial_t)(\sum_{i=1}^{\infty}v_it^{2i})=0$. Just as before we have $[\tau^2,t]-[\tau,[\tau,t]]=0$. Therefore $(m^*\circ m^*)(t)=0$. Further

$$(m^*\circ m^*)(\tau)=m^*(\sum_{i=1}^{\infty}w_it^{2i}+\tau^2)$$

$$=(\sum_{i=1}^{\infty}v_it^{2i}\partial_t)(\sum_{i=1}^{\infty}w_it^{2i})+(\sum_{i=1}^{\infty}w_it^{2i}\partial_\tau)(\tau^2)+[\tau,\sum_{i=1}^{\infty}w_it^{2i})].$$

Arguing as before we see that the first term in the last expression is zero whereas the second and third cancel each other out. Therefore $(m^*\circ m^*)(\tau)=0$ and we are done. $\square$

Remark 5.3. For an odd d the differential on the underlying complex $A=\{\Sigma^dR\to R\}$ is zero and therefore its homology is fixed. For d even the differential is given by multiplication with $u_1=x\in R_e$. The element u_1 plays a special role among u_i's fixing the homology of the Moore algebra. We will be interested mostly in the case when x is a nonzero divisor in R in which case $H_*(A)=R/x$.

Remark 5.4. The universal odd Moore algebra has an ideal generated by the element y. This is a nonunital A_∞-algebra over R_o such that $m_{2i}[y]^{2i}=v_i[y]$ and $m_{2i-1}=0$. This A_∞-algebra was introduced in the early nineties by M.Kontsevich. It turns out to be related to Morita-Miller-Mumford classes in the cohomology of moduli spaces of algebraic curves, cf. [**14**]. It would be interesting to understand whether our more general constructions can yield new information about cohomologies of these moduli spaces.

We see, therefore, that an arbitrary even Moore algebra A over a ring R is specified by the collection $\{u_i^A\}\in R$ where u_i^A is the image of u_i under the classifying map $R_e\to R$. In that case the A_∞ structure on A is the following derivation m_A^* of the algebra $(T\Sigma A)^*=R\langle\langle\tau,t\rangle\rangle$:

$$m_A^*=\sum_{i=1}^{\infty}u_i^At^i\partial_\tau+ad\tau-\tau^2\partial_\tau.$$

Similarly an odd Moore algebra over R is determined by the collection $\{v_i^A,w_i^A\in R\}$, the images of v_i and w_i under the classifying map $R_o\to R$. The A_∞ structure on A is the following derivation m_A^* of the algebra $(T\Sigma A)^*$:

$$m_A^*=\sum_{i=1}^{\infty}v_i^At^{2i}\partial_t+\sum_{i=1}^{\infty}w_i^At^{2i}\partial_\tau-\tau^2\partial_\tau+ad\tau.$$

We see that an even (odd) Moore algebra is completely characterized by a power series $u^A(t) := \sum_{i=1}^{\infty} u_i^A t^i$ (by a pair of power series $(v^A(t), w^A(t)) := (\sum_{i=1}^{\infty} v_i^A t^{2i}, \sum_{i=1}^{\infty} w_i^A t^{2i})$ respectively). We will call these power series *characteristic power series* for corresponding Moore algebras.

Remark 5.5. The notion of a characteristic power series is similar to that of a formal group law. Further the universal (even or odd) Moore algebra is analogous to the universal formal group law over the Lazard ring (which is also a polynomial ring in infinitely many variables). The Moore algebras corresponding to different points of R_e or R_o could still be weakly equivalent (note that this is exactly what happens also for formal groups). Moreover we have certain infinite-dimensional Lie groups acting on R_o and R_e whose orbits correspond to weakly equivalent Moore algebras. These actions are far from being free which means that there are moduli stacks rather than moduli spaces of Moore algebras. We will see in the next section that in the even case the corresponding group is just the group of formal power series in one variable with vanishing constant term. This again forces one to think of the analogy with formal groups.

6. Classification problem

It is an interesting and nontrivial problem to classify Moore algebras over a given ring up to a (unital) weak equivalence. In this paper we will consider only the even case. An even Moore algebra A of degree d has the characteristic series

$$u^A(t) = u(t) = \sum_{i=1}^{\infty} u_i t^i \tag{6.1}$$

Here $|t| = -(d+2)$ and $|u_i| = i(d+2) - 2$ from which we conclude that $|u(t)| = -2$. Conversely any such power series determines an even Moore algebra. It is easy to see that if $f = (f_1, f_2, \ldots) : BA \to BC$ is a weak equivalence between two even Moore algebras then f_1 is an isomorphism so f is invertible (even though A and C need not be minimal). It follows that the set of weak equivalence classes of even Moore algebras coincides with the set of orbits of the group $\overline{Aut}(R\langle\langle \tau, t\rangle\rangle)$ on the set of (unital) A_∞ structures on A which could be identified with the set of power series (6.1). According to Remark 3.17 the group $\overline{Aut}(R\langle\langle \tau, t\rangle\rangle)$ is the group of formal power series $f(t) = f_1 t + f_2 t^2 + \ldots$, where f_1 is invertible and the group operation is composition. (Notice that the condition that $\overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ consists of morphisms of zero degree imposes some restrictions on f_i, namely $|f_i| = (i-1)(d+2)$.) The action is given by substitution of power series. To summarize we have the following

Theorem 6.1. *The set of equivalence classes of even Moore algebras over R is in $1-1$ correspondence with the set of orbits of the group $\overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ acting on the set of formal power series with coefficients in R of degree -2 with vanishing constant term. The action of the group element $f(t)$ on the power series $u(t)$ is given by the formula $u(t) \to u(f(t))$.*

Remark 6.2. Suppose that the ring R is 2-periodic, i.e. it possesses an invertible element v of degree 2. In that case the group $\overline{Aut}(R\langle\langle \tau, \mathbf{t}\rangle\rangle)$ is isomorphic to the group of formal power series with coefficients in R_0, the zeroth component of R and having vanishing constant term. The set of characteristic series becomes the set of all power series with coefficients in R_0 without constant term and the action is given by substitution as above. In other words there are no degree restrictions on the coefficients of power series.

Just as for formal groups it seems hopeless to try to make the classification over an arbitrary ring. The restriction that we place on R is that we assume that R is either a (graded) field or a (graded) complete discrete valuation ring. We refer the reader to the book [**25**] by Serre for an account on discrete valuation rings. In this book the ungraded rings are treated but passage to the graded case is automatic.

Next we introduce the notion of the height of a formal power series which will be one of the invariants of the associated Moore algebra.

Definition 6.3. Let $u(t) = \sum_{i=1}^{\infty} u_i t^i$ be a formal power series without a constant term. Then we say that $u(t)$ has height n if u_n is the first nonzero coefficient of $u(t)$. The height of the characteristic series of an even Moore algebra A is called the height of A.

Proposition 6.4. *Let R be a graded field of characteristic zero, A and C be two even Moore algebras over R with characteristic series $u^A(t) = \sum_{i=1}^{\infty} u_i^A t^i$ and $u^C(t) = \sum_{i=1}^{\infty} u_i^C t^i$. Then A is weakly equivalent to C if and only if $n =$height$(A) =$height(C) and $r^n u_n^A = u_n^C$ for some $r \in R_0$. Thus the equivalence class of an even Moore algebra of degree d is determined by a pair (n, r) where n is the height and $r \in R_0^\times / {R_0^\times}^n$ is an element in $R_0^\times$ modulo the subgroup of nth powers.*

Proof. Let A have height n. Then $u^A(t) = \sum_{i=n}^{\infty} u_i^A t^i$. First we prove that there exists a power series $h(t)$ such that $u^A(h(t)) = u_n^A t^n$. Let $k_1 \in \mathbb{Z}$ the smallest integer for which $u_{k_1}^A \neq 0$ and $k_1 > n$. If no such integer exists then $u^A(t)$ is already in the desired form $u^A(t) = u_n^A t^n$. Otherwise consider the polynomial

$$h_1(t) = t - \frac{t^{k_1-(n-1)} u_{k_1}^A}{n u_n^A}.$$

Then by Taylor's formula

$$\begin{aligned} u^A \circ h_1(t) &= u^A(t) - [\frac{d}{dt} u^A(t)] \frac{t^{k_1-(n-1)} u_{k_1}^A}{n u_n^A} \mod (t^{k_1+1}) \\ &= u_n^A t^n \mod (t^{k_1+1}). \end{aligned}$$

Now let $k_2 > n$ be the smallest integer for which the coefficient at t^{k_2} in $u^A \circ h_1(t)$ is nonzero. Clearly $k_2 > k_1$. Then just as before we could find $h_2(t)$ for which $u^A \circ h_1 \circ h_2(t) = u_n^A t^n \mod (t^{k_2})$. Continuing this process we construct the sequence $h_1, h_2, \ldots$ of polynomials of the form $h_i = t + a_i t^{k_i}$ for some $a_i \in R$. The sequence $\{h_1 \circ h_2 \circ \ldots \circ h_i\}$ clearly converges in the t-adic topology and denoting its limit by $h(t)$ we obtain $u^A(h(t)) = u_n^A t^n$.

In other words we proved that A is weakly equivalent to the Moore algebra A' having the characteristic series $u^{A'}(t) = u_n^A t^n$. Similarly C is equivalent to a Moore algebra C' with characteristic series $u^{C'}(t) = u_m^C t^m$. The rest is clear: A' and C' are equivalent if and only if

1. $n = m$ and
2. $u^{A'}(rt) = r^n u_n^A t^n = u^{C'}(t) = u_n^C t^n$ for some $r \in R_0^\times$

which means that $r^n u_n^A = u_n^C$. □

Remark 6.5. The assumption that R has characteristic 0 could be replaced with the assumption that height(A)=height(C) does not divide *char*(R). The proof is the same verbatim.

Remark 6.6. Notice that the statement of Theorem 6.4 is vacuous in the case of height 1. Indeed, an even Moore algebra having the characteristic series $\sum_{i=1}^\infty u_i t^i$ with u_1 invertible is trivial since the underlying complex $\Sigma^d R \to R$ is contractible. To get a nontrivial even Moore algebra of height 1 we have to have nonzero non-invertible (homogeneous) elements in the ground ring R. This is the situation that arises in the study of MU-modules and MU-algebras in stable homotopy theory, cf. [**28**] and [**18**]. In order to obtain reasonable classification results we need to impose certain conditions on R.

Definition 6.7. Let R be a (graded) complete discrete valuation ring with uniformizer π and $u(t) = \sum_{i=1}^\infty u_i t^i$ is a power series with coefficients in R. We will call $u(t)$ *trivial* if $u(t) = \pi t$ and *canonical* if there exists an n for which

1. $u_1 = \pi$;
2. π divides $u_2, u_3, \ldots, u_{n-1}$;
3. u_n is invertible;
4. $u_{n+1} = u_{n+2} = \ldots = 0$.

Remark 6.8. A canonical power series $u(t)$ could be defined equivalently as $tP(t)$ where $P(t)$ is an Eisenstein polynomial.

Proposition 6.9. *Let R be a graded complete discrete valuation ring with residue field of characteristic* 0 *and uniformizer π. Let A be an even Moore algebra having characteristic series $u^A(t) = \sum_{i=1}^\infty u_i^A t^i$ where $u_1^A = r\pi$, $r \in R^\times$. Then A is weakly equivalent to the algebra having either trivial or canonical characteristic series. Moreover two Moore algebras having canonical characteristic series are weakly equivalent if and only if these series coincide.*

Remark 6.10. Recall, that a complete discrete valuation ring R with residue field R/π of characteristic zero is isomorphic to the formal power series ring $R/\pi[[T]]$, in particular R is a vector space over R/π.

Proof. In the interests of readability we suppress the superscript A and will write $u(t) = \sum_{i=1}^\infty u_i t^i$ for $u^A(t) = \sum_{i=1}^\infty u_i^A t^i$. First we could assume that $u_1 = \pi$ or else use the substitution $u(t) \to u(r^{-1}t)$ to reduce $u(t)$ to the desired form. Next suppose that all coefficients of $u(t)$ are divisible by π. Then clearly using substitutions $u(t) \to u(t + a_n t^n)$ for suitable n and a_n we could eliminate these coefficients one by one and reduce $u(t)$ to the trivial form. Now assume that not all u_i are divisible by π

and denote by u_k the first such. In other words $u(t) = \sum_{i=1}^{k} u_i t^i \mod (t^{k+1})$ where $u_i = 0 \mod (\pi)$ for $i = 1, 2, \dots, k-1$ and u_k is invertible in R.

If for $i > k$ the coefficients u_i are zero we are done since $u(t)$ is already in the canonical form. If not let $l(u)$ be the maximal integer l for which $u_i = 0 \mod (\pi^l)$, $i = k+1, k+2, \dots$. Notice that $l(u)$ could be zero. Our first step would be to find an appropriate substitution $u(t) \to u(h^1(t))$ such that $l(u(h)) > l(u)$. Set

$$s_1 := min\{i : i > k, \pi^{l(u)} \text{ divides } u_i \text{ but } \pi^{l(u)+1} \text{ does not divide } u_i\}.$$

Then we have

$$u(t) = \sum_{i=1}^{k} u_i t^i + u_{s_1} t^{s_1} \mod (t^k \pi^{1(u)+1} + t^{s_1+1}\pi^{l(u)}).$$

Let $h_1(t) := t - \frac{u_{s_1}}{k u_k} t^{s_1-(k-1)}$ (recall that u_k is invertible). Then Taylor's formula implies that

$$u \circ h_1(t) = \sum_{i=1}^{k} v_i t^i \mod (t^k \pi^{1(u)+1} + t^{s_1+1}\pi^{l(u)})$$

where $v(t) := \sum_{i=1}^{k} v_i t^i$ is a canonical polynomial. Notice that $l(v) \geqslant l(u)$. If $l(v) > l(u)$ then our first step is completed. Assuming that $l(v) = l(u)$ set

$$s_2 := min\{i : i > k, \pi^{l(v)} \text{ divides } v_i \text{ but } \pi^{l(v)+1} \text{ does not divide } v_i\}.$$

Observe that $l(u) = l(v)$ implies $s_2 > s_1$. It follows that

$$v(t) = \sum_{i=1}^{k} v_i t^i + u_{s_2} t^{s_2} \mod (t^k \pi^{1(u)+1} + t^{s_2+1}\pi^{l(u)}).$$

Then just as before set $h_2(t) := t - \frac{v_{s_2}}{k v_k} t^{s_2-(k-1)}$ and consider the series $w(t) := v(h_2(t)) = u \circ h_1 \circ h_2$. Continuing in this way we construct a sequence of power series $\{u \circ h_1 \circ h_2 \circ \dots \circ h_n\}$. This is clearly a Cauchy sequence in the t-adic topology and converges (or perhaps stops at a finite stage) to a power series $u^1(t)$ having the property that $l(u^1) > l(u)$. Notice that $u^1(t) = u(h^1(t))$ where $h(t) = h_1 \circ h_2 \circ \dots$. Moreover we have $h^1(t) = t \mod (\pi^{l(u)})$. This completes the first step. It is clear how to proceed further. Repeating the above procedure we find power series $h^2(t)$ and $u^2(t) := u^1(h^2(t))$ such that $h^1(t) = t \mod (\pi^{l(u^1)})$ and $l(u^2) > l(u^1)$. The sequence $\{h^1 \circ h^2 \circ \dots \circ h^n\}$ is a Cauchy sequence in the π-adic topology and converges to $h(t)$. Then $u(h(t))$ is a canonical polynomial.

We still need to prove that two even Moore algebras having different canonical polynomials cannot be equivalent. In other words we have to show that if $u(t) = \sum_{i=1}^{n} u_i t^i$ and $v(t) = \sum_{i=1}^{m} v_i t^i$ are two canonical polynomials and $h(t) = \sum_{i=1}^{\infty} h_i t^i$ is such that

$$u(h(t)) = v(t) \tag{6.2}$$

then $u(t) = v(t)$. Indeed since $u_1 = v_1 = \pi$ the equality (6.2) implies that $h_1 = 1$ so we have $h(t) = t + \tilde{h}(t)$ where $\tilde{h}(t) = 0 \mod (t^2)$. Further from (6.2) we obtain $u_n(h(t))^n = v_m t^m \mod (\pi)$. It follows that $m = n$, $u_n = v_m \mod (\pi)$

and $(h(t))^n = t^n \mod (\pi)$. Since the residue field R/π has characteristic 0 the last equality implies $h(t) = t \mod (\pi)$ or equivalently $\tilde{h}(t) = 0 \mod (\pi)$. Now suppose that $\tilde{h} \neq 0$ and let k be the unique integer for which π^k divides $\tilde{h}(t)$ and π^{k+1} does not divide $\tilde{h}(t)$. By Taylor's formula

$$u(h(t)) = u(t + \tilde{h}(t)) = u(t) + u'(t)\tilde{h}(t) \mod (\pi^{k+1}).$$

Since $u_k \neq 0 \mod (\pi)$ the series $u'(t)\tilde{h}(t) \mod (\pi^{k+1})$ will necessarily have nonzero terms of order $> k$ in t. This is a contradiction with our assumption that $u(h(t)) = v(t)$ is a canonical polynomial of degree k. With this our proposition is proved. □

Remark 6.11. One would naturally like to know to whether Proposition 6.9 remains true if the residue field R/π has characteristic p. Suppose that is the case and let A be an even Moore algebra having characteristic series $u^A(t) = \sum_{i=1}^{\infty} u_i^A t^i$ with $u_1 = r\pi, r \in R^\times$. If all coefficients $u_i, i = 2, 3, \ldots$ of $u^A(t)$ are divisible by π then exactly as in the proof of Proposition 6.9 one shows that A is equivalent to a Moore algebra having characteristic series $u(t) = t$. If not, let u_k^A be the first invertible coefficient in $u(t)$. If p does not divide k then the proof of Proposition 6.9 carries over verbatim to show that $u^A(t)$ can be reduced to a canonical form and this canonical form is unique. Therefore in this case we have an exact analogue of Proposition 6.9. If p does divide k the classification seems to be much more subtle.

7. Cohomology of Moore algebras

In this section we compute Hochschild cohomology of Moore algebras of even degree subject to the condition that the first coefficient of its characteristic series is a nonzero divisor and discuss their connection with totally ramified extensions of local fields. It would be interesting to calculate Hochschild cohomology for *odd* Moore algebras. In principle the same method should apply, however in order to get a sensible answer one has to place some restrictions on characteristic series and it is not immediately clear what these restrictions should be.

Proposition 7.1. *Let A be the Moore algebra of even degree over R with characteristic series $u(t) = u^A(t)$. Let us assume that the coefficient $u_1 = u_1^A$ of $u(t)$ is not a zero divisor in R. Then there is an isomorphism of R-modules $HH^*(A, A) \cong R[[t]]/(u'(t))$ where $u'(t)$ denotes the derivative of the power series $u(t)$.*

Proof. We will compute $HH^*(A, A)$ as the homology of the operator $[?, m^*]$ on the space of normalized (continuous) derivations of $T\Sigma A^* = R\langle\langle \tau, t\rangle\rangle$. Recall that $m^* = u(t)\partial_\tau + ad\tau - \tau^2\partial_\tau$. Let ξ be a normalized derivation so $\xi = A(t)\partial_\tau + B(t)\partial_t$. Then completely automatic calculations show that $[\xi, m^*] = u'(t)B(t)\partial_\tau$. The condition that u_1 is not a zero divisor in R implies that $u'(t)$ is not a zero divisor in $R[[t]]$. Therefore the kernel of the operator $[?, m^*]$ consists of derivations of the form $A(t)\partial_\tau$ whereas its image is precisely the derivations of the form $u'(t)B(t)\partial_\tau$ and we are done. □

Let us now look at the Hochschild cohomology of an even Moore algebra A from the point of view of the associated dga $\tilde{A}$. Without loss of generality we may suppose that $\tilde{A}$ is a cell R-module. Again, our standing assumption is that u_1^A is not a zero divisor in R so the internal homology of A is R/u_1^A. We have the classical Hochschild (bi)complex $C^*(\tilde{A}, \tilde{A})$:

$$\tilde{A} \to Hom(\tilde{A}, \tilde{A}) \to \ldots \to Hom(\tilde{A}^{\otimes n}, \tilde{A}) \to \ldots \tag{7.1}$$

Associated to this bicomplex is the spectral sequence with the E^1-term $E^1_{*n} = H_*(Hom(\tilde{A}^{\otimes n}, \tilde{A}))$. Since $\tilde{A}$ is weakly equivalent to a finite cell R-module the natural map $Hom(\tilde{A}, R) \otimes \tilde{A} \to Hom(\tilde{A}, \tilde{A})$ is a homology isomorphism. We have the following sequence of homology isomorphisms:

$$Hom(\tilde{A}^{\otimes 2}, \tilde{A}) \simeq Hom(\tilde{A}, R) \otimes Hom(\tilde{A}, \tilde{A})$$
$$\simeq Hom(\tilde{A}, R) \otimes \tilde{A} \otimes_{\tilde{A}} Hom(\tilde{A}, \tilde{A}) \simeq Hom(\tilde{A}, \tilde{A}) \otimes_{\tilde{A}} Hom(\tilde{A}, \tilde{A}).$$

More generally we have the following homology isomorphism

$$Hom(\tilde{A}^{\otimes n}, \tilde{A}) \simeq Hom(\tilde{A}, \tilde{A}) \otimes_{\tilde{A}} Hom(\tilde{A}, \tilde{A}) \otimes_{\tilde{A}} \ldots \otimes_{\tilde{A}} Hom(\tilde{A}, \tilde{A}) \text{ (n times)}.$$

Further a straightforward computation shows

$$H_*Hom(\tilde{A}, \tilde{A}) = Ext^*_R(R/u_1^A, R/u_1^A) = \Lambda_{R/u_1^A}(z)$$

where $\Lambda_{R/u_1^A}(z)$ denotes the exterior algebra over R/u_1^A on one generator z of degree $-|u_1^A| - 1 = -d - 1$. So the E_1-term of our spectral sequence has the form

$$H_*(A) = R/u_1^A \to \Lambda_{R/u_1^A}(z) \to \ldots \to (\Lambda_{R/u_1^A}(z))^{\otimes n} \to \ldots$$

This is the usual cobar complex for the Hopf algebra $H_*Hom(\tilde{A}, \tilde{A}) = \Lambda_{R/u_1^A}(z)$ and its homology is

$$E_2^{**} = Ext^{**}_{\Lambda_{R/u_1^A}(z)}(R/u_1^A, R/u_1^A) = R/u_1^A[[t]] \tag{7.2}$$

where t has degree $-d-2$. For dimensional reasons $E^2 = E^3 = \ldots = E^\infty$. Next notice that the the spectral sequence E_1^{**} is multiplicative via the pairing

$$Hom(\tilde{A}^i, \tilde{A}) \otimes Hom(\tilde{A}^j, \tilde{A}) \to Hom(\tilde{A}^{i+j}, \tilde{A} \otimes \tilde{A}) \to Hom(\tilde{A}^{i+j}, \tilde{A})$$

where the second map is induced by the multiplication $\tilde{A} \otimes \tilde{A} \to \tilde{A}$. This pairing turns E_1^{**} into a graded ring and it follows that (7.2) is in fact an isomorphism of rings.

So we proved the following

Proposition 7.2. *The Hochschild cohomology ring of an even Moore algebra of degree d over R with $u_1^A \in R$ a nonzero divisor is a complete filtered ring whose associated graded ring is a formal power series algebra over R/u_1^A on one generator in degree $-d-2$.*

Remark 7.3. The arguments above would be considerably simpler if we knew that the spectral sequence of Proposition 4.9 were multiplicative. Unfortunately this is not known yet.

Now let R be a (graded) discrete valuation ring with the uniformizer $\pi = u_1$ and consider the unit map $f : R \to H^*(A, A)$ where A is as in Proposition 7.2. Since the filtered ring $H^*(A, A)$ has a formal power series ring over the field $R/(\pi)$ for its associated graded ring we conclude that $H^*(A, A)$ has no zero divisors. Therefore the map f is either an injection or its kernel is the maximal ideal $(\pi) \subset R$. Furthermore the ring $H^*(A, A)$ is itself a graded discrete valuation ring. Using Proposition 7.1 we see that the kernel of f is (π) if and only if $u'(t) = 0 \mod (\pi)$. The last equality is equivalent to $u(t) = 0 \mod (\pi)$ if $char(R/\pi) = 0$ or to $u(t) = v(t^p) \mod (\pi)$ for some power series $v(t)$ if $char(R/\pi) = p$.

Now suppose that f is injective. Then Proposition 7.2 implies that $u'(t)$ is not divisible by π which means that there exists $n \in \mathbb{Z}$ for which nu_n is invertible in R. Consider the smallest such n; it obviously equals the height of the series $u(t)$ reduced $\mod (\pi)$. By the Weierstrass Preparation Theorem the ring $H^*(A, A)$ is free of rank n over R. Therefore we obtain the following

Corollary 7.4. *Let R be a graded discrete valuation ring with uniformizer π and residue field R/π of characteristic p. Let A be an even Moore algebra over R with characteristic series $u(t) = \sum_{i=1}^{\infty} u_i^A t^i$ where $u_1^A = \pi$. Then*

(i) The ring $H^*(A, A)$ is either an R/π-algebra or a totally ramified extension of R whose ramification index equals the height of the series $u(t) \mod (\pi)$.

(ii) The ring $H^*(A, A)$ is an R/π-algebra if and only if $u(t) = v(t^p) \mod (\pi)$ for some polynomial $v(t)$.

Remark 7.5. Varying $u(t)$ we could get ramified extensions of arbitrary index that is coprime to p. In particular if $charR/\pi \neq 2$ the inclusion $f : R \hookrightarrow H^*(A, A)$ could be an isomorphism. A_∞-algebras having the property that f is an isomorphism are analogous to central separable algebras which were studied extensively in ring theory, cf. [**2**] and we hope to return to them in the future.

We conclude this section with a few simple examples illustrating our results. Let $R = \hat{\mathbb{Z}}_p[v, v^{-1}]$ where $\mathbb{Z}_p$ is the ring of p-adic integers for $p \neq 2$ and v is a formal Laurent variable of degree 2. Consider two A_∞ structures m^1 and m^∞ on the complex $A = \{R \xrightarrow{p} R\}$. These will in fact be differential graded algebra structures, i.e. $m_i^1 = m_i^\infty = 0$ for $i > 2$. Namely, set $m_2^\infty[y|y] = 0$ and $m_2^1[y|y] = v[1]$. In other words (A, m^∞) is just the exterior algebra on y in degree 1 with differential $dy = p$ while (A, m^1) is the dga generated by y with the same differential $dy = p$ but with the relation $y^2 = v$. Then Hochschild cohomology of (A, m^∞) is just the algebra $R/p[[t]] = \mathbb{F}_p[v, v^{-1}][[t]]$ while Hochschild cohomology of (A, m^1) is the ring R itself (the ramification index equals 1 in this case). The notations m^1 and m^∞ suggest that there are also m^n's for finite n. These indeed exist and could be obtained by setting $m_i^n[y]^{\otimes i} = 0$ for $i \neq n$ and $m_n^n[y]^{\otimes n} = v^n[1]$. The Hochschild cohomology of (A, m^n) realizes a totally ramified extension of the p-adic integers of index n which is not divisible by p.

Concluding remarks. It should be noted that our present approach to the moduli problem is rather ad hoc and it would be valuable to consider it from the more general point of view. Here we mention the (still unpublished) preprint of M. Schlessinger and J. Stasheff [**27**] where this program is carried out for rational homotopy types. These authors effectively study the *commutative* A_∞ structures on a complex with fixed homology ring H over the field of rationals. They replace H with its multiplicative resolution ΛZ, and consider the graded Lie algebra $Der(\Lambda Z)$ of derivations of H. Then it turns out that the moduli space under consideration is represented by the standard construction $A(Der(\Lambda Z))$ which computes homology of the Lie algebra $Der(\Lambda Z)$. This simple and elegant approach is very appealing and we feel that it is possible to extend it in the context of A_∞-algebras. The role of $Der(\Lambda Z)$ should be played by the Hochschild complex $C^*(H, H)$.

It is now clear that the set of homotopy types of dga's with a fixed homology algebra is only π_0 of the 'true' moduli space. The other invariants are picked up by monoids of homotopy self-equivalences corresponding to different path components of the moduli space. This point of view is developed in [**3**]. However in this context the problem of computing π_0 differs sharply from that of computing the higher homotopy groups. Indeed, in [**19**] we showed that, essentially, higher homotopy groups of mapping spaces could be reduced to (a version of) Hochschild cohomology. In fact in the cited reference the result is obtained for S-algebras but the arguments are still valid for dga's.

Also relevant to this problem is the recent paper by V. Hinich [**11**] where homotopy invariant deformation theory was constructed in the abstract setting of an algebra over an operad. However the approach in the cited reference is restricted by working in characteristic 0 and considering connected algebras only.

Another problem is to extend our results to the category of R-algebras in the sense of [**4**]. In the simplest case, which is already highly nontrivial, one is asked to classify the structures of KU-algebra structures on KU/p. Here KU is the spectrum of topological K-theory which is known to be a commutative S-algebra. The related problem is to compute $THH(KU/p, KU/p)$, the topological Hochschild cohomology of KU/p. We saw that in the algebraic case we obtain tamely ramified extensions of the p-adics. Perhaps in the topological case one could get wildly ramified extensions?

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