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NON-ABELIAN COHOMOLOGY VIA PARITY QUASI-COMPLEXES

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Abstract

The homotopy category of parity quasi-complexes is introduced. The homotopy structure is compatible with the nonabelian homology of parity quasi-complexes.

Parity contracting homotopies are defined, determining the parity free resolutions in a canonical way, enabling the nonabelian bar construction.

In this way, the even/odd grouping of the simplicial maps in the cocycle conditions of nonabelian cohomology is explained.

1. Introduction

The article develops the approach to nonabelian cohomology via canonical parity resolutions proposed by the author in [**11**].

The empirical even-odd grouping of the simplicial maps yielding parity differentials is obtained from a homotopy structure on the category of parity quasi-complexes (PQC) on some concrete, possibly non-additive, category.

The homotopy structure is compatible with the categorical structure and the parity homology functor descends to the corresponding homotopy category. Canonical resolutions are constructed, by adapting classical techniques from relative homological algebra. Namely, parity contracting homotopies (PCH) are introduced and the canonical PCH corresponding to the free object determines the parity differentials, as usual.

Section 2 recalls some of the main approaches to non-abelian cohomology, with emphasis on those using resolutions, attempting to define an analog of the bar construction.

Section 3 recalls parity quasi-complexes and the corresponding homology functor (Definitions 3.1, 3.2)). The homotopy structure relevant to non-abelian cohomology is defined (3.3) and proved compatible with the categorical structure, yielding the associated homotopy category (Theorem 3.1). Finally, the compatibility with non-abelian homology is proved (Corollary 3.2). Specializing the definition of homotopy, parity contracting homotopies are introduced (Definition 3.5).

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Section 4 introduces the non-abelian bar resolution (Definition 4.1) as a canonical parity quasi-complex associated to a contracting parity homotopy (Theorem 4.1).

The article concludes mentioning some present limitations of the proposed approach, discussing future developments.

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2. Trends in non-abelian cohomology

The study of group extensions with given abstract kernel was completed by Eilenberg-Mac Lane in 1946, settling the question in dimension $n \leqslant 2$ [**5**].

Previous attempts to find the right definition of non-abelian cohomology in higher dimensions (n-step extensions), are of two types. One approach looks for a non abelian analog of the bar construction, in order to define derived functors using resolutions. The other approach uses "objects" representing cocycles via monodromy.

As an example of the complexity of the defining equations for cocycle and coboundary conditions within the first approach, we will mention those in dimension three, in two versions. One version is due to P. Dedecker, 1969 [**4**]. A non-abelian cochain is actually a pair of classical cochains (λ, k). Additional maps $(\alpha, \phi, h, ...)$ are involved in the condition for two non-abelian cocycles to be cohomologous:

$$\lambda'(s,t) = \alpha(s) \cdot^{\phi(s)} \alpha(t) \cdot \lambda(s,t) \cdot^{\rho h(s,t)} \alpha(st)^{-1} \cdot \rho a(s,t)^{-1}$$

$$\begin{aligned} k'(s,t,u) = &{}^{\alpha(s)\phi(s)}a(t,u) \cdot^{\phi(s)}_{\alpha(s)} L(t,u) \cdot^{[{}^{\phi(s)}\rho h(t,u)]} a(s,tu) \cdot \\ &k(s,t,u) \cdot^{\rho h(s,t)} a(st,u)^{-1} \cdot a(s,t)^{-1} \end{aligned}$$

The other version we will mention is due to L. Breen, 1992 [**1**]. A non-abelian cochain is now a quadruple (x, z, λ, a) and the condition for two cocycles $a(k,l,m)$ and $a'(k,l,m)$ to be cohomologous is (loc. cit. p.488):

$$\begin{aligned} &{}^{z(k)^{x'(k)} z(l)^{x'(k)x'(l)} z(m)} a'(k,l,m)^{z(k)} ({}^{x'(k)} a(l,m)) \{\lambda(k), z(l,m)\}^{x(k)} {}^{z(l,m)} a(k,lm) \\ &= {}^{z(k)^{x'(k)} z(l)} \{\lambda'(k,l), z(m)\}^{-1} a(k,l)^{z(k,l)} a(kl,m) a(k,l,m) \end{aligned}$$

It is a mixture of classical 1, 2 and 3 cochains. We will not go into details regarding these conditions.

The other approach to non-abelian cohomology involves a change of point of view, considering objects and concepts representing cocycles.

As a typical example we mention the notions of torsors, gerbes and liens studied in the work of Giraud, 1971 [**8**], and more generally the technique of descent, Grothendieck, 1959.

We may think of a G-torsor over a space X as being a principal G-bundle with a discrete structure group G. Closely related notions are local systems and principal bundles with flat connections.

A gerbe may be thought of as a categorification of the notion of a torsor. Now the "base" is a site (category with a Grothendieck topology) and the "total space" is a fibered category, with fibers groupoids verifying some additional conditions.

Our approach belongs to the first type and exploits the "parity" features exhibited by the cocycle and coboundary equations in low dimensions. This empirical fact was generalized in [**11**], being thought of as an incarnation of the key idea of parity, present for example in super-symmetry, duality, completions etc.. Non-abelian cohomology based on parity quasi-complexes may be thought of as implementing the sphere spectrum using the language of cobordism theory. It is a shadow of n-category theory.

Although this attempt was initiated independently in [**11**] and then used to investigate the relation with the cohomology of monoidal categories [**10**] via categorification [**13**], the idea that parity is important in the description of non abelian cohomology was known, including its relevance to n-categories. Street introduced parity complexes in [**17**] and studied the algebra of simplices with applications to category theory [**18**]. Davidov defined parity quasi-complexes of natural transformations to characterize cohomology of monoidal categories [**3**] (see also [**16**]).

We will "explain" the parity structure of the parity differentials as emerging from a *parity* contracting homotopy, which determines the non abelian bar differentials in the same way the contracting homotopy of a free resolution determines the differential [**15**] (Theorem 7.2, p.271).

3. Homotopy category of parity quasi-complexes

We begin with a recollaction on parity quasi-complexes, as introduced in [**11**]. Then a homotopy structure will be introduced in the corresponding category. It is compatible with the categorical structure and the NA-homology functor, which descends to the homotopy category of parity quasi-complexes, as first noted in [**12**].

3.1. Parity quasi-complexes and nonabelian homology functor

Fix a concrete category $\mathcal{C}$ for simplicity (strict monoidal, with finite limits and colimits), so that we can talk about group objects, equalizers and coequalizers. The category of its group objects is denoted by $\mathcal{G}$. A typical example is the pair of categories of sets and groups.

Definition 3.1. Let $Ch^{\pm}(\mathcal{C})$ be the category of sequences $\{\partial_n^+, \partial_n^-\}_{n\in\mathbb{Z}}$ of pairs of morphisms of $\mathcal{C}$, called *parity quasicomplexes* (PQC):

$$(C, \partial^{\pm}) : \qquad \begin{tikzcd} \dots C_n \arrow[r, bend left, "\partial_n^+"] \arrow[r, bend right, "\partial_n^-"'] \arrow[r, phantom, "\partial_n \Uparrow"] & C_{n-1} \dots \end{tikzcd} \tag{1}$$

A *morphism of PQC* is a map $f \in Hom_0(X, Y)$, i.e. a family of morphisms $f_n : X_n \to Y_n$, commuting with the parity differentials $\partial^{\pm}$.

A PQC $(X, \partial^{\pm})$ should rather be thought of as sequence of 2-morphisms $\partial_n : \partial_n^- \to \partial_n^+$, being denoted also by (X, ∂). The relation to ω-categories steaming from interpreting the parity differentials as source and target maps, will be investigated elsewhere.

We will be interested in the homology of monoids of $\mathcal{C}$. The corresponding category will be denoted by $\mathcal{G}$. Although non commutative, the additive notation will be used for the multiplication of an object of $\mathcal{G}$.

Definition 3.2. The n^{th} *NA-homology space* of a PQC of monoids in $\mathcal{C}$ is defined as the pointed set of *cycles* $Z_n(X)$, i.e. the equalizer of ∂_n^+ and ∂_n^-, modulo the equivalence relation defined below:

$$H_n^N(X) = equal(\partial^+, \partial^-)/\sim .$$

Two elements $x, y \in X_n$ are *homologous*, denoted by $x \overset{c}{\sim} y$ (or simply $x \sim y$), iff there exists $c \in X_{n+1}$ such that $\partial^+ c + x = y + \partial^- c$:

$$\begin{tikzcd} \cdot \arrow[r, "\partial^- c"] \arrow[d, "x"'] & \cdot \arrow[d, "y"] \\ \cdot \arrow[r, "\partial^+ c"'] & \cdot \end{tikzcd}$$

A PQC is called *exact* if it has trivial homology.

Since a morphism of PQCs commutes with both differentials, it induces as usual a morphism on homology.

Proposition 3.1. *A morphism $f : (X, \partial) \to (Y, \partial)$ of PQCs induces in a functorial way a morphism of pointed sets between the corresponding NA-homology spaces: $H^N(f) : H^N(X) \to H^N(Y)$.*

A morphism f of PQC is a *quasi-isomorphism* if it induces in homology an isomorphism $H(f)$ in the underlying category $\mathcal{C}$.

Parity quasicomplexes exhibit "curvature", i.e. in general $\partial^2 \neq 0$. The usual notion of contracting homotopy is not sufficient, as the following lemma demonstrates.

Lemma 3.1. *An augmented sequence of groups (X, ∂), having a contracting homotopy s in $Sets_*$ with image generating the groups X_i, is a complex: $\partial^2 = 0$.*

Proof. The usual proof [**15**], p.268, yields in the nonabelian case:

$$\partial\partial s + \partial = \partial + s\partial\partial.$$

Since it is a positive sequence, there inductively follows $\partial^2 s = 0$. Under the assumtion that the image of s generates the corresponding X_i, $\partial^2 = 0$ follows, concluding the proof. ☐

The notion of homotopy appropriate for PQC should not "mix the positive and negative boundaries": it should be a $\mathbb{Z}_2$-graded version.

3.2. The homotopy structure

A useful homotopy structure should behave well with respect to composition of morphisms of PQCs and functors, yielding a 2-categorical structure. Moreover, in order to define derived functors using resolutions, one needs that homotopic maps induce the same map in homology, so that the homology functor descends to the corresponding homotopy category.

To achieve this in the context of parity quasi-complexes, it is natural to extend the principle of "separation of signs" to homotopies and define a $\mathbb{Z}_2$-graded analog.

Definition 3.3. Let $f, g : (X, \partial^{\pm}) \to (Y, \partial^{\pm})$ be morphisms of PQCs of monoids in $\mathcal{C}$. f is *parity homotopic* to g, denoted $f \xrightarrow{s} g$, iff for all n X_n is generated by the set of elements $x \in E_n$ satisfying the following relations:

$$f(x) + s(\partial^-(x)) = \partial^+(s(x)) + g(x) \tag{2}$$

$$s(\partial^+(x)) = \partial^-(s(x)). \tag{3}$$

Remark 3.1. The above addition is the "natural addition" of morphisms ($+_Y \circ (f \otimes g) \circ \Delta_X$). The "free addition" through adjunction ([**7**], p.211), consisting in extending uniquely a map defined on free generators, will be used to define the parity differentials. They satisfy the parity homotopy relations only on generators, leading to the courser relation defined above. Note that this "technical" issue is typical in the noncommutative case, e.g. within the theory of distributively generated near-rings and pseudohomomorphisms ([**6, 7**], [**14**] p.313).

There is a natural "vertical" composition of homotopies.

Lemma 3.2. *Parity homotopy is transitive.*

Proof. If

$$f + s\partial^- = \partial^+ s + g, \quad s\partial^+ = \partial^- s$$

and

$$g + t\partial^- = \partial^+ t + h, \quad t\partial^+ = \partial^- t,$$

then

$$f + (s+t)\partial^- = \partial^+ s + g + t\partial^- = \partial^+(s+t) + h$$

and

$$(s+t)\partial^+ = \partial^-(s+t).$$

□

Parity homotopy is distributive with respect to composition of morphisms.

Lemma 3.3. *If $f, g : X \to Y$ are parity homotopic $s : f \to g$ and $h : Y \to Z$ ($h : Z \to X$), then $hs : hf \to hg$ ($sh : fh \to gh$) is a parity homotopy between hf and hg (fh and gh).*

Proof. We will prove left distributivity. Left multiply equations 2 and 3 by h. Since h commutes with the parity differentials, the statement follows. □

The compatibility with composition of PQC morphisms follows.

Corollary 3.1. *Let $f, g : X \to Y$ and $f', g' : Y \to Z$ be morphisms of PQCs. If $f \to g$ and $f' \to g'$ then $f'f \to g'g$.*

Proof. The parity homotopy relation $\sim$ is left and right distributive with respect to composition of PQC morphisms. Then $f'f \sim g'f \sim g'g$. □

The above facts can be summarized as the following theorem.

Theorem 3.1. *$Ch^{\pm}(\mathcal{C})$ has a natural structure of a 2-category.*

Definition 3.4. Its *1-truncation*, consisting of the same objects but equivalence classes of morphisms modulo homotopy, is called the associated *homotopy category* ([**9**], p.126).

The compatibility between the non-abelian homology functor and the parity homotopy structure is investigated next.

Lemma 3.4. *Parity homotopic morphisms $f \to g : X \to Y$ induce the same map in homology: $H^N(f) = H^N(g)$.*

Proof. If x is a cycle, i.e. $\partial^- x = \partial^+ x$, and $f \to g$, then:

$$\partial^- sx = s\partial^+ x = s\partial^- x,$$

so that:

$$f(x) + \partial^- sx = \partial^+ sx + g(x)$$

i.e. $f(x)$ and $g(x)$ are homologous cycles in Y. □

This allows us to factor the homology functor through the canonical projection onto the homotopy category.

Corollary 3.2. *The non-abelian homology functor H^N induces a canonical homology functor on the homotopy category of PQCs over $\mathcal{G}$.*

We will investigate next resolutions of objects in $\mathcal{G}$.

3.3. Parity contracting homotopies and resolutions

Consider the parity analog of contracting homotopies.

Definition 3.5. A *parity contracting homotopy* (PCH) is a map $s \in End_1(X)$, such that $0 \xrightarrow{s} id_X$:

$$s\partial^- = \partial^+ s + id_X \tag{4}$$

$$s\partial^+ = \partial^- s \tag{5}$$

Corollary 3.3. *A PQC possessing a parity contracting homotopy is exact, i.e. has trivial homology.*

Proof. Using Lemma 3.4, yields $0 = H^N(0_X) = H^N(id_X) = id_{H^N(X)}$. □

Augmentations are defined in a similar way as for complexes.

Definition 3.6. An augmented PQC $\epsilon : X \to G$ is a non-negative PQC X together with a morphism $\epsilon : X_0 \to G$.

An augmented PQC $\epsilon : X \to G$ is called a *PQC-resolution of G* if it is an exact PQC, when extending X as a PQC by defining $\partial_0^+ = \epsilon$ and $\partial_0^- = 0$.

In an exact PQC, every cycle is a boundary having a canonical decomposition into a "positive" and a "negative " part (compare [**17**], p.317).

Proposition 3.2. *If x is a cycle of an exact PQC, then it decomposes canonically into a positive and a negative boundary:*

$$x = \partial^+(-sx) - \partial^-(-s)x.$$

Proof. As before, $\partial^- x = \partial^+ x$ in combination with (4):

$$s\partial^- x = \partial^+ sx + x, \quad s\partial^+ x = \partial^- sx,$$

yields $\partial^- sx = \partial^+ sx + x$. Rearranging the terms, the above relation follows. □

We will prove in section 4 that the noncommutative bar resolution has a parity contracting homotopy.

4. Non-abelian bar construction

The results from relative homological algebra are adapted to the noncommutative case, following [**15**]. The major difference is that the noncommutative case requires a $\mathbb{Z}_2$-grading of the usual bar resolution, i.e. a "separation" of the positive and negative terms. The additive notation is used for clarity [**15**], p.114.

The results are formulated in the context of the category of G-groups $\mathcal{G}$ [**6**], consisting of groups with a group action and equivariant group homomorphisms. The free constructions specified below are similar to those of [**6**] for the category of $G - groups$.

Definition 4.1. Let G be a group.

The *nonabelian bar resolution* of G is the PQC $(B(G), \partial^\pm)$:

$$\cdots \qquad \begin{tikzcd} B_2 \arrow[r, bend left, "\partial_2^+"] \arrow[r, bend right, "\partial_2^-"'] & B_1 \arrow[l, dotted, "e_1"] \arrow[r, bend left, "\partial_1^+"] \arrow[r, bend right, "\partial_1^-"'] & B_0 \arrow[l, dotted, "e_0"] \arrow[r, bend left, "\epsilon"] \arrow[r, bend right, "0"'] & \mathbb{Z} \arrow[l, dotted, "e_{-1}"] \arrow[r] & 0 \end{tikzcd} \tag{6}$$

of G-groups $B_n(G)$ and G-equivariant group homomorphisms $\partial_n^\pm$, together with the augmentation ϵ and the group homomorphisms e_n, defined as follows.

For positive n, $B_n = \mathcal{F}(\mathcal{U}(G^n))$ is the free G-group with (G-)generators $[x_1|...|x_n]$ all n-tuples of elements $x_1, ..., x_n$ of G. Operation on a generator with an element $x \in G$ yields an element $x[x_1|...|x_n]$ in B_n, so B_n may be described as the free group generated by all $x[x_1|...|x_n]$. Finally B_0 is the free G-group on one generator $[\ ]$, so it is isomorphic to $\mathbb{Z}G$, and $B_{-1} = \mathbb{Z}$. To obtain a parity quasi-complex, assume $B_n = 1$ for negative n.

The group homomorphisms e_n are defined by:

$$e_{-1}(1) = -[\ ]\ , \quad e_n(x[x_1|...|x_n]) = -[x|x_1|...|x_n], \quad n \geqslant 0.$$

Similar to the abelian case ([**15**], theorem 6.3, p.268), the PQC structure is determined, if we require e_n to be a parity contracting homotopy. The negative sign included in the definition of e_n will reverse the order of the odd simplicial maps in the formula providing negative differential ∂^-.

Theorem 4.1. *There is a unique structure of a parity quasi-complex* $(B, \partial^{\pm})$ *on the graded object* B*, admitting* e *as a parity contracting homotopy.*

The parity differentials satisfy the following structural equations on group generators:

$$\partial_n^+ = \overrightarrow{\sum_{\substack{0 \leqslant i \leqslant n \\ i \ even}}} \partial_n^i \qquad \partial_n^- = \overleftarrow{\sum_{\substack{0 \leqslant i \leqslant n \\ i \ odd}}} \partial_n^i \tag{7}$$

Proof. Since the image of e_i generates B_{i+1}, the relations 4 inductively define $\partial^{\pm}$.

To prove the stated formulas, note that on group generators, applying e after a simplicial map ∂^i yields $\partial^{i+1}s$:

$$\partial_{n+1}^{k+1} e_n = e_{n-1}\partial_n^k, k \geqslant 0, \qquad \partial_{n+1}^0 e_n = -id_n.$$

Then, on G-group generators, we have:

$$\partial_{n+1}^+([x, y_1, ...y_n]) = -\partial_{n+1}^+ e_n(x[y]) = -[e_{n-1}\partial_n^- - id_n(x[y])]$$

and

$$\begin{aligned} e_{n-1}\partial_n^-(x[y]) &= ... + e\partial_n^3(x[y]) + e\partial_n^1(x[y]) && (8) \\ &= ... + \partial_n^4 e(x[y]) + \partial_n^2 e(x[y]) && (9) \\ &= -(\partial_n^2 + \partial_n^4 + ...)[x, y]. && (10) \end{aligned}$$

Since $id_n = \partial_n^0 e_{n-1}$, the formula for ∂_{n+1}^+ is established.

The computations for ∂_{n+1}^- are similar, and will be omitted. □

The above G-equivariant group homomorphisms $\partial^{\pm}$ (compare [**11**], p.6), are expressed in terms of the usual simplicial maps defined on G-generators by:

$$\begin{aligned} &\partial_n^0[x_1|...|x_n] = x_1[x_2|...|x_n], \qquad \partial_n^n[x_1|...|x_n] = [x_1|...|x_{n-1}] && (11) \\ &\partial_n^i[x_1|...|x_n] = [x_1|...|x_i x_{i+1}|...|x_n], \ i = 1, ..., n-1 \end{aligned}$$

In particular $s_0(x[\,]) = [x]$ and:

$$\partial_1^+[x] = x[\,], \quad \partial_1^-[x] = [\,], \qquad \partial_2^+[x|y] = x[y] + [x], \quad \partial_2^-[x|y] = [xy]$$

Definition 4.2. The G-equivariant group homomorphism defined on generators by:

$$\partial_n([x_1, ..., x_n]) = \partial_n^+([x_1, ..., x_n]) - \partial_n^-([x_1, ..., x_n]),$$

will be called the differential of the PQC $(X, \partial^{\pm})$.

Since it possesses a parity contracting homotopy, the nonabelian bar construction is a resolution of the group G.

5. Conclusions and further developments

A homotopy structure was defined for the category of PQC of monoids of a concrete category $\mathcal{C}$ with equalizers. It was proved to be compatible with the categorical structure, with the nonabelian homology functor and with contracting homotopies.

A canonical PQC resolution for groups was constructed, as a consequence of the existence of a natural parity contracting homotopy. In this way enough free resolutions exist.

At this point we should recall that the homology of PQC generalizes the well-known cycle and cohomologous equations in low dimensions $0 \leqslant n \leqslant 2$. All that is left in order to establish a nonabelian cohomology for G-groups, is a Comparison Theorem for free or projective resolutions (see [**15**], p.87). This issue will not be discussed within this article, except for mentioning that it is natural to expect that PQC must satisfy an additional condition in order to enable such a result, e. g. being "complexes" in some sense. The obvious requirement that parity differentials be class functions relative to the homology equivalence relation is apparently too strong a requirement. This requirement is not satisfied by the nonabelian bar resolution, even as in low dimensions.

This difficulty seams to reside in the limitations of the bar construction itself, which needs to be generalized to include additional "separators" as in [**2**] (see Examples p.4116). This shows the need for a categorical interpretation in the spirit of [**18**], the above generalization of the bar construction [**2**] being a shadow of a construction within the framework of n-categories.

Finally, we will only claim herein that, the above direction of developing the present approach will yield applications to TQFTs obtained from state-sum constructions, which would also provide a testing ground for the development of the theory (within a future article).

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