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SYMBOL LENGTHS IN MILNOR K-THEORY

KARIM JOHANNES BECHER AND DETLEV W. HOFFMANN

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Abstract

Let F be a field and p a prime number. The p-symbol length of F, denoted by $\lambda_p(F)$, is the least integer l such that every element of the group K_2F/pK_2F can be written as a sum of $\leqslant l$ symbols (with the convention that $\lambda_p(F) = \infty$ if no such integer exists). In this article, we obtain an upper bound for $\lambda_p(F)$ in the case where the group $F^\times/F^{\times p}$ is finite of order p^m. This bound is $\lambda_p(F) \leqslant \frac{m}{2}$, except for the case where $p = 2$ and F is real, when the bound is $\lambda_2(F) \leqslant \frac{m+1}{2}$. We further give examples showing that these bounds are sharp.

1. Introduction

Let F be a commutative field. Let $F^\times$ denote the multiplicative group of F. We recall the definition of the groups K_nF $(n \geqslant 1)$ of Milnor K-theory, as defined in [**15**]. K_1F is nothing but the multiplicative group $F^\times$ in additive notation. In order to distinguish between addition in K_1F and in F, we write $\{x\} \in K_1F$ for $x \in F^\times$, so that we have $\{x\} + \{y\} = \{xy\}$ in K_1F for $x, y \in F^\times$. For $n \geqslant 2$, the group K_nF is defined as the quotient of the group $(K_1F)^{\otimes n}$ modulo the subgroup generated by the elementary tensors $\{x_1\} \otimes \cdots \otimes \{x_n\} \in (K_1F)^{\otimes n}$ where $x_1, \dots, x_n \in F^\times$ and $x_i + x_{i+1} = 1$ for some $i < n$.

Let now be $n \geqslant 1$ and let H denote the group K_nF or some quotient of it. For $x_1, \dots, x_n \in F^\times$, we denote the canonical image of $\{x_1\} \otimes \cdots \otimes \{x_n\} \in (K_1F)^{\otimes n}$ in H by $\{x_1, \dots, x_n\}$ and we call such an element a *symbol in* H. This notation differs from Milnor's original one where the same symbol is denoted by $l(x_1) \cdots l(x_n)$. Obviously, H is generated by its symbols. Note further that the zero element of H is a symbol. We may then ask whether there exists an integer $l \geqslant 0$ such that every element of H can be written as a sum of l symbols. If this is the case then we denote by $\lambda(H)$ the least such integer l; otherwise we set $\lambda(H) = \infty$. We call the value $\lambda(H) \in \mathbb{N} \cup \{\infty\}$ the *symbol length of* H.

Quotients of K_nF of particular interest are $K_nF/l\,K_nF$ where l is a positive integer; we shall abbreviate this quotient by $K_n^{(l)}F$. In the case where $l = 2$, we may also use Milnor's notation k_nF for $K_nF/2\,K_nF$.

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Let now p be a prime number. We define $\lambda_p(F) = \lambda(K_2^{(p)}F)$ and call this the *p-symbol length* of the field F.

Lenstra showed that if F is a global field, then K_2F consists of symbols (cf. [**10**]); it follows then that $\lambda_p(F) = 1$ for all primes p. The same is true if F is a local field. Whether the rational function field in two variables over the complex numbers $F = \mathbb{C}(X, Y)$ also satisfies $\lambda_p(F) = 1$ for all primes p is a striking open question; at least for $p \leqslant 3$ the answer is positive, by a theorem of Artin (cf. [**1**]). A recent result of Saltman implies that $\lambda_2(F) = 2$ if F is the function field of a curve over a q-adic local field where $q \neq 2$ (cf. [**19**]).

In this article, we establish an upper bound for $\lambda_p(F)$ under the assumption that $F^\times/F^{\times p}$ is finite and we show further that it is generally the best possible.

Theorem 1.1. *Suppose that* $|F^\times/F^{\times p}| = p^m$ *where* $m \in \mathbb{N}$*. Then*

$$\lambda_p(F) \leqslant \begin{cases} \left[\frac{m}{2}\right] & \text{if } p \neq 2 \text{ or if } F \text{ is nonreal,} \\ \left[\frac{m+1}{2}\right] & \text{if } F \text{ is real and } p = 2. \end{cases}$$

Here we use the notation $[x]$ for the integral part of $x \in \mathbb{R}$. Recall that the field F is nonreal if and only if -1 is a sum of squares in F.

The proof of Theorem 1.1 is divided into several parts, which will occupy the next two sections. In Section 2, we relate the K-groups modulo p to certain exterior power spaces and, under the condition that $p \neq 2$ or that -1 is a square in F, we deduce the estimate $\lambda_p(F) \leqslant [\frac{m}{2}]$ from a known fact on alternating spaces (2.3). In a similar way we get the weaker bound $\lambda_p(F) \leqslant [\frac{m+1}{2}]$ for $p = 2$ (2.8). We will therefore be left with the task to exclude that $\lambda_2(F)$ be equal to $\frac{m+1}{2}$ unless F is a real field. This will be done in the third section by an argument involving quadratic form theory (3.5).

Finally we show in the fourth section that in all cases, according to whether F is real or not, and whether p equals 2 or not, the estimate stated above is best possible (4.2). To do so, we give examples where F is a field containing a primitive p-th root of unity, such that $|F^\times/F^{\times p}| = p^m$ and such that there is a simple F-division algebra which is a product of l symbol algebras of degree p, where $l = [\frac{m+1}{2}]$ if $p = 2$ and if F is real, and where $l = [\frac{m}{2}]$ otherwise; the existence of such a division algebra implies indeed that $\lambda_p(F) \geqslant l$. We will further apply a similar argument to show that for the rational function field in $m+1$ variables over a field F containing a primitive p-th root of unity, we always have $\lambda_p(F(X_0, \dots, X_m)) \geqslant m$ (4.5). This improves the bound $\lambda_p(F(X_0, \dots, X_m)) \geqslant [\frac{m+1}{2}]$, shown in [**7**, Proposition 3] (under a stronger hypothesis).

2. K-groups modulo p and exterior powers

Let k denote a commutative field, V a vector space over k and n a positive integer. Let $\Lambda^n V$ denote the exterior power of degree n over V. This is a vector space over k generated by elements $v_1 \wedge \cdots \wedge v_n$, where $v_1, \dots, v_n \in V$, subject to the relations of

k-multilinearity as well as to the relation that $v_1 \wedge \cdots \wedge v_n = 0$ whenever $v_i = v_{i+1}$ for some $i < n$. An element of $\Lambda^n V$ which is of the shape $v_1 \wedge \cdots \wedge v_n$ is called a *pure n-vector*.

Suppose now that V has finite dimension m. There exists a least integer N such that every element of $\Lambda^n V$ is a sum of N pure n-vectors. If the dimension of V is m then we denote this integer by $N(k; m, n)$ (since it depends only on k, n and m). For $n = 2$ a classical result tells us that $N(k; m, 2) = [\frac{m}{2}]$, independently of the field k (cf. [**3**, §5, Corollaire 2]).

Let p be a prime number and n a positive integer. Let $\mathbb{F}_p$ denote the finite field with p elements. The group $K_n^{(p)}F$, associated to the field F, is endowed with a natural structure as a vector space over $\mathbb{F}_p$. Note that for $m \in \mathbb{N}$, equality $|F^\times / F^{\times p}| = p^m$ means that $K_1^{(p)}F$ as a vector space over $\mathbb{F}_p$ has dimension m.

Let $x_1, \ldots, x_n$ be elements of $F^\times$. If $x_i = x_{i+1}$ for some $i < n$ then one has

$$\{x_1, \ldots, x_n\} \;=\; \{x_1, \ldots, x_i, -1, x_{i+2}, \ldots, x_n\}$$

in $K_2^{(p)}F$ by [**15**, Lemma 1.2]; in particular, this symbol is of order at most 2. Suppose now that $-1 \in F^{\times p}$, that is either $p \neq 2$ or $p = 2$ and $-1 \in F^{\times 2}$. This implies that any symbol in $K_n^{(p)}F$ with a repeated coefficient is zero. Therefore, if $-1 \in F^{\times p}$ then the natural $\mathbb{F}_p$-homomorphism $(K_1^{(p)}F)^n \longrightarrow K_n^{(p)}F$ induces a natural $\mathbb{F}_p$-homomorphism

$$\Lambda^n(K_1^{(p)}F) \longrightarrow K_n^{(p)}F$$

which maps a pure n-vector $\{x_1\} \wedge \cdots \wedge \{x_n\}$ to the symbol $\{x_1, \ldots, x_n\}$ and which is therefore surjective. From this we conclude:

Lemma 2.1. *Suppose that $p \neq 2$ or that $-1 \in F^{\times 2}$. If $|F^\times / F^{\times p}| = p^m$ then $\lambda(K_n^{(p)}F) \leqslant N(\mathbb{F}_p; m, n)$.* □

Unfortunately, the exact value of $N(\mathbb{F}_p; m, n)$ is not known in general when $n \geqslant 3$ (cf. [**6**] on this problem and approximative results).

We may further observe that the bound in (2.1) is best possible for all m, n and p. Indeed, one will have $\lambda(K_n^{(p)}F) = N(\mathbb{F}_p; m, n)$ if $-1 \in F^{\times p}$, $|F^\times / F^{\times p}| = p^m$ and if the natural $\mathbb{F}_p$-homomorphism $\Lambda^n(K_1^{(p)}F) \longrightarrow K_n^{(p)}F$ is bijective, and examples where these conditions hold will be given in (2.5).

Proposition 2.2. *Suppose that $p \neq 2$ or that $-1 \in F^{\times 2}$. The following are equivalent:*

(i) *for any $n \geqslant 2$, the natural $\mathbb{F}_p$-homomorphism $\Lambda^n(K_1^{(p)}F) \longrightarrow K_n^{(p)}F$ is an isomorphism;*

(ii) *the natural $\mathbb{F}_p$-homomorphism $\Lambda^2(K_1^{(p)}F) \longrightarrow K_2^{(p)}F$ is an isomorphism;*

(iii) *whenever $a, b \in F^\times$ are such that the symbol $\{a, b\} \in K_2^{(p)}F$ is zero then the elements $\{a\}$ and $\{b\}$ in $K_1^{(p)}F$ are $\mathbb{F}_p$-linearily dependent;*

(iv) *for any $a \in F^\times \setminus F^{\times p}$ there exists $r \geqslant 0$ such that $a^r(1-a) \in F^{\times p}$.*

Proof. The implications ($i \Rightarrow ii \Rightarrow iii \Rightarrow iv$) are obvious.

Assume now that (iv) holds. Then it follows for $n \geqslant 2$ that a pure n-vector $\{x_1\} \wedge \cdots \wedge \{x_n\} \in \Lambda^n(K_1^{(p)}F)$ is zero as soon as $x_i + x_{i+1} = 1$ for some $i < n$. Since this corresponds to the defining relation for $K_n^{(p)}F$ as a quotient of $(K_1^{(p)}F)^{\otimes n}$, we conclude that the natural surjective $\mathbb{F}_p$-homomorphism $\Lambda^n(K_1^{(p)}F) \longrightarrow K_n^{(p)}F$ is in fact bijective. This shows that (iv) implies (i). □

In the case where $p \neq 2$ and F contains a primitive p-th root of unity, the conditions (i)–(iv) are satisfied if and only if F is a *p-rigid field*, as defined in [**23**, p. 772].

Proposition 2.3. *Suppose that* $|F^\times / F^{\times p}| = p^m$, *and further that* $p \neq 2$ *or that* $-1 \in F^{\times 2}$. *Then*

(a) *every element of* $K_2^{(p)}F$ *is a sum of* $[\frac{m}{2}]$ *symbols,*

(b) *every element of* $K_{m-1}^{(p)}F$ *is a symbol,*

(c) $K_n^{(p)}F = 0$ *for any* $n > m$,

(d) *the equivalent conditions in* (2.2) *hold if and only if* $K_m^{(p)}F \neq 0$, *and in this case one has* $K_m^{(p)}F \cong \mathbb{Z}/p\,\mathbb{Z}$.

Proof. (a) By (2.1) every element of $K_2^{(p)}F$ can be written as a sum of $N(\mathbb{F}_p; m, 2)$ symbols, and this number is known to be equal to $[\frac{m}{2}]$.

(b) It is easy to see that $N(k; m, m-1) = 1$ for any field k. Therefore we have $\lambda(K_{m-1}^{(p)}F) \leqslant 1$ by (2.1).

(c) $K_n^{(p)}F$ can be considered as a quotient of the space $\Lambda^n(K_1^{(p)}F)$, which vanishes as soon as n exceeds m, the dimension of $K_1^{(p)}F$.

(d) Since $K_1^{(p)}F$ is of dimension m, one has $\Lambda^m(K_1^{(p)}F) \cong \mathbb{Z}/p\,\mathbb{Z}$. Therefore the first condition in Proposition 2.2 implies that $K_m^{(p)}F \cong \mathbb{Z}/p\,\mathbb{Z}$.

Assume now that the equivalent conditions in Proposition (2.2) do not hold for F. In particular, since (iii) does not hold, there are $a_1, a_2 \in F^\times$, $\mathbb{F}_p$-linearly independent modulo p-th powers, such that $\{a_1, a_2\} = 0$ in $K_2^{(p)}F$. We may choose $a_3, \ldots, a_m \in F^\times$ such that $\{a_1\}, \ldots, \{a_m\}$ form an $\mathbb{F}_p$-basis of $K_1^{(p)}F$. Then the pure n-vector $\{a_1\} \wedge \cdots \wedge \{a_m\}$ in $\Lambda^m(K_1^{(p)}F)$ is nonzero, while the symbol $\{a_1, \ldots, a_m\}$ in $K_m^{(p)}F$ is zero. Hence the natural surjection $\Lambda^m(K_1^{(p)}F) \longrightarrow K_m^{(p)}F$ is not injective. As $\Lambda^m(K_1^{(p)}F) \cong \mathbb{Z}/p\,\mathbb{Z}$, it follows that $K_m^{(p)}F = 0$. □

Lemma 2.4. *Suppose* $p \neq 2$ *or that* $-1 \in F^{\times 2}$, *and that* F *is* p*-henselian with respect to a discrete valuation* v *with residue field* $\bar{F}$ *with* $\mathrm{char}(\bar{F}) \neq p$. *If the equivalent conditions in* (2.2) *hold for* $\bar{F}$ *then they hold for* F *as well.*

Proof. Since F is p-henselian and $p \neq \mathrm{char}(\bar{F})$, we know that for any $u \in F$ with $v(u) = 0$ we have $u \in F^{\times p}$ if and only if $\overline{u} \in \bar{F}^{\times p}$. We show that Condition (iv) in (2.2) holds for the field F if it holds for $\bar{F}$.

Let a be an element of F which is not a p-th power in F. We are looking for an integer r such that $a^r - a^{r+1}$ is a p-th power in F. Using that $-1 \in F^{\times p}$ and possibly replacing a by $-a^{-1}$, we may assume that $v(a) \geqslant 0$. If $v(a) > 0$ then $a^r - a^{r+1}$ is a p-th power if and only of a^r is a p-th power, hence we are done with $r = p$. On the other hand, if $v(a) = 0$ then we saw that $\overline{a}$ cannot be a p-th power in $\bar{F}$; hence, by hypothesis, there exists $r \in \mathbb{Z}$ such that $\overline{a^r(1-a)}$ is a p-th power in $\bar{F}$ and it follows that $a^r(1-a)$ is a p-th power in F. □

Examples 2.5. In each of the following cases, the field F satisfies the equivalent conditions of (2.2):

- $F = C(\!(X_1)\!) \dots (\!(X_m)\!)$ where C is an algebraically closed field,
- $F = R(\!(X_1)\!) \dots (\!(X_m)\!)$ where $p \neq 2$ and R is a real closed field,
- F is a *non-p-adic m-dimensional local field*, i.e., there exists a sequence of fields $F_0, \dots, F_m = F$ where F_0 is finite with $\mathrm{char}(F_0) \neq p$ and, for $1 < i \leqslant m$, the field F_i is complete with respect to a discrete valuation with residue field F_{i-1}.

Indeed, this follows from the basic case $m = 0$, using the above lemma. Note further that $|F^\times / F^{\times p}| = p^m$ in the first two cases. In the third case one has $|F^\times / F^{\times p}| = p^{m+1}$ if p divides $(\mathrm{char}(F_0) - 1) = |F_0^\times|$ and $|F^\times / F^{\times p}| = p^m$ otherwise.

The remainder of this section together with the following section is devoted to the study of the case where $p = 2$. Then, without the hypothesis that $-1 \in {F^\times}^2$, we cannot relate $K_n^{(p)}F$ to the exterior power space $\Lambda^n(K_1^{(p)}F)$ over $\mathbb{F}_p$ as before. We therefore are going to define a kind of "twisted exterior power space" over the field with two elements.

In the sequel, the letter k will only be used to denote the Milnor K-groups modulo 2 of a field, that is $k_nF = K_n^{(2)}F$ $(n \geqslant 0)$.

Let V be a vector space over the field $\mathbb{F}_2$, ε a fixed element of V and n a positive integer. We shall write $\Lambda_\varepsilon^n V$ for the vector space over $\mathbb{F}_2$ generated by elements $v_1 \wedge \cdots \wedge v_n$, with $v_1, \dots, v_n \in V$, subject to the relations of multilinearity and symmetry (i.e. $v_1 \wedge \cdots \wedge v_n$ does not depend on the order of the coefficients v_i) and further to the relation that $v_1 \wedge \cdots \wedge v_n = 0$ whenever $v_{i+1} = v_i + \varepsilon$ for some $i < n$. Note that if $\varepsilon = 0$, then $\Lambda_\varepsilon^n V$ is just the exterior power space $\Lambda^n V$.

We call *pure n-vectors in* $\Lambda_\varepsilon^n V$ the elements of the form $v_1 \wedge \cdots \wedge v_n$. From now on we restrict to the case where $n = 2$. For given $x \in V$, we shall say that a pure 2-vector is of the shape $x \wedge *$ if it can be written as $x \wedge y$ for some $y \in V$.

Lemma 2.6. *Let V be a vector space over $\mathbb{F}_2$ with a special element $\varepsilon \in V$. Let ξ be an element of $\Lambda_\varepsilon^2 V$ and let $v_1, \dots, v_l, w_1, \dots, w_l \in V$ be such that*

$$\xi = v_1 \wedge w_1 + \cdots + v_l \wedge w_l \,.$$

(a) *If v is a nontrivial sum of some of the elements $v_1, \dots, v_l$ then ξ can be written as a sum of l pure 2-vectors where the first is of the shape $v \wedge *$.*

(b) *If v is a nontrivial sum of some of the elements $v_1, \dots, v_l, w_1, \dots, w_l$ then ξ can be written as a sum of l pure 2-vectors where the first is of the shape $v \wedge *$ or $(v + \varepsilon) \wedge *$.*

(c) *If $v_1, \ldots, v_l, w_1, \ldots, w_l$, ε are linearly dependent then ξ can be written as a sum of l pure 2-vectors where the first is of the shape $\varepsilon \wedge *$.*

Proof. (a) If $l = 1$ this is trivial. In the case $l = 2$ one uses the equality

$$v_1 \wedge w_1 + v_2 \wedge w_2 = (v_1 + v_2) \wedge w_1 + v_2 \wedge (w_1 + w_2)\,.$$

The general case follows from this by induction on l.

(b) Using the relations

$$v_i \wedge w_i = w_i \wedge v_i = (v_i + w_i + \varepsilon) \wedge w_i$$

one readily sees that the statement follows from (a).

(c) Suppose that $v_1, \ldots, v_l, w_1, \ldots, w_l$, ε are linearly dependent. Then one of the elements 0 and ε can be written as a nontrivial sum of some of the elements $v_1, \ldots, v_l, w_1, \ldots, w_l$. Observing that a pure 2-vector of the shape $0 \wedge *$ is zero, hence equal to $\varepsilon \wedge 0$, we conclude by (b) that ξ can be written as a sum of l pure 2-vectors where the first is of the shape $\varepsilon \wedge *$. □

Proposition 2.7. *Let V be a vector space over $\mathbb{F}_2$ of dimension m and let $\varepsilon \in V$. Then any element of $\Lambda^2_\varepsilon V$ is a sum of $[\frac{m+1}{2}]$ pure 2-vectors. If $\xi \in \Lambda^2_\varepsilon V$ is not a sum of $[\frac{m}{2}]$ pure 2-vectors, then*

$$\xi = \varepsilon \wedge \varepsilon + v_1 \wedge w_1 + \cdots + v_n \wedge w_n\,,$$

where $v_1, w_1, \ldots, v_n, w_n \in V$ and $2n+1 = m$; moreover, under these circumstances, $(v_1, w_1, \ldots, v_n, w_n, \varepsilon)$ is a basis of V.

Proof. Suppose that $\xi \in \Lambda^2_\varepsilon V$ is a sum of l but not of $l-1$ pure 2-vectors where $l > \frac{m}{2}$. In a representation of ξ as a sum of l pure 2-vectors, the $2l$ coefficients are necessarily linearly dependent. By (2.6(c)), we may write $\xi = \varepsilon \wedge w + \xi'$ with $w \in V$ and with $\xi' \in \Lambda^2_\varepsilon V$ equal to a sum of $l-1$ pure 2-vectors. Assume that w is different from ε. Then, for dimension reasons, at least one of the elements 0, ε, w and $w+\varepsilon$ can be written as a nontrivial sum of the coefficients of any representation of ξ' as a sum of $l-1$ pure 2-vectors. Using the fact that $0 \wedge * = \varepsilon \wedge 0$, it follows from (2.6(b)) that ξ' can be written as a sum of $l-1$ pure 2-vectors where the first one is of the shape $x \wedge y$ with $y \in V$ and x equal to one of the elements w, ε and $w+\varepsilon$. Now $\varepsilon \wedge w + x \wedge y$ cannot be equal to a pure 2-vector, since otherwise ξ would be equal to a sum of $l-1$ pure 2-vectors. Hence, by the relations in $\Lambda^2_\varepsilon V$, we must have $x = w + \varepsilon$ and therefore

$$\varepsilon \wedge w + x \wedge y = \varepsilon \wedge (x + \varepsilon) + y \wedge x \;=\; \varepsilon \wedge \varepsilon + (y + \varepsilon) \wedge x\,.$$

From this we conclude that we can write

$$\xi \;=\; \varepsilon \wedge \varepsilon + v_1 \wedge w_1 + \cdots + v_{l-1} \wedge w_{l-1}$$

with $v_1, w_1, \ldots, v_{l-1}, w_{l-1} \in V$. Since ξ is not a sum of less than l pure 2-vectors, we conclude from (2.6(c)) that $v_1, w_1, \ldots, v_{l-1}, w_{l-1}, \varepsilon$ are linearly independent. Then, since $2l - 1 \geqslant m$, these elements form a basis of V, in particular, if we put $n = l-1$, then $2n+1 = m$. □

Since k_1F has an obvious structure of a vector space over $\mathbb{F}_2$, we may apply the above results to k_1F together with the special element $\varepsilon = \{-1\}$. By the relations defining k_nF, there exists a surjective $\mathbb{F}_2$-homomorphism

$$\Lambda^n_{\{-1\}}(k_1F) \longrightarrow k_nF$$

which maps a pure n-vector $\{x_1\} \wedge \cdots \wedge \{x_n\}$ to a symbol $\{x_1, \ldots, x_n\}$. From this together with the last proposition we obtain immediately:

Corollary 2.8. *Suppose that $|F^\times/F^{\times 2}| = 2^m$. Then every element of k_2F can be written as a sum of $[\frac{m+1}{2}]$ symbols.* □

Remark 2.9. Note that (2.1) and (2.8) immediately imply following weaker version of Theorem 1.1: if $|F^\times/F^{\times p}| = p^m$ then

$$\lambda_p(F) \leqslant \begin{cases} \left[\frac{m}{2}\right] & \text{if } p \neq 2 \text{ or } -1 \in F^{\times 2}, \\ \left[\frac{m+1}{2}\right] & \text{if } p = 2 \text{ and } -1 \notin F^{\times 2}. \end{cases}$$

In order to establish Theorem 1.1 entirely, it remains to show that $\lambda_2(F) \leqslant [\frac{m}{2}]$ holds for any nonreal field F with $|F^\times/F^{\times 2}| = 2^m$. This will be accomplished with (3.5) in the following section.

3. K-groups modulo 2 and quadratic forms

For the following definitions and facts, we suppose that the characteristic of F is different from 2. We shall use the standard notations in quadratic form theory as established in [**8**] and [**20**]. However, we use a different convention for Pfister forms: if $a_1, \ldots, a_n \in F^\times$ then $\langle\langle a_1, \ldots, a_n\rangle\rangle$ will denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. Let $W(F)$ denote the Witt ring of F and IF the fundamental ideal consisting of the classes of quadratic forms of even dimension. Let further $I^nF = (IF)^n$ and $\bar{I}^nF = I^nF/I^{n+1}F$ for any $n \geqslant 0$. The ideal I^nF is additively generated by the n-fold Pfister forms over F.

Milnor has defined for any $n \geqslant 1$ a homomorphism $s_n : k_nF \longrightarrow \bar{I}^nF$, mapping a symbol $\{x_1, \ldots, x_n\}$ to the class of the Pfister form $\langle\langle x_1, \ldots, x_n\rangle\rangle$ and determined by this property. Milnor showed that $s_2 : k_2F \to \bar{I}^2F$ is an isomorphism [**15**, Theorem 4.1.]. It has been proven recently that $s_n : k_nF \longrightarrow \bar{I}^nF$ is an isomorphism for any $n \geqslant 1$ [**18**, Theorem 4.1].

We write $\pm F^{\times 2}$ for the union $F^{\times 2} \cup -F^{\times 2}$. Recall that F is a *real euclidean field*, if $F^{\times 2}$ is an ordering of F, i.e. if F is real and $F^\times = \pm F^{\times 2}$.

Proposition 3.1. *Suppose that $|F^\times/F^{\times 2}| = 2$ and $n \geqslant 2$. If F is nonreal then $k_nF = 0$, otherwise F is real euclidean and the unique nonzero element in k_nF is $\{-1, \ldots, -1\}$.*

Proof. By the hypothesis we have $F^\times = F^{\times 2} \cup aF^{\times 2}$ for some $a \in F^\times$. Suppose that k_nF is nontrivial. It is obvious that then the unique nonzero element of k_nF is the symbol $\{a, \ldots, a\}$, which can also be written as $\{-1, \ldots, -1, a\}$. As this symbol is nontrivial, -1 cannot be a square and a cannot be a sum of two squares in F.

Hence $aF^{\times 2} = -F^{\times 2}$ and -1 is not a sum of two squares. Therefore, $F^\times = \pm F^{\times 2}$ and every sum of two squares is again a square in F. We conclude that F is a real euclidean field. To complete the proof we remind that, if F is any real field, then the symbol $\{-1,\dots,-1\} \in k_nF$ is nontrivial. $\square$

In the sequel we will focus on the study of k_2F. If φ is a quadratic form in I^2F then we write $\overline{\varphi}$ for its class in $\bar{I}^2F$. The following statement is folklore, at least in the context where k_2F is replaced by $\mathrm{Br}_2(F)$, the 2-torsion of the Brauer group of F, and $s_2 : k_2F \to \bar{I}^2F$ by the inverse of the homomorphism $c : \bar{I}^2F \to \mathrm{Br}_2(F)$, induced by the Clifford invariant (cf. [**5**, Lemma 2.2.] and the references given there). We wish to give a self-contained proof here in our setting, where the statement does not depend on Merkurjev's result [**11**] that $c : \bar{I}^2F \to \mathrm{Br}_2(F)$ is an isomorphism.

Lemma 3.2. *Suppose that* $\mathrm{char}(F) \neq 2$. *Let* $m \geqslant 1$ *and* $\xi \in k_2F$. *A necessary and sufficient condition that* ξ *be a sum of* m *symbols is that the class* $s_2(\xi) \in \bar{I}^2F$ *contain a quadratic form of dimension* $2m+2$.

Proof. We first prove by induction on m that the condition is necessary. We write $\xi = \xi' + \{a,b\}$ where $\xi' \in k_2F$ is a sum of $m-1$ symbols and $a,b \in F^\times$. Now, if $m = 1$ then we have $\xi' = 0$ and $\xi = \{a,b\}$, hence $s_2(\xi) = \overline{\langle\langle a,b\rangle\rangle}$, and the dimension of $\langle\langle a,b\rangle\rangle$ is $4 = 2m+2$. Suppose now that $m > 1$. By induction hypothesis we have $s_2(\xi') = \overline{\varphi'}$ for a quadratic form φ' of dimension $2(m-1)+2 = 2m$. We decompose φ' into $\varphi'' \perp \langle r\rangle$, where φ'' is a quadratic form of dimension $2m-1$ and $r \in F^\times$. As $s_2(\{a,b\}) = \overline{-r\langle\langle a,b\rangle\rangle}$ we obtain

$$s_2(\xi) \;=\; s_2(\xi') + s_2(\{a,b\}) = \overline{\varphi' \perp -r\langle\langle a,b\rangle\rangle}\,.$$

But the form $\varphi' \perp -r\langle\langle a,b\rangle\rangle$ is Witt equivalent to $\varphi'' \perp \langle ra, rb, -rab\rangle$. Hence, if we put $\varphi = \varphi'' \perp \langle ra, rb, -rab\rangle$ then we have $s_2(\xi) = \overline{\varphi}$ and $\dim\varphi = 2m+2$.

To show that the condition is sufficient, we use again induction on m. Let ψ be a quadratic form in I^2F of dimension $2m+2$ such that $s_2(\xi) = \overline{\psi}$. We decompose ψ into $\psi' \perp \langle c,d,e\rangle$ with ψ' a quadratic form of dimension $2m-1$ and $c,d,e \in F^\times$. Since ψ is Witt equivalent to $\psi' \perp \langle -cde\rangle \perp e\langle\langle -ce,-de\rangle\rangle$, we have

$$s_2(\xi) \;=\; \overline{\psi' \perp \langle -cde\rangle} + \overline{e\langle\langle -ce,-de\rangle\rangle} \;=\; \overline{\psi' \perp \langle -cde\rangle} + s_2(\{-ce,-de\})\,.$$

We write $\xi = \xi' + \{-ce,-de\}$, with $\xi' \in k_2F$. Since s_2 is an isomorphism, it follows that $s_2(\xi') = \overline{\psi' \perp \langle -cde\rangle}$, where the quadratic form $\psi' \perp \langle -cde\rangle$ has dimension $2m = 2(m-1)+2$ and trivial discriminant. Hence, if $m = 1$ then $\psi' \perp \langle -cde\rangle$ must be the hyperbolic plane, thus $\xi' = 0$ by injectivity of s_2 and $\xi = \{-ce,-de\}$. If $m > 1$ then by induction hypothesis ξ' is a sum of $m-1$ symbols, which implies that ξ is a sum of m symbols. $\square$

Lemma 3.3. *Suppose that* $\mathrm{char}(F) \neq 2$. *Let* $a \in F^\times$. *If* a *can be written in* F *as a sum of three squares (or less), then there exists* $b \in F^\times$ *such that in* k_2F *we have* $\{-1,-1\} = \{-a,-b\}$.

Proof. If a is a sum of three squares in F, then there is some $b \in F^\times$ such that $\langle\langle -1,-1\rangle\rangle = \langle 1,1,1,1\rangle = \langle 1,a,b,ab\rangle = \langle\langle -a,-b\rangle\rangle$; hence $s_2(\{-1,-1\}) = s_2(\{-a,-b\})$ and thus $\{-1,-1\} = \{-a,-b\}$, by injectivity of s_2. $\square$

Proposition 3.4. *Suppose that* $|F^\times/F^{\times 2}| = 2^m$ *where* $m > 1$ *and that there is an element* $\xi \in k_2F$ *which cannot be written as a sum of* $[\frac{m}{2}]$ *symbols. Then there exist* $a_1, b_1, \ldots, a_n, b_n \in F^\times$*, where* $2n+1 = m$*, such that in* k_2F *we have*

$$\xi = \{-1,-1\} + \{-a_1,-b_1\} + \cdots + \{-a_n,-b_n\}\,.$$

Furthermore, given any such representation of ξ *as a sum of* n *symbols, the elements* $\{-1\}, \{a_1\}, \{b_1\}, \ldots, \{a_n\}, \{b_n\}$ *form an* $\mathbb{F}_2$*-basis of* k_1F *and none of the elements* $a_1, b_1, \ldots, a_n, b_n$ *is a sum of squares in* F*.*

Proof. By (2.3(a)) and by the hypothesis on ξ, the element -1 cannot be a square in F, in particular, $\mathrm{char}(F) \neq 2$.

Let θ be an element of $\Lambda^2_{\{-1\}}(k_1F)$ which maps to ξ under the canonical homomorphism $\Lambda^2_{\{-1\}}(k_1F) \longrightarrow k_2F$. The hypothesis implies that θ is not a sum of $[\frac{m}{2}]$ pure 2-vectors in $\Lambda^2_{\{-1\}}(k_1F)$. Since k_1F has dimension m, we conclude by (2.7) that m is odd and that θ is a sum of $\frac{m+1}{2}$ pure 2-vectors where the first one is $\{-1\}\wedge\{-1\}$. Hence we may choose elements $a_1, b_1, \ldots, a_n, b_n \in F^\times$, where $2n+1 = m$, such that

$$\theta = \{-1\}\wedge\{-1\} + \{-a_1\}\wedge\{-b_1\} + \cdots + \{-a_n\}\wedge\{-b_n\}.$$

Further, using $\{-1\} + \{-x\} = \{x\}$ in k_1F for any $x \in F^\times$, we conclude from the second part of (2.7) that the elements $\{-1\}, \{a_1\}, \{b_1\}, \ldots, \{a_n\}, \{b_n\}$ form an $\mathbb{F}_2$-basis of k_1F.

From the above representation of θ we obtain immediately that

$$\xi = \{-1,-1\} + \{-a_1,-b_1\} + \cdots + \{-a_n,-b_n\}.$$

The proof will be complete if we can show for any $l \in \mathbb{N}$ and any representation of ξ as above, that none of the elements $a_1, b_1, \ldots, a_n, b_n$ can be a sum of l squares in F. Suppose that this statement is true for a certain positive integer l. To show that it still holds after replacing l by $l+1$, we may assume that a_1 is a sum of $l+1$ squares in F, in order to derive a contradiction.

Since $\{-1,-1\} + \{-a_1,-b_1\}$ cannot be equal to a symbol, a_1 cannot be a sum of three squares in F by Lemma 3.3, hence $l \geqslant 3$. By our assumption we may write $a_1 = c + e$ with elements $c, e \in F^\times$, where c is a sum of two squares and e is a sum of $l-1$ but not a sum of fewer squares in F.

As $l - 1 \geqslant 2$, the element e is not a square in F. Since there exists an element of F which is a sum of $l+1$ but not of l squares in F, the form $l \times \langle 1\rangle$ over F is necessarily anisotropic. It follows that $e \notin -F^{\times 2}$. As a_1 is not a sum of l squares in F, we also obtain that $e \notin a_1F^{\times 2}$. Further, by the equality $a_1 = (a_1^2 - ea_1)/c$ and since c is a sum of 2 squares in F while this is not the case for a_1, we see that $-ea_1$ is not a square in F, i.e. $e \notin -a_1F^{\times 2}$. Until here we have shown that $e \notin \pm F^{\times 2} \cup \pm a_1F^{\times 2}$.

In k_2F we have the equality $\{-a_1, e\} = \{-c, a_1e\}$. Since c and e are sums of l squares in F, we get from the induction hypothesis that the symbol $\{-a_1,-b_1\}$ cannot be equal neither to $\{-a_1,-e\}$ nor to $\{-a_1,e\}$. Hence the symbols $\{-a_1,-eb_1\}$ and $\{-a_1, eb_1\}$ are both nonzero. We conclude that $e \notin \pm b_1F^{\times 2} \cup \pm a_1b_1F^{\times 2}$.

Since $\{-1\}, \{a_1\}, \{b_1\}, \ldots, \{a_n\}, \{b_n\}$ form an $\mathbb{F}_2$-basis of k_1F, we may write $e = xy$ where

$$x \in \pm F^{\times 2} \cup \pm a_1 F^{\times 2} \cup \pm b_1 F^{\times 2} \cup \pm a_1 b_1 F^{\times 2}$$

and where y is equal to 1 or to a nontrivial product of some of the elements $a_2, b_2, \ldots, a_n, b_n$. Since $e \notin \pm F^{\times 2} \cup \pm a_1 F^{\times 2} \cup \pm b_1 F^{\times 2} \cup \pm a_1 b_1 F^{\times 2}$ as we have shown above, we have $y \neq 1$. Therefore (2.6(b)) shows that in $\Lambda^2_{\{-1\}}(k_1F)$ the element $\{-a_2\} \wedge \{-b_2\} + \cdots + \{-a_n\} \wedge \{-b_n\}$ can be rewritten as a sum of $(n-1)$ pure 2-vectors where the first is of the shape $\{\pm y\} \wedge \{*\}$. Hence we may suppose that $a_2 = \pm y$. By the relations in $\Lambda^2_{\{-1\}}(k_1F)$ and since $e = xy = \pm a_2 x$ where x is as described above, we can rewrite $\{-a_1\} \wedge \{-b_1\} + \{-a_2\} \wedge \{-b_2\}$ as a sum of two pure 2-vectors which is either of the shape $\{-a_1\} \wedge \{*\} + \{\pm e\} \wedge \{*\}$ or of the shape $\{-a_1\} \wedge \{\pm e\} + \{*\} \wedge \{*\}$. So we may actually suppose that e is equal to one of $\pm b_1$ or $\pm a_2$. However, by the induction hypothesis, neither b_1 nor a_2 can be a sum of l squares in F. Hence e is equal either to $-a_2$ or to $-b_1$. In particular, since $a_1 = c+e$ where c is a sum of 2 squares in F, the quadratic form $\langle 1, 1, -a_1, -b_1, -a_2 \rangle$ is isotropic.

We put $\zeta = \{-1, -1\} + \{-a_1, -b_1\} + \{-a_2, -b_2\} \in k_2F$. In $\bar{I}^2F$ we compute

$$\begin{aligned} s_2(\zeta) &= \overline{\langle\langle -1, -1 \rangle\rangle} + \overline{(-1) \cdot \langle\langle -a_1, -b_1 \rangle\rangle} + \overline{(-1) \cdot \langle\langle -a_2, -b_2 \rangle\rangle} \\ &= \overline{\langle 1, 1, -a_1, -b_1, -a_1 b_1, -a_2, -b_2, -a_2 b_2 \rangle}\,. \end{aligned}$$

By the above, the 8-dimensional form $\langle 1, 1, -a_1, -b_1, -a_1b_1, -a_2, -b_2, -a_2b_2 \rangle$ is isotropic, hence Witt equivalent to a 6-dimensional form φ. Then $s_2(\zeta) = \overline{\varphi}$, thus ζ is equal to a sum of two symbols by (3.2). But then ξ can be written as a sum of $n < \frac{m}{2}$ symbols, in contradiction to the hypothesis. □

Corollary 3.5. *If F is a nonreal field such that $|F^\times / F^{\times 2}| = 2^m$ then every element of k_2F can be written as a sum of $[\frac{m}{2}]$ symbols.*

Proof. For $m = 1$ this is clear by (3.1). Assume that $m > 1$ and that there exists an element in k_2F which cannot be written as a sum of $[\frac{m}{2}]$ symbols. Then we must have $-1 \notin F^{\times 2}$ by (2.9), and by the last proposition there exist elements in F which are not sums of squares. Therefore F is real. □

The last corollary completes the proof of Theorem 1.1 (see Remark 2.9).

If the field F is nonreal, then one denotes by $s(F)$ the least positive integer s such that -1 is a sum of s squares over F, otherwise one puts $s(F) = \infty$. The invariant $s(F)$ is called the *level of* F. By a famous result due to Pfister, the integers occurring as the level of some field are precisely the powers of 2. However, it is still an unsolved problem whether there is a nonreal field F of level greater than 4 which has finite square class group $F^\times / F^{\times 2}$. (For further information on this problem, see [**2**] and the references given there.) If the answer to this question turned out to be negative then one could simplify the proof of Proposition 3.4.

We will see in the next section that, whenever there exist fields of finite level s and with finite square class group, then for m sufficiently large there is such a field L such that, in addition to $s(L) = s$, one has $|L^\times / L^{\times 2}| = 2^m$ and $\lambda_2(L) = [\frac{m}{2}]$ (see

Corollary 4.3). This shows in particular that the estimate given in Corollary 3.5 is generally the best possible.

4. Symbols and central simple algebras

Let p be a prime number. We suppose for the moment that the field F contains a primitive p-th root of unity ω; in particular, $\mathrm{char}(F) \neq p$. Given $a, b \in F^\times$ we denote by $(a,b)_{F,\omega}$ the central simple F-algebra of degree p generated by two elements α and β which are subject to the relations $\alpha^p = a$, $\beta^p = b$ and $\beta\alpha = \omega\alpha\beta$. We call this a *symbol algebra of degree p over F*. In the case $p = 2$, we have $\omega = -1$ and, hence, we recover the definition of an F-quaternion algebra; we then write $(a,b)_F$ instead of $(a,b)_{F,-1}$.

Let $\mathrm{Br}(F)$ denote the Brauer group of F and by $\mathrm{Br}_p(F)$ its p-torsion subgroup. There exists a canonical group homomorphism $K_2^{(p)}F \longrightarrow \mathrm{Br}_p(F)$ which maps a symbol $\{a,b\} \in K_2^{(p)}F$, with $a, b \in F^\times$, to the class of $(a,b)_{F,\omega}$ in $\mathrm{Br}_p(F)$. By the Merkurjev–Suslin theorem (cf. [**14**] or [**13**]), this is actually an isomorphism, but we shall not use this fact in the sequel.

Lemma 4.1. *Suppose that the field F contains a primitive p-th root of unity. Let $a_1, b_1, \dots, a_n, b_n \in F^\times$. If $(a_1,b_1)_{F,\omega} \otimes_F \cdots \otimes_F (a_n,b_n)_{F,\omega}$ is a division algebra then the element $\{a_1,b_1\} + \cdots + \{a_n,b_n\}$ in $K_2^{(p)}F$ cannot be written as a sum of less than n symbols.*

Proof. We put $\xi := \{a_1,b_1\} + \cdots + \{a_n,b_n\}$. Let D denote the central F-division algebra whose class in $\mathrm{Br}(F)$ is the image of ξ under the canonical homomorphism $K_2^{(p)} \longrightarrow \mathrm{Br}_p(F)$. If ξ is a sum of m symbols in $K_2^{(p)}F$, then the degree of D is at most p^m. On the other hand, if $(a_1,b_1)_{F,\omega} \otimes_F \cdots \otimes_F (a_n,b_n)_{F,\omega}$ is a division algebra, then it is F-isomorphic to D, which then must be of degree p^n. These facts together imply the statement. □

Theorem 4.2. *Suppose that* $\mathrm{char}(F) \neq p$ *and that* $|F^\times/F^{\times p}| = p^n$*. Let* $m \geqslant 2n-1$ *and* $L = F(\!(X_1)\!)\dots(\!(X_{m-n})\!)$*. Then* $|L^\times/L^{\times p}| = p^m$ *and*

$$\lambda_p(L) = \begin{cases} \left[\frac{m}{2}\right] & \text{if } p \neq 2 \text{ or if } F \text{ is nonreal,} \\ \left[\frac{m+1}{2}\right] & \text{if } p = 2 \text{ and } F \text{ is real.} \end{cases}$$

Proof. It is well-known that $|L^\times/L^{\times p}| = p^{m-n} \cdot |F^\times/F^{\times p}| = p^n$ To prove the remaining claims we may assume that m is equal either to $2n-1$ or to $2n$. Indeed, if $m > 2n$ then we replace n by $n' = [\frac{m}{2}]$ and F by $F' = F(\!(X_1)\!)\dots(\!(X_{n'-n})\!)$, observing that $|F'^\times/F'^{\times p}| = p^{n'}$ and that L is the iterated power series field in $m - n'$ variables over F'.

Let ω denote a primitive p-th root of unity. Recall that $F(\omega)/F$ is a field extension of degree dividing $p-1$. Therefore any irreducible polynomial of degree p over F will stay irreducible over $F(\omega)$. For any $a \in F^\times \setminus F^{\times p}$, the polynomial $X^p - a$ is irreducible over F [**9**, Chapter VIII, Theorem 9.1.], hence also over $F(\omega)$. This shows that the canonical homomorphism $F^\times/F^{\times p} \longrightarrow F(\omega)^\times/F(\omega)^{\times p}$ is injective. The hypothesis therefore implies $|F(\omega)^\times/F(\omega)^{\times p}| \geqslant p^n$.

By Kummer theory (cf. [**9**, Chapter VIII, § 8], for example), we may choose elements $a_1, \dots, a_n \in F^\times$ such that $F(\omega, a_1^{1/p}, \dots, a_n^{1/p})$ is an extension of $F(\omega)$ of degree p^n. Note that $L(\omega) = F(\omega)((X_1))\dots((X_{m-n}))$.

We assume now that $m = 2n$. By [**22**, Proposition 2.10], the central simple $L(\omega)$-algebra $(a_1, X_1)_{L(\omega),\omega} \otimes_{L(\omega)} \cdots \otimes_{L(\omega)} (a_n, X_n)_{L(\omega),\omega}$ is a division algebra. Therefore, the element $\{a_1, X_1\}+\cdots+\{a_n, X_n\}$ in $K_2^{(p)}L(\omega)$ is not a sum of less than n symbols. It follows that in $K_2^{(p)}L$ this element cannot be a sum of less than n symbols either. This together with Theorem 1.1 shows that $\lambda_p(L) = n = [\frac{m}{2}] = [\frac{m+1}{2}]$.

Assume next that $m = 2n-1$ and that $p \neq 2$ or F is nonreal. The same argument as before shows that the element $\{a_1, X_1\} + \cdots + \{a_{n-1}, X_{n-1}\} \in K_2^{(p)}L$ is not a sum of less than $n-1 = [\frac{m}{2}]$ symbols. This together with (1.1) yields $\lambda_p(L) = [\frac{m}{2}]$.

In the remaining case, F is a real field, $p = 2$ and $m = 2n-1$. We may then choose the elements $a_1, \dots, a_n \in F^\times$ as above in such a way that $a_1, \dots, a_{n-1}$ become squares in some real closure E of F. The F-quaternion algebra $(-1,-1)_F$ does not split over the field $F(\sqrt{a_1}, \dots, \sqrt{a_{n-1}})$, which is contained in E. Hence, by [**22**, Proposition 2.10], the product $(-1,-1)_L \otimes_L (a_1, X_1)_L \otimes_L \cdots \otimes_L (a_{n-1}, X_{n-1})_L$ is a division algebra. So, by (4.1), the element $\{-1,-1\}+\{a_1, X_1\}+\cdots+\{a_{n-1}, X_{n-1}\}$ in $K_2^{(p)}L$ cannot be written as a sum of less than n symbols. By (1.1) we conclude that $\lambda_2(L) = n = [\frac{m+1}{2}]$. □

Corollary 4.3. *Suppose that* $\mathrm{char}(F) \neq 2$ *and that* $|F^\times/F^{\times 2}| = 2^n$. *Let* $m \geqslant 2n-1$ *and* $L = F((X_1))\dots((X_{m-n}))$. *Then* $|L^\times/L^{\times 2}| = 2^m$, $s(L) = s(F)$ *and*

$$\lambda_2(L) = \begin{cases} \left[\frac{m}{2}\right] & \text{if } F \text{ is nonreal,} \\ \left[\frac{m+1}{2}\right] & \text{otherwise.} \end{cases}$$

Proof. It is well-known that L has the same level as F; the rest of the statement is contained in the theorem. □

Examples 4.4. In each of the following situations, L is a nonreal field with

$$|L^\times/L^{\times 2}| = 2^m \quad \text{and} \quad \lambda_2(L) = \left[\tfrac{m}{2}\right].$$

(1) $L = \mathbb{C}((X_1))\dots((X_m))$, where $m \geqslant 0$. The level of L is 1 in this case.

(2) $L = \mathbb{F}_3((X_1))\dots((X_{m-1}))$, where $m \geqslant 1$ and where $\mathbb{F}_3$ denotes the field with three elements. In this case, $s(L) = 2$.

(3) $L = \mathbb{Q}_2((X_1))\dots((X_{m-3}))$, where $m \geqslant 5$ and where $\mathbb{Q}_2$ is the field of dyadic numbers. On the other hand, a direct but tedious calculation shows that one has $\lambda_2(F) = 1$ for any field F of level 4 with $|F^\times/F^{\times 2}| \leqslant 2^4$. (This can also be seen from the results in [**21**].)

The key idea in the proof of the following statement goes back to Nakayama [**17**].

Proposition 4.5. *Suppose that* F *contains a primitive* p*-th root of unity* ω. *Let* L *be an extension of* $F(X_0, \dots, X_m)$ *contained in* $F(X_0)((X_1))\dots((X_m))$. *There is a* *tensor product of m symbol algebras of degree p over L which is a division algebra. In particular, one has $\lambda_p(L) \geqslant m$.*

Proof. We choose distinct monic irreducible polynomials $f_1, \dots, f_m \in F[X_0]$. They represent $\mathbb{F}_p$-linearly independent classes in $F(X_0)^\times / F(X_0)^{\times p}$. Hence, [**22**, Proposition 2.10] implies that $(f_1, X_1)_{L,\omega} \otimes_L \cdots \otimes_L (f_m, X_m)_{L,\omega}$ is a central division algebra over $L = F(X_0)((X_1))\dots((X_m))$. It is now clear that the same holds if L is any subfield of $F(X_0)((X_1))\cdots((X_m))$ containing $F(X_0, \dots, X_m)$; in particular, $\lambda_p(L) \geqslant m$, by (4.1). □

In view of the inequality of the proposition, it seems natural to ask:

Question 4.6. *Let m be a positive integer, p a prime, and C an algebraically closed field with $\mathrm{char}(F) \neq p$. Let $F = C(X_0, \dots, X_m)$ be the rational function field in m variables over C. Do we have $\lambda_p(F) = m$?*

Assume now that $m = 1$. Already here, the problem is unsolved and very interesting. Showing that $\lambda_p(F) = 1$ would imply, via the Merkurjev–Suslin Theorem, that any central division algebra over $F = C(X_0, X_1)$ of exponent p is cyclic (equivalently, a symbol algebra of degree p). This is true at least for $p \leqslant 3$ (cf. [**1**]). By a very recent result of de Jong, any central division algebra of exponent p over F is of degree p (cf. [**4**]), but it is not presently known whether such an algebra is necessarily cyclic when $p \geqslant 5$.

An important and rather open problem in the study of symbol lengths is their behaviour under field extensions. First of all, it would be interesting to know whether $\lambda_p(L) < \lambda_p(F)$ is possible for a prime p and a finite extension L/F of degree prime to p (cf. [**7**, Conjecture 3] for $p = 2$). Furthermore, no upper bound is known for $\lambda_p(L)$ in terms of $\lambda_p(F)$ and the degree of the (finite) extension L/F.

To finish, let us indicate what can happen with respect to the latter problem in the case $p = 2$. In our examples, the base field F will have finitely many square classes. A far more systematic discussion of the behaviour of the 2-symbol length λ_2 under field extensions L/F of degree 2 and 3 can be found in [**16**].

Examples 4.7. (1) Let F_0 be the quadratic closure of $\mathbb{Q}$ and L_0/F_0 an extension of degree 3. For given $m \geqslant 0$, let $F = F_0((X_1))\dots((X_m))$ and $L = L_0((X_1))\dots((X_m))$. Then one has $[L : F] = [L_0 : F_0] = 3$ and it turns out that $\lambda_2(F) = \left[\frac{m}{2}\right]$, by (4.2), whereas $\lambda_2(L) = m$.

As for the last claim, one may argue as follows. By [**8**, Chapter VII, Appendix, Corollary 3], L_0 has infinitely many square classes. Let $a_1, \dots, a_m \in L_0^\times$ be $\mathbb{F}_2$-independent in $L_0^\times / L_0^{\times 2}$. Then [**22**, Proposition 2.10] shows that the L-algebra $(a_1, X_1)_L \otimes_L \cdots \otimes_L (a_n, X_n)_L$ is a division algebra. Using (4.1), it follows that $\lambda_2(L) \geqslant m$. On the other hand, a straightforward computation shows that any sum of $m+1$ symbols in $k_2 L$ can be written as a sum of an element defined over L_0 plus m symbols of the shape $\{*, X_i\}$, $1 \leqslant i \leqslant m$. However, $k_1 F_0 = 0$ and $[L_0 : F_0] = 3$ imply that $k_2 L_0 = 0$, by [**12**, Lemma 2], because $k_1 F_0 = 0$ and $[L_0 : F_0] = 3$. One conludes that $\lambda_2(L) = m$.

(2) Let F_0 be a nonreal field such that $|F_0^\times/F_0^{\times 2}| = 2^m$ for given $m \geqslant 1$, and such that $k_2F_0 = 0$. Let L_0/F_0 be a quadratic extension. Then it follows from [**8**, Chapter VII, Theorem 3.4.] that $|L_0^\times/L_0^{\times 2}| = 2^{2m-1}$. We put $F = F_0(\!(X_1)\!)\dots(\!(X_{n-m})\!)$ and $L = L_0(\!(X_1)\!)\dots(\!(X_{n-m})\!)$ for given $n \geqslant 3m-1$. Then $[L:F] = [L_0:F_0] = 2$, and (4.2) yields $\lambda_2(F) = [\frac{n}{2}]$ and $\lambda_2(L) = [\frac{n+m-1}{2}]$. In particular, for $n = 3m-1$ we get $\lambda_2(F) = [\frac{3m-1}{2}]$ and $\lambda_2(L) = 2m-1$, so $\lambda_2(L) = \frac{4\lambda_2(F)+(-1)^m}{3}$.

References

[1] M. Artin. Brauer-Severi varieties (notes by A. Verschoren), *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)* (ed. F. van Oystaeyen and A. Verschoren), 194–210, LMN 917, Springer, Berlin, 1982.

[2] K.J. Becher. On the Number of Square Classes of a Field of Finite Level. *Documenta Math.*, Extra Volume, Quadratic Forms LSU: 65–84, 2001.

[3] N. Bourbaki. *Éléments de mathématique. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 9: Formes sesquilinéaires et formes quadratiques.* Hermann, Paris, 1959.

[4] A.J. de Jong. The period-index problem for the Brauer group of an algebraic surface, *Preprint*, `http://www-math.mit.edu/~dejong/papers/perind.dvi`, 2002.

[5] D.W. Hoffmann. Dimensions of Anisotropic Indefinite Quadratic Forms, I. *Documenta Math.*, Extra Volume, Quadratic Forms LSU: 183–200, 2001.

[6] B. Kahn. Sommes de tenseurs décomposables. *Prépublication*, 1991.

[7] B. Kahn. Comparison of some field invariants. *J. Algebra*, 232(2):485–492, 2000.

[8] T.Y. Lam. *The algebraic theory of quadratic forms.* W. A. Benjamin, Inc., Reading, MA, 1973.

[9] S. Lang. *Algebra.* Addison-Wesley, Reading, MA, 2nd ed., 1984.

[10] H.W. Lenstra. K_2 of a global field consists of symbols. *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, 69–73. LNM 551, Springer, Berlin, 1976.

[11] A.S. Merkurjev. On the norm residue symbol of degree 2. *Dokl. Akad. Nauk SSSR*, 261(3):542–547, 1981.

[12] A.S. Merkurjev. Brauer groups of fields. *Comm. Algebra*, 11(22):2611–2624, 1983.

[13] A.S. Merkurjev. K_2 of fields and the Brauer group. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, 529–546. AMS, Providence, RI, 1986.

[14] A.S. Merkurjev and A.A. Suslin. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(5):1011–1046, 1135–1136, 1982. (Engl. transl.: *Math. USSR Isvestiya* 38(1992), 215–221.)

[15] J. Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1970.

[16] J. Mináč and A. Wadsworth, The u-invariant for algebraic extensions, *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 333–358, AMS, Providence, RI, 1995.

[17] T. Nakayama. Über die direkte Zerlegung einer Divisionsalgebra. *Jap. J. Math.*, 12:65–70, 1935.

[18] D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for Milnor's K-theory with applications to quadratic forms. *K-theory Preprint Archives*, `http://www.math.uiuc.edu/K-theory/0454/`, 2000.

[19] D.J. Saltman, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc., 12(1): 25–47, 1997. Correction in: J. Ramanujan Math. Soc., 13(2): 125–129, 1998.

[20] W. Scharlau. *Quadratic and Hermitian forms.* Grundlehren 270, Springer, Berlin, 1985.

[21] L. Szczepanik. Fields and quadratic form schemes with the index of radical not exceeding 16. *Ann. Math. Silesianae* 1(13):23–46, 1985.

[22] J.-P. Tignol. Algèbres indécomposables d'exposant premier. *Adv. in Math.*, 65(3):205–228, 1987.

[23] R. Ware. Galois groups of maximal p-extensions, *Trans. Amer. Math. Soc.*, 333(2):721–728, 1992.

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Karim Johannes Becher `becher@maths.ucd.ie`

Universität Konstanz
Fachbereich Mathematik und Statistik, D204
78457 Konstanz
Germany

Detlev W. Hoffmann `detlev.hoffmann@nottingham.ac.uk`

School of Mathematical Sciences
University of Nottingham
University Park
Nottingham NG7 2RD
UK