Homology, Homotopy and Applications, vol.6(1), 2004, pp.341–350

NOTE ON THE RATIONAL COHOMOLOGY OF THE FUNCTION SPACE OF BASED MAPS

YASUSUKE KOTANI

(*communicated by Hvedri Inassaridze*)

Abstract

In this paper, for a formal, path connected, finite-dimensional CW-complex X of finite type and a q-connected space Y of finite type with $q \geqslant \dim X$, we determine the necessary and sufficient condition for the rational cohomology algebra $H^*(\mathcal{F}_*(X,Y);\mathbb{Q})$ of the function space $\mathcal{F}_*(X,Y)$ of based maps to be free.

1. Introduction

Let $\mathcal{F}(X,Y)$ and $\mathcal{F}_*(X,Y)$ be function spaces of free maps and based maps from a space X to a space Y respectively. Then $\mathcal{F}(X,Y)$ and $\mathcal{F}_*(X,Y)$ are path connected if X is a path connected, finite-dimensional CW-complex of finite type and Y is a q-connected space with $q \geqslant \dim X$.

A commutative graded algebra $A = \{A^p\}_{p\geqslant 0}$ satisfying $A^0 = \mathbb{Q}$ is said to be free if A is isomorphic to a free commutative graded algebra $\wedge V$ on a graded vector space V.

A commutative cochain algebra (A,d) satisfying $H^0(A) = \mathbb{Q}$ is said to be formal if (A,d) and $(H(A),0)$ are connected by a chain of quasi-isomorphisms. A path connected space X is said to be formal if the commutative cochain algebra $A_{\mathrm{PL}}(X)$ of rational polynomial differential forms on X is formal.

It is known that, for an arbitrary n-connected space Y with $n \geqslant 1$, the rational cohomology algebra

$$H^*(\Omega^n Y;\mathbb{Q}) = H^*(\mathcal{F}_*(S^n,Y);\mathbb{Q})$$

of the n-fold loop space $\Omega^n Y$ of Y is free, and that spheres S^n are formal.

In this paper, for a formal, path connected, finite-dimensional CW-complex X of finite type and a q-connected space Y of finite type with $q \geqslant \dim X$, we consider the condition for the rational cohomology algebra $H^*(\mathcal{F}_*(X,Y);\mathbb{Q})$ of the function space $\mathcal{F}_*(X,Y)$ of based maps to be free.

Let $H^*(X;\mathbb{Q}) = \{H^p(X;\mathbb{Q})\}_{p\geqslant 0}$ be the rational cohomology algebra for a path connected space X with the cup product

$$\cup\colon H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}).$$

Received October 23, 2003, revised July 12, 2004; published on August 6, 2004.
2000 Mathematics Subject Classification: Primary 55P62; Secondary 54C40.
Key words and phrases: rational cohomology, function spaces, free commutative graded algebras, formal spaces.

Recall that the *rational cup length* $\mathrm{cup}(X;\mathbb{Q})$ of X is defined by

$$\sup\{n \in \mathbb{Z} \mid f_1 \cup \cdots \cup f_n \neq 0 \text{ for } f_1, \ldots, f_n \in H^+(X;\mathbb{Q})\},$$

where $H^+(X;\mathbb{Q}) = \{H^p(X;\mathbb{Q})\}_{p>0}$.

Let $(\wedge V, d)$ be a Sullivan algebra. Elements in $\wedge V$ of the form $v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$ are said to have word length k. Then the differential d decomposes uniquely as the sum

$$d = d_0 + d_1 + d_2 + \cdots$$

of derivations d_i raising the word length by i. (cf. [**3**, Section 12(a)]). Now, we define the *differential length* $\mathrm{dl}(\wedge V, d)$ of $(\wedge V, d)$ by the least integer m such that $d_{m-1} \neq 0$. If $d_i = 0$ for all $i \geqslant 0$, that is, $d = 0$, we define $\mathrm{dl}(\wedge V, 0) = \infty$. We also define the differential length $\mathrm{dl}(Y)$ of a simply connected space Y of finite type by that of a minimal Sullivan model for Y. Then we can establish

Theorem 1.1. *The differential length of a simply connected space of finite type is independent of a choice of minimal Sullivan models. Thus it is a rational homotopy invariant.*

Our main theorem is as follows.

Theorem 1.2. *Let X be a formal, path connected, finite-dimensional* CW*-complex of finite type and Y a q-connected space of finite type with $q \geqslant \dim X$. Then $H^*(\mathcal{F}_*(X,Y);\mathbb{Q})$ is free if and only if* $\mathrm{cup}(X;\mathbb{Q}) < \mathrm{dl}(Y)$.

This paper is organized as follows. In Section 2, we recall the construction of a minimal Sullivan model for $\mathcal{F}(X,Y)$ due to E. H. Brown, Jr. and R. H. Szczarba [**2**, Thoerem 1.9]. Moreover, we describe a minimal Sullivan model for $\mathcal{F}_*(X,Y)$ is obtained by that for $\mathcal{F}(X,Y)$ using the evaluation fibration, which is established by K. Kuribayashi [**4**, Theorem 3.6]. The proofs of Theorems are given in Section 3 and 4 respectively. In Section 5, we give some examples.

The author would like to express his deepest gratitude to Professor K. Kuribayashi for permitting to use some results in [**4**, Section 3], and also to Professor T. Yamaguchi for helpful suggestions.

2. Minimal Sullivan models for $\mathcal{F}(X,Y)$ and $\mathcal{F}_*(X,Y)$

Let X and Y be as in Theorem 1.2. Then the construction of a minimal Sullivan model for $\mathcal{F}(X,Y)$ due to E. H. Brown, Jr. and R. H. Szczarba [**2**, Theorem 1.9] is described as follows.

Let $m_Y\colon (\wedge V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(Y)$ be a minimal Sullivan model for Y. Let $H_*(X;\mathbb{Q}) = \{H_p(X;\mathbb{Q})\}_{p\geqslant 0}$ be the rational homology coalgebra for X with the coproduct

$$\Delta\colon H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q}) \otimes H_*(X;\mathbb{Q}).$$

Let $\wedge V \otimes H_*(X;\mathbb{Q})$ be a graded vector space with grading $|v \otimes c| = |v| - |c|$ for $v \in \wedge V$ and $c \in H_*(X;\mathbb{Q})$. Let $\wedge(\wedge V \otimes H_*(X;\mathbb{Q}))$ be the free commutative graded algebra on $\wedge V \otimes H_*(X;\mathbb{Q})$ with the differential $d \otimes \mathrm{id}$, and let I be the ideal in $\wedge(\wedge V \otimes H_*(X;\mathbb{Q}))$ generated by $1 \otimes 1 - 1$ and all elements of the form

$$v' \wedge v'' \otimes c - \sum (-1)^{|v''||c'_j|}(v' \otimes c'_j) \wedge (v'' \otimes c''_j) \tag{2.1}$$

for $v', v'' \in \wedge V$ and $c \in H_*(X;\mathbb{Q})$ with $\Delta c = \sum c'_j \otimes c''_j$. Then $(d \otimes \mathrm{id})(I) \in I$ ([**2**, Theorem 3.3]) and the composition map

$$\rho\colon \wedge(V \otimes H_*(X;\mathbb{Q})) \hookrightarrow \wedge(\wedge V \otimes H_*(X;\mathbb{Q})) \to \wedge(\wedge V \otimes H_*(X;\mathbb{Q}))/I$$

is an isomorphism of graded algebras ([**2**, Theorem 3.3]). Let δ be the differential on $\wedge(V \otimes H_*(X;\mathbb{Q}))$ given by $\delta = \rho^{-1}(d \otimes \mathrm{id})\rho$. Then, by [**2**, Theorem 1.9], $\mathcal{F}(X,Y)$ has a minimal Sullivan model of the form

$$(\wedge(V \otimes H_*(X;\mathbb{Q})), \delta).$$

Next, let us consider the evaluation fibration

$$\mathcal{F}_*(X,Y) \to \mathcal{F}(X,Y) \xrightarrow{ev_*} Y,$$

where ev_* is the evaluation map at the basepoint of X. Let $i\colon (\wedge V, d) \hookrightarrow (\wedge(V \otimes H_*(X;\mathbb{Q})), \delta)$ be the inclusion map defined by $i(v) = v \otimes 1$ for $v \in V$. From the consideration in [**4**, Section 3], we have a commutative diagram

$$\begin{tikzcd}
A_{\mathrm{PL}}(Y) \arrow[r, "A_{\mathrm{PL}}(ev_*)"] & A_{\mathrm{PL}}(\mathcal{F}(X,Y)) \\
(\wedge V, d) \arrow[u, "m_Y", "\simeq"'] \arrow[r, hook, "i"] & (\wedge(V \otimes H_*(X;\mathbb{Q})), \delta), \arrow[u, "m", "\simeq"']
\end{tikzcd}$$

where $m\colon (\wedge(V \otimes H_*(X;\mathbb{Q})), \delta) \xrightarrow{\simeq} A_{\mathrm{PL}}(\mathcal{F}(X,Y))$ is a minimal Sullivan model for $\mathcal{F}(X,Y)$ described above. Thus the inclusion map i is viewed as a model for the evaluation map ev_*.

Let J be an ideal of $\wedge(V \otimes H_*(X;\mathbb{Q}))$ generated by $v \otimes 1$ for $v \in V$. Let $\overline{\delta}$ be the differential on $\wedge(V \otimes H_*(X;\mathbb{Q}))/J$ induced from δ on $\wedge(V \otimes H_*(X;\mathbb{Q}))$. Then, by [**3**, Proposition 15.5] and [**4**, Theorem 3.6], $\mathcal{F}_*(X,Y)$ has a minimal Sullivan model of the form

$$(\wedge(V \otimes H_*(X;\mathbb{Q}))/J, \overline{\delta}) = (\wedge(V \otimes H_+(X;\mathbb{Q})), \overline{\delta}),$$

where $H_+(X;\mathbb{Q}) = \{H_p(X;\mathbb{Q})\}_{p>0}$.

3. Proof of Theorem 1.1

It is known that minimal Sullivan models for a simply connected space of finite type are all isomorphic, and that the isomorphism class of a minimal Sullivan model for a simply connected space of finite type is a rational homotopy invariant. Hence, for the proof of Theorem 1.1, it is sufficient to prove the following.

Proposition 3.1. *Let $(\wedge V, d)$ and $(\wedge V', d')$ be isomorphic Sullivan algebras. Then* $\mathrm{dl}(\wedge V, d) = \mathrm{dl}(\wedge V', d')$.

Proof. Let $f\colon (\wedge V, d) \xrightarrow{\cong} (\wedge V', d')$ be an isomorphism of differential graded algebras.

First, suppose that $\mathrm{dl}(\wedge V', d') = \infty$, that is, $d' = 0$. Then, since $fd = d'f = 0$ and f is an isomorphism, we have $d = 0$. Thus $\mathrm{dl}(\wedge V, d) = \infty$.

Next, suppose that $\mathrm{dl}(\wedge V', d') = m < \infty$, that is, $d'_i = 0$ for $0 \leqslant i < m-1$ and $d'_{m-1} \neq 0$. Then, since f is an isomorphism, for an arbitrary element $v \in V$, there exists an element $v' \in V'$ such that

$$f(v) = v' + (\text{higher terms}).$$

Now, assume that dv has terms of the form $v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$ and $k \leqslant m-1$. Then $f(dv)$ has terms of the form

$$f(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k) = v'_1 \wedge \cdots \wedge v'_k + (\text{higher terms})$$

for $v'_1, \ldots, v'_k \in V'$ and $k \leqslant m-1$. However, $d'f(v) = f(dv)$ has no such terms because $d'_i = 0$ for $0 \leqslant i < m-1$. It is a contradiction. Hence we have $d_i = 0$ for $0 \leqslant i < m-1$ since d is a derivation. So we get the inequality $\mathrm{dl}(\wedge V, d) \geqslant \mathrm{dl}(\wedge V', d')$. Since f^{-1} is also an isomorphism, we get the inverse inequality. Thus $\mathrm{dl}(\wedge V, d) = \mathrm{dl}(\wedge V', d') = m$. ◻

4. Proof of Theorem 1.2

Let X and Y be as in Theorem 1.2. Let $(\wedge V, d)$ be a minimal Sullivan model for Y and $H_*(X;\mathbb{Q})$ the rational homology coalgebra for X. Then, as described in Section 2, $\mathcal{F}_*(X,Y)$ has a minimal Sullivan model of the form

$$(\wedge(V \otimes H_+(X;\mathbb{Q})), \overline{\delta}),$$

where $\overline{\delta}$ is induced from $\delta = \rho^{-1}(d \otimes \mathrm{id})\rho$ on $\wedge(V \otimes H_*(X;\mathbb{Q}))$ by reducing elements contained in the ideal J generated by $v \otimes 1$ for $v \in V$.

It is easy to see that $H^*(\mathcal{F}_*(X,Y);\mathbb{Q}) \cong H(\wedge(V \otimes H_+(X;\mathbb{Q})), \overline{\delta})$ is free if and only if $\overline{\delta} = 0$, and that $\overline{\delta} = 0$ if and only if $\delta(\wedge(V \otimes H_+(X;\mathbb{Q}))) \in J$. Hence, for the proof of Theorem 1.2, it is sufficient to prove the following.

Proposition 4.1. (1). *If* $\mathrm{cup}(X;\mathbb{Q}) < \mathrm{dl}(Y)$, *then* $\delta(\wedge(V \otimes H_+(X;\mathbb{Q}))) \in J$ *or equivalently* $\overline{\delta} = 0$.

(2). *If* $\mathrm{cup}(X;\mathbb{Q}) \geqslant \mathrm{dl}(Y)$, *then* $\delta(\wedge(V \otimes H_+(X;\mathbb{Q}))) \notin J$.

Thus we need to explain the differential δ in detail. Let Δ be the coproduct on $H_*(X;\mathbb{Q})$. Then the reduced coproduct

$$\overline{\Delta}\colon H_+(X;\mathbb{Q}) \to H_+(X;\mathbb{Q}) \otimes H_+(X;\mathbb{Q})$$

is defined by $\overline{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c$ for $c \in H_+(X;\mathbb{Q})$. Moreover, the k-th coproduct $\Delta^{(k)}$ and the k-th reduced coproduct $\overline{\Delta}^{(k)}$ are defined inductively by $\Delta^{(0)} = \overline{\Delta}^{(0)} = \mathrm{id}$, $\Delta^{(1)} = \Delta$, $\overline{\Delta}^{(1)} = \overline{\Delta}$ and

$$\Delta^{(k)} = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \Delta^{(k-1)}\colon H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})^{\otimes k+1},$$

$$\overline{\Delta}^{(k)} = (\overline{\Delta} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \overline{\Delta}^{(k-1)}\colon H_+(X;\mathbb{Q}) \to H_+(X;\mathbb{Q})^{\otimes k+1},$$

where $H^{\otimes k+1}$ denotes the $(k+1)$-times tensor product of H.

Let $H^*(X;\mathbb{Q})$ be the rational cohomology algebra for X with the cup product $\cup$. Since X is of finite type, $H^*(X;\mathbb{Q})$ with $\cup$ and $H_*(X;\mathbb{Q})$ with Δ are dual each other. Hence we have immediately

Lemma 4.2. *If* $\operatorname{cup}(X;\mathbb{Q}) = n$, *then* $\overline{\Delta}^{(k-1)} \neq 0$ *for* $0 < k \leqslant n$ *and* $\overline{\Delta}^{(k-1)} = 0$ *for all* $k > n$.

Let $B_{H_*} = \{c_0 = 1, c_1, c_2, \dots\}$ be a basis for $H_*(X;\mathbb{Q})$ with $0 < |c_1| \leqslant |c_2| \leqslant \cdots$. Then, for an arbitrary element $c_j \in B_{H_*}$ and $k \geqslant 2$, we may denote

$$\Delta^{(k-1)} c_j = \sum \mu_{j_1,\dots,j_k} c_{j_1} \otimes \cdots \otimes c_{j_k},$$

where $0 \neq \mu_{j_1,\dots,j_k} \in \mathbb{Q}$ and $c_{j_1}, \dots, c_{j_k} \in B_{H_*}$. By the definition of the reduced coproduct, we have immediately

Lemma 4.3. $\overline{\Delta}^{(k-1)} c_j = 0$ *if and only if there exists an integer* s *such that* $c_{j_s} = 1$ *in each term of* $\Delta^{(k-1)} c_j$.

Moreover, since the cup product $\cup$ is associative and commutative, so is the coproduct Δ, that is, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and $\tau\Delta = \Delta$, where τ is defined by $\tau(c \otimes c') = (-1)^{|c||c'|} c' \otimes c$. Hence we have immediately

Lemma 4.4. $\mu_{j_1,\dots,j_s,j_{s+1},\dots,j_k} = (-1)^{|c_{j_s}||c_{j_{s+1}}|} \mu_{j_1,\dots,j_{s+1},j_s,\dots,j_k}$.

Let $B_V = \{v_1, v_2, \dots\}$ be a basis for V with $0 < |v_1| \leqslant |v_2| \leqslant \cdots$. Then, if $dv_i = v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $v_i \in B_V$ and $\Delta^{(k-1)} c_j = \sum \mu_{j_1,\dots,j_k} c_{j_1} \otimes \cdots \otimes c_{j_k}$ for $c_j \in B_{H_*}$, we have

$$\delta(v_i \otimes c_j) = \sum (-1)^{\varepsilon(i_1,j_1;\dots;i_k,j_k)} \mu_{j_1,\dots,j_k} (v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_k} \otimes c_{j_k}), \qquad (4.1)$$

where the sign $(-1)^{\varepsilon(i_1,j_1;\dots;i_k,j_k)}$ is determined by (2.1), that is,

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \otimes c_j = \sum (-1)^{\varepsilon(i_1,j_1;\dots;i_k,j_k)} \mu_{j_1,\dots,j_k} (v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_k} \otimes c_{j_k})$$

in the graded algebra $\wedge(\wedge V \otimes H_*(X;\mathbb{Q}))/I$. More precisely, $\varepsilon(i_1,j_1;\dots;i_k,j_k)$ is given by

Lemma 4.5. $\varepsilon(i_1,j_1;\dots;i_k,j_k) = \sum_{l=1}^{k-1} (|v_{i_{l+1}}| + \cdots + |v_{i_k}|)|c_{j_l}|$

Proof. We prove by induction on k. Let $k = 2$. Then, if $\Delta c_j = \sum \mu_{j_1,j_2} c_{j_1} \otimes c_{j_2}$ for $c_j \in B_{H_*}$, we have

$$v_{i_1} \wedge v_{i_2} \otimes c_j = \sum (-1)^{|v_{i_2}||c_{j_1}|} \mu_{j_1,j_2} (v_{i_1} \otimes c_{j_1}) \wedge (v_{i_2} \otimes c_{j_2}),$$

and so $\varepsilon(i_1,j_1;i_2,j_2) = |v_{i_2}||c_{j_1}|$.

Let $k \geqslant 3$ and assume that the formula is true until $k-1$. Since $\Delta^{(k-1)} = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \Delta^{(k-2)}$, if $\Delta^{(k-1)} c_j = \sum \mu_{j_1,\dots,j_k} c_{j_1} \otimes \cdots \otimes c_{j_k}$ for $c_j \in B_{H_*}$, we can denote

$$\Delta^{(k-2)} c_j = \sum \mu_{j'_1,j_3,\dots,j_k} c_{j'_1} \otimes c_{j_3} \otimes \cdots \otimes c_{j_k}$$

with $\Delta c_{j'_1} = \sum \mu'_{j_1,j_2} c_{j_1} \otimes c_{j_2}$ and $\mu_{j_1,\dots,j_k} = \mu'_{j_1,j_2}\mu_{j'_1,j_3,\dots,j_k}$. Then, by putting $v_{i'_1} = v_{i_1} \wedge v_{i_2}$, we have

$$\begin{aligned}
&v_{i_1} \wedge \cdots \wedge v_{i_k} \otimes c_j \\
&= v_{i'_1} \wedge v_{i_3} \wedge \cdots \wedge v_{i_k} \otimes c_j \\
&= \sum (-1)^{\varepsilon(i'_1,j'_1;i_3,j_3;\dots;i_k,j_k)} \mu_{j'_1,j_3,\dots,j_k} (v_{i'_1} \otimes c_{j'_1}) \wedge (v_{i_3} \otimes c_{j_3}) \wedge \cdots \wedge (v_{i_k} \otimes c_{j_k}).
\end{aligned}$$

Furthermore, since

$$v_{i'_1} \otimes c_{j'_1} = v_{i_1} \wedge v_{i_2} \otimes c_{j'_1} = \sum (-1)^{|v_{i_2}||c_{j_1}|} \mu'_{j_1,j_2} (v_{i_1} \otimes c_{j_1}) \wedge (v_{i_2} \otimes c_{j_2}),$$

we have

$$\begin{aligned}
&v_{i_1} \wedge \cdots \wedge v_{i_k} \otimes c_j \\
&= \sum (-1)^{\varepsilon(i'_1,j'_1;i_3,j_3;\dots;i_k,j_k)+|v_{i_2}||c_{j_1}|} \mu_{j_1,\dots,j_k} (v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_k} \otimes c_{j_k}),
\end{aligned}$$

and so

$$\begin{aligned}
&\varepsilon(i_1,j_1;\dots;i_k,j_k) \\
&= \varepsilon(i'_1,j'_1;i_3,j_3;\dots;i_k,j_k) + |v_{i_2}||c_{j_1}| \\
&= (|v_{i_3}| + \cdots + |v_{i_k}|)|c_{j'_1}| + \sum_{l=3}^{k-1} (|v_{i_{l+1}}| + \cdots + |v_{i_k}|)|c_{j_l}| + |v_{i_2}||c_{j_1}| \\
&= \sum_{l=1}^{k-1} (|v_{i_{l+1}}| + \cdots + |v_{i_k}|)|c_{j_l}|
\end{aligned}$$

because $|c_{j'_1}| = |c_{j_1}| + |c_{j_2}|$. □

Now we can prove Proposition 4.1.

Proof of Proposition 4.1. Notice that $\mathrm{cup}(X;\mathbb{Q}) < \infty$ since X is finite-dimensional.

First, suppose that $\mathrm{dl}(Y) = \infty$. Then, since $d = 0$, we have $\delta = \rho^{-1}(d \otimes \mathrm{id})\rho = 0$, and so $\overline{\delta} = 0$.

Next, suppose that $\mathrm{dl}(Y) = m < \infty$. Fix a basis $B_{H_*} = \{c_0 = 1, c_1, c_2, \dots\}$ for $H_*(X;\mathbb{Q})$ with $0 < |c_1| \leqslant |c_2| \leqslant \cdots$ and a basis $B_V = \{v_1, v_2, \dots\}$ for V with $0 < |v_1| \leqslant |v_2| \leqslant \cdots$. Then, for an arbitrary element $v_i \in B_V$, we may denote

$$dv_i = \sum_{k \geqslant m} \lambda_{i_1,\dots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

where $0 \neq \lambda_{i_1,\dots,i_k} \in \mathbb{Q}$ and $v_{i_1}, \dots, v_{i_k} \in B_V$ with $i_1 \leqslant \cdots \leqslant i_k$.

(1). For an arbitrary element $c_j \in B_{H_*}$ with $c_j \neq 1$ and $k \geqslant m$, we may denote

$$\Delta^{(k-1)} c_j = \sum \mu_{j_1,\dots,j_k} c_{j_1} \otimes \cdots \otimes c_{j_k},$$

where $0 \neq \mu_{j_1,\dots,j_k} \in \mathbb{Q}$ and $c_{j_1}, \dots, c_{j_k} \in B_{H_*}$. Since $\mathrm{cup}(X;\mathbb{Q}) < \mathrm{dl}(Y) = m$, by Lemma 4.2, $\overline{\Delta}^{(k-1)} c_j = 0$ for $k \geqslant m$, and so, by Lemma 4.3, there exists an integer

s such that $c_{j_s} = 1$ in each term of $\Delta^{(k-1)} c_j$ for $k \geqslant m$. Hence we have

$$\begin{aligned}&\delta(v_i \otimes c_j)\\ &= \sum_{k \geqslant m} (-1)^{\varepsilon(i_1,j_1;\dots;i_k,j_k)} \lambda_{i_1,\dots,i_k} \mu_{j_1,\dots,j_k} (v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_k} \otimes c_{j_k}) \in J\end{aligned}$$

for an arbitrary element $v_i \in B_V$. Thus $\delta(\wedge(V \otimes H_+(X;\mathbb{Q})) \in J$ since δ is a derivation.

(2). Since $\mathrm{cup}(X;\mathbb{Q}) \geqslant \mathrm{dl}(Y) = m$, by Lemma 4.2, there exists an element $c_j \in B_{H_*}$ such that $c_j \neq 1$ and

$$\overline{\Delta}^{(m-1)} c_j = \sum \mu_{j_1,\dots,j_m} c_{j_1} \otimes \cdots \otimes c_{j_m} \neq 0.$$

Since $\mathrm{dl}(Y) = m$, there exists an element $v_i \in B_V$ such that dv_i has a term of the form $\lambda_{i_1,\dots,i_m} v_{i_1} \wedge \cdots \wedge v_{i_m}$ with $\lambda_{i_1,\dots,i_m} \neq 0$ and $i_1 \leqslant \cdots \leqslant i_m$. Then $\delta(v_i \otimes c_j)$ has terms of the form

$$\sum (-1)^{\varepsilon(i_1,j_1;\dots;i_m,j_m)} \lambda_{i_1,\dots,i_m} \mu_{j_1,\dots,j_m} (v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_m} \otimes c_{j_m})$$

with $c_{j_s} \neq 1$ for $1 \leqslant s \leqslant m$.

If $i_1 < \cdots < i_m$, we see that each term $(v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_m} \otimes c_{j_m})$ cannot be canceled by other terms.

If $i_s = i_{s+1}$ for some s, $|v_{i_s}|$ must be even. Then we have

$$\begin{aligned}&(v_{i_s} \otimes c_{j_s}) \wedge (v_{i_s} \otimes c_{j_{s+1}})\\ &= (-1)^{(|v_{i_s}|-|c_{j_s}|)(|v_{i_s}|-|c_{j_{s+1}}|)} (v_{i_s} \otimes c_{j_{s+1}}) \wedge (v_{i_s} \otimes c_{j_s})\\ &= (-1)^{|c_{j_s}||c_{j_{s+1}}|} (v_{i_s} \otimes c_{j_{s+1}}) \wedge (v_{i_s} \otimes c_{j_s})\end{aligned}$$

and, by Lemma 4.5,

$$\begin{aligned}&\varepsilon(i_1,j_1;\dots;i_s,j_s;i_s,j_{s+1};\dots;i_m,j_m)\\ &\quad - \varepsilon(i_1,j_1;\dots;i_s,j_{s+1};i_s,j_s;\dots;i_m,j_m)\\ &= (|v_{i_s}| + |v_{i_{s+2}}| + \cdots + |v_{i_k}|)|c_{j_s}| + (|v_{i_{s+2}}| + \cdots + |v_{i_k}|)|c_{j_{s+1}}|\\ &\quad - (|v_{i_s}| + |v_{i_{s+2}}| + \cdots + |v_{i_k}|)|c_{j_{s+1}}| - (|v_{i_{s+2}}| + \cdots + |v_{i_k}|)|c_{j_s}|\\ &= |v_{i_s}|(|c_{j_s}| - |c_{j_{s+1}}|) \equiv 0 \bmod 2.\end{aligned}$$

Hence, by considering the coefficients with Lemma 4.4, we see that each term $(v_{i_1} \otimes c_{j_1}) \wedge \cdots \wedge (v_{i_m} \otimes c_{j_m})$ cannot be canceled by other terms. (For example, see Example 3 in Section 5).

Thus there exists an element $v_i \otimes c_j \in \wedge(V \otimes H_+(X;\mathbb{Q}))$ such that $\delta(v_i \otimes c_j) \notin J$. □

5. Some examples

Since $\mathrm{cup}(X;\mathbb{Q}) < \infty$ if X is finite-dimensional and $\mathrm{dl}(Y) > 1$ for any simply connected space Y of finite type, we have

Proposition 5.1. *Let X be a formal, path connected, finite-dimensional* CW*-complex of finite type and Y a q-connected space of finite type with $q \geqslant \dim X$.*

Then, if Y has a minimal Sullivan model of the form $(\wedge V, 0)$ or all cup products on $H^+(X;\mathbb{Q})$ are trivial, $H^*(\mathcal{F}_*(X,Y);\mathbb{Q})$ is always free.

Example 1. The following spaces have a minimal Sullivan model with a trivial differential:

- odd dimensional spheres,
- path connected H-spaces of finite type (cf. [**3**, Section 12(a), Example 3]),
- classifying spaces of path connected topological groups of finite type (cf. [**3**, Proposition 15.15]),
- Eilenberg-MacLane spaces of type (π, n) with $n \geqslant 1$, π is Abelian and $\pi \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional (cf. [**3**, Section 15(b), Example 2]),
- a product of above spaces. □

Example 2. The following spaces are formal and all cup products on the positive dimensional rational cohomology algebra are trivial:

- spheres,
- suspensions of spaces (cf. [**3**, Proposition 13.9]),
- co-H-spaces,
- a wedge of above spaces.

Note that a co-H-space is rationally homotopy equivalent to a wedge of spheres (cf. [**1**, Section 7]), and a wedge of formal spaces is also formal. □

A product of spheres $S^{i_1} \times \cdots \times S^{i_n}$ is an $(i_1 + \cdots + i_n)$-dimensional CW-complex and a formal space with $\mathrm{cup}(S^{i_1} \times \cdots \times S^{i_n};\mathbb{Q}) = n$.

It is known that the n-th James reduced product space $J_n(S^{2i})$ of a $2i$-dimensional sphere S^{2i} is a $2ni$-dimensional CW-complex which has the rational cohomology

$$H^*(J_n(S^{2i});\mathbb{Q}) = \mathbb{Q}[c]/(c^{n+1})$$

with $|c| = 2i$, and has a minimal Sullivan model of the form

$$(\wedge(v,\theta), d\theta = v^{n+1})$$

with $|v| = 2i$. Hence we have

Proposition 5.2. (1). *Let Y be a q-connected space of finite type with $q \geqslant i_1 + \cdots + i_n$. Then*

$$H^*(\mathcal{F}_*(S^{i_1} \times \cdots \times S^{i_n}, Y);\mathbb{Q})$$

is free if and only if $\mathrm{dl}(Y) > n$.

(2). *Let Y be a $2ni$-connected space of finite type. Then*

$$H^*(\mathcal{F}_*(J_n(S^{2i}), Y);\mathbb{Q})$$

is free if and only if $\mathrm{dl}(Y) > n$.

(3). *Let X be a formal, path connected, p-dimensional* CW*-complex of finite type with $p < 2i$. Then*

$$H^*(\mathcal{F}_*(X, J_n(S^{2i})); \mathbb{Q})$$

is free if and only if $\mathrm{cup}(X; \mathbb{Q}) < n + 1$.

Example 3. $H^*(\mathcal{F}_*(S^1 \times S^3, S^6); \mathbb{Q})$ is not free.

Notice that $\mathrm{dl}(S^6) = 2 = \mathrm{cup}(S^1 \times S^3; \mathbb{Q})$. A basis for $H_*(X; \mathbb{Q})$ is given by $\{1, c_1, c_3, c_4\}$ with $|c_j| = j$, $\overline{\Delta} c_1 = \overline{\Delta} c_3 = 0$ and

$$\overline{\Delta} c_4 = \mu_{1,3} c_1 \otimes c_3 + \mu_{3,1} c_3 \otimes c_1,$$

where $\mu_{1,3} = (-1)^{1 \cdot 3} \mu_{3,1} = -\mu_{3,1}$. A minimal Sullivan model for S^6 is given by $(\wedge(v_6, v_{11}), d)$ with $|v_i| = i$, $dv_6 = 0$ and $dv_{11} = {v_6}^2$. By applying the construction described in Section 2, $\mathcal{F}_*(S^1 \times S^3, S^6)$ has a minimal Sullivan model of the form

$$(\wedge(\{v_6, v_{11}\} \otimes \{c_1, c_3, c_4\}), \overline{\delta}).$$

Then, by the formula (4.1) and Lemmas 4.4 and 4.5, we have

$$\begin{aligned}
&\delta(v_{11} \otimes c_4) \\
&= (-1)^{6 \cdot 4} (v_6 \otimes c_4) \wedge (v_6 \otimes 1) + (-1)^{6 \cdot 0} (v_6 \otimes 1) \wedge (v_6 \otimes c_4) \\
&\quad + (-1)^{6 \cdot 1} \mu_{1,3} (v_6 \otimes c_1) \wedge (v_6 \otimes c_3) + (-1)^{6 \cdot 3} \mu_{3,1} (v_6 \otimes c_3) \wedge (v_6 \otimes c_1) \\
&= (v_6 \otimes c_4) \wedge (v_6 \otimes 1) + (-1)^{(6-0)(6-4)} (v_6 \otimes c_4) \wedge (v_6 \otimes 1) \\
&\quad + \mu_{1,3} (v_6 \otimes c_1) \wedge (v_6 \otimes c_3) + (-1)^{(6-3)(6-1)+1} \mu_{1,3} (v_6 \otimes c_1) \wedge (v_6 \otimes c_3) \\
&= 2(v_6 \otimes c_4) \wedge (v_6 \otimes 1) + 2\mu_{1,3} (v_6 \otimes c_1) \wedge (v_6 \otimes c_3),
\end{aligned}$$

and so $\overline{\delta}(v_{11} \otimes c_4) = 2\mu_{1,3} (v_6 \otimes c_1) \wedge (v_6 \otimes c_3) \neq 0$. □

References

[1] M. Arkowitz, *Co-H-spaces*, Handbook of algebraic topology, 1143–1173, North-Holland, 1995.

[2] E. H. Brown, Jr. and R. H. Szczarba, *On the rational homotopy type of function spaces*, Trans. Amer. Math. Soc. **349** (1997), 4931–4951.

[3] Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics **205**, Springer-Verlag, 2001.

[4] K. Kuribayashi, *Rational model for the evaluation map and iterated cyclic homology*, preprint.

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/
or by anonymous ftp at
ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2004/n1a18/v6n1a18.(dvi,ps,pdf)

Yasusuke KOTANI `kotani@math.kochi-u.ac.jp`

Department of Mathematics
Faculty of Science
Kochi University
Kochi 780-8520, Japan