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MONOTONE-LIGHT FACTORIZATION FOR KAN FIBRATIONS OF SIMPLICIAL SETS WITH RESPECT TO GROUPOIDS.

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Abstract

We show that the class of second order covering maps of simplicial sets in the sense of R. Brown and G. Janelidze is a part of a factorization system for the class of Kan fibrations of simplicial sets.

1. Introduction

Reflective factorization systems (E, M) on a category C are constructed out of reflective subcategories X in C satisfying mild additional conditions. As shown by A. Carboni, G. Janelidze, G. M. Kelly, and R. Pare [**2**], under some further conditions such a pair (E, M) also produces a new, so-called monotone-light, factorization system $(E', M^\star)$ in C, with $M^\star$ being defined as the class of covering maps in the sense of categorical Galois theory.

However a more general context of Galois theory, involves, besides of a reflective subcategory X, a certain class of maps F of C. It is therefore reasonable to investigate a suitable more general notion of a monotone-light factorization for a given class F. And it turns out that much of the considerations of [**2**] can be adopted to such relative cases as well.

In the present paper we describe the (relative) monotone-light construction, and apply it to the situation studies by R. Brown and G. Janelidze in [**3**], where C is the category of simplicial sets, X is the category of groupoids and F is the class of Kan fibrations of simplicial sets; and in this way we obtain a factorization system for Kan fibrations.

2. Monotone-light factorization

It is known that every full reflective subcategory X of a category C, under some weak conditions, gives rise to a factorization system (E, M) (for the class of all morphisms in C), where E is the class of maps inverted by the reflection $I\colon C \to X$. Further as in [**2**], the pair $(E', M^\star)$, where E' is the stabilization of E and $M^\star$ is the localization of M, may also be a factorization system; it is then called a monotone-light factorization system.

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Furthermore, the class M coincides then with the class of trivial covering maps of the Galois theory, and its localization $M^\star$ coincides with the class of all covering maps.

Below we give a more precise description of the monotone-light factorization construction. Moreover, we consider a case which besides of the reflective subcategory involves a certain class of maps F in the category C. Accordingly we deal with the following kinds of factorization systems for the given class F:

Given a category C and a class of morphisms F in C containing identities, closed under composition, and pullback stable, by a factorization system for F we mean a pair of classes of maps (E, M) with:

1. E and M both contain identities, and are closed under composition with isomorphisms;
2. Every map f in F can be written as me with m in M and e in E; whereupon me is called (E, M) factorization of f;
3. M is contained in F;
4. For each m in M and e in E and for every pair of morphisms u, v with $mu = ve$, there is a unique map w with $mw = v$ and $we = u$.

The ground data for which we consider monotone-light construction will be:

C, a category with pullbacks;

X, a full reflective subcategory in C, with inclusion $H\colon X \to C$ and reflection $I\colon C \to X$;

F, a class of morphisms in C containing identities, closed under composition, and pullback stable.

We can say that (C, X, H, I, F) is a Galois structure (compare [**2**], section 1).

A map f in F is called a trivial covering with respect to the given Galois structure if the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HI(A) \arrow[d, "HIf"] \\ B \arrow[r, "\eta_B"] & HIB \end{tikzcd}$$

where η is the unit of the reflection, is a pullback.

Now we restate what was said in the beginning of this section: a Galois structure (C, X, H, I, F) with F the class of all maps usually gives a factorization system (E, M) for the class of all maps of C; and under some conditions M is the class of trivial covering maps.

The fact that the class of trivial coverings is a part of a factorization system remains true for the general Galois structure. That is, if the Galois structure (C, X, H, I, F) is admissible (e.g. see [**3**], section 1), then the class of trivial coverings M (with respect to this structure), together with the class E consisting of all morphisms inverted by I, forms a factorization system (E, M) for the class F.

Now we set: A morphism in F is said to be locally in M if a pullback of it along some effective F-descent map (see [**2**]; one may think of (F-)descent maps in a category as "good surjections") is in M. By $M^\star$ we denote the class of all maps locally in M; and say that $M^\star$ is localization of M. As a result, as soon as M is the class of trivial coverings, the localization $M^\star$ is nothing but the class of covering maps with respect to the given Galois structure.

In the same manner we set: A morphism is said to be stably in E if every pullback of it along an F is in E. By E' we denote the class of all maps stably in E; we say as well that E' is stabilization of E.

Our interest in localization and stabilization relates to the fact that the pair $(E', M^\star)$ may also constitute a factorization system for F; in this case it is called a monotone-light factorization for F, and we say that F admits a(the) monotone-light factorization.

Remark 2.1 Localization-stabilization can as well be considered for factorization systems not necessarily obtained from the Galois structure. In fact, [**2**] provides a necessary and sufficient condition for a localization-stabilization of an arbitrary factorization system itself to be a factorization system ([**2**], 6.10). However [**2**] restricts to the case where F is the class of all maps.

In order to obtain our main result for the Kan fibrations (see Theorem 3.2 below) we will use the following result that can be proved with the same arguments as its absolute version in [**2**]:

Observation on stabilizing objects 2.2. *Let (C, X, H, I, F) be an admissible Galois structure. The class F admits monotone-light factorization if for each object B of C there is an effective F-descent map $p\colon C \to B$ with C a stabilizing object; the latter means that in (E, M) factorization $h = me$, of any morphism $h\colon X \to C$ with codomain C, e is stably in E.*

3. Monotone-light factorization for Kan fibrations

Now consider the following Galois structure (C, X, H, I, F):

C is the category of simplicial sets;

X is the category of groupoids;

$H\colon X \to C$ is the canonical inclusion called the nerve functor, which has the fundamental groupoid functor $I = \pi_1\colon C \to X$ as its left adjoint;

F is the class of fibrations in the sense of Kan;

for details see Gabriel, Zisman [**3**].

Covering maps (trivial covering maps) with respect to this structure are called second order coverings (second order trivial coverings respectively) of simplicial sets.

C has a full subcategory with objects the Kan complexes (see [**3**]). Since each groupoid is a Kan complex, the above described Galois structure can be restricted to the Galois structure (C', X, H', I', F), where:

C' is category of the Kan complexes;

X is the category of groupoids;

$H'\colon X \to C'$ is the inclusion;

$I'\colon C' \to X$ restriction of I to C';

F is the class of Kan fibrations in C'.

This structure is admissible by Theorem 3.1 of [**1**].

We will show that in this case the condition of stabilizing objects is satisfied, and thus F' admits monotone-light factorization. First we prove:

Proposition 3.1. *If E is a Kan complex with $\pi_2(E) = 0$, then it is a stabilizing object.*

Proof. For a factorization $f = me$ of a Kan fibration $f\colon A \to E$ with m a second order trivial covering and $I(e)$ an isomorphism, given a pullback

$$\begin{tikzcd}
C \times_D A \arrow[r, "pr1"] \arrow[d, "e'"] & A \arrow[d, "e"] \\
C \arrow[r, "k"] & D \arrow[d, dotted, "m"] \\
& E
\end{tikzcd}$$

with k a Kan fibration, we need to prove that $I(e')$ is an isomorphism. Since functors I and H do not change vertices of simplicial sets, it suffices to show that for each vertex (c, a) of $C \times_D A$ the map $\pi_1(C \times_D A, (c, a)) \to \pi_1(C, c)$ is an isomorphism of groups.

The pullback square above gives a commutative diagram with exact rows:

$$\begin{tikzcd}
\pi_2(A,a) \arrow[r] \arrow[d] & \pi_1(c \times F,(c,a)) \arrow[r] \arrow[d] & \pi_1(C \times_D A,(c,a)) \arrow[r] \arrow[d] & \pi_1(A,a) \arrow[r] \arrow[d] & \pi_0(c \times F,(c,a)) \arrow[d] \\
\pi_2(D,e(a)) \arrow[r] & \pi_1(F,c) \arrow[r] & \pi_1(C,c) \arrow[r] & \pi_1(D,e(a)) \arrow[r] & \pi_0(F,c)
\end{tikzcd}$$

were F is the fiber of k on $e(a)$. We observe:

All arrows not involving π_0 are group homomorphisms;

$\pi_1(A, a) \to \pi_1(D, e(a))$ is an isomorphism for $I(e)$ is an isomorphism;

$(c \times F, (c, a)) \to (F, c)$ is an isomorphism and hence so are the maps induced on π_1 and π_0;

$\pi_2(D, e(a)) = 0$ for $\pi_2(E) = 0$ and m is a Kan fibration with fiber a groupoid since it is a second order trivial covering.

By the five lemma $\pi_1(C \times_D A, (c, a)) \to \pi_1(C, c)$ is an isomorphism.

The exact sequence for a fibration used above is described in [**3**]; we can use it here since A and D are Kan complexes and k and $pr1$ are Kan fibrations. □

Theorem 3.2. *Kan fibrations admit monotone-light factorization.*

Proof. We can use Observation 2.2. Since effective F'-descent maps in C' are the same as surjective Kan fibrations [1, Proposition 3.2], it remains only to note that for each Kan complex B there is a surjective fibration $p\colon E \to B$ with all connected components of E contractible and hence having $\pi_2(E) = 0$. □

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