Homology, Homotopy and Applications, vol.6(1), 2004, pp.109–135

ON THE HOMOTOPY TYPE OF A CHAIN ALGEBRA

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(*communicated by James Stasheff*)

Abstract

Let R be a P.I.D and let A be a dga over R. It is well-known that the graded homology modules $H_*(A)$ and $Tor_*^A(R,R)$ alone do not suffice (in general) to determine the homotopy type of the dga A. J.H. Baues had built a more precise invariant, the "certain" exact sequence of Whitehead associated with A. Whitehead had built it for CW-complexes. In this work we explore this sequence to show how it can be used to classify the homotopy types of A.

1. Introduction

The classification of homotopy types is a classical and fundamental task in homotopy theory. There are only a few explicit results in the literature (see [**4**], [**6**], [**7**], [**8**], [**9**], [**12**], [**17**]) within the context of finite polyhedra and (see [**4**], [**5**], [**6**], [**11**]) within the context of differential graded algebras and differential graded Lie algebras. Recall that the existence of the models of Adams-Hilton [**1**], Anick [**2**], [**3**] and Quillen [**16**] justify the geometrical interest of these types of algebras.

Let R be a principal ideal domain $P.I.D$. We denote by $\mathbf{DGA}_*(\mathbf{flat})$ the category of R-flat, differential, graded, associative, augmented and connected algebras (dga). Recall that $\mathbf{DGA}_*(\mathbf{flat})$ is a cofibration category [**5, 6**], so an object cylinder is well defined in this category; hence we can define the notion of homotopy.

In [**18**] J.H.C. Whitehead has introduced a "certain" exact sequence associated with a simply connected CW-complex and derived the classification of 4-dimensional simply connected CW-complexes. After these results J.H. Baues proved, in [**5**], that the Whitehead sequence exists also for a dga A and showed that this sequence can establish a classification of the homotopy types of 4-dimensional simply connected dgas.

Let $V=(V_i)_{i\geqslant 0}$ be a graded module, with $V_0=0$. The tensor algebra $T(V)$ is the graded algebra whose underlying graded module is the graded sum $\underset{n\geqslant 0}{\oplus} V^{\otimes n}$ (here $V^{\otimes 0}=0$). We say that a dga A is free if, forgetting differentials, one has

The author gratefully acknowledges the referee for helpful comments and corrections. The author also would like to thank Professor James Stasheff for his help regarding the publication of this paper.
Received July 7, 2003, revised March 3, 2004; published on April 5, 2004.
2000 Mathematics Subject Classification: 55Q15, 55U40.
Key words and phrases: Differential graded algebra, Whitehead exact sequence, Detecting functor, Homotopy type.

$A \cong T(V)$ for some V. In this case the differential on A is determined by its restriction $V \to T(V)$ and the graded module of indecompsable of A is isomorphic to V.
A dga-morphism $f : A \to B$ is called a quasi-isomorphism if the induced morphism $H_*(f) : H_*(A) \to H_*(B)$ is an isomorphism.

In [**5**] Baues showed that for each object A of $\mathbf{DGA}_*(\mathbf{flat})$ there exists a quasi-isomorphism $(T(V), \partial) \to A$. The tensor algebra $(T(V), \partial)$ is called a model of A. It is not unique up to isomorphism but it is unique up to homotopy.

Let $(T(V), \partial$ be a model of A, then the Whitehead exact sequence associated with A is the following long exact sequence:

$$\cdots \to Tor^A_{n+2}(R,R) \xrightarrow{b_{n+2}} \Gamma_n \longrightarrow H_n(A) \longrightarrow Tor^A_{n+1}(R,R) \xrightarrow{b_{n+1}} \cdots$$

Recall that $Tor^A_*(R,R) = H_*(s^{-1}V, d)$, where s^{-1} is the desuspension graded homomorphism and where d is the linear part of the differential ∂ (see [**4, 5**]).
Recall also that Γ_n is defined by setting:

$$\Gamma_n = \ker(H_n(T(V_{\leqslant n})) \longrightarrow V_n).$$

In this paper we study the problem of the classification of homotopy types in the category $\mathbf{DGA}_*(\mathbf{flat})$. How we can recognize the homotopy type of an object in $\mathbf{DGA}_*(\mathbf{flat})$ and how we can compute the number of homotopy types of a certain object in $\mathbf{DGA}_*(\mathbf{flat})$ satisfying some given data?
Our tool to address this problem is the Whitehead exact sequence. Indeed this sequence can be taken apart to give the following characteristic elements which we call the Γ-system associated with A:
- a graded module $Tor^A_{n\geqslant 2}(R,R)$,
- a family of homomorphisms $(b_{n+2})_{n\geqslant 2}$, where $b_{n+2} : Tor^A_{n+2}(R,R) \to \Gamma_n$,
- a family of extensions $\big(i_*(\pi_n) \in \mathrm{Ext}(Tor^A_{n+1}(R,R), \operatorname{coker} b_{n+2})\big)_{n\geqslant 2}$, where $\pi_n \in \mathrm{Ext}(\ker_{n+1}, \mathrm{coker b_{n+2}})$ denotes the extension represented by the short exact sequence $\operatorname{coker} b_{n+2} \rightarrowtail H_n(A) \twoheadrightarrow \ker b_{n+1}$ and where $\ker b_{n+1} \overset{i}{\hookrightarrow} \mathrm{Tor}^A_{n+1}(R,R)$.

We start by defining an algebraic category denoted $\mathbf{\Gamma}$. Roughly speaking each object of this category is the Γ-system associated with a certain free dga. Then we define a surjective function:

$$F : \mathbf{ObDGA}_*(\mathbf{flat})/_{\simeq} \to \mathbf{Ob\Gamma},$$

by using the Γ-system associated with A. But unfortunately the function F is not a functor since it is not natural with respect to dga morphisms. Therefore, to make this correspondence natural, we introduce the subcategory $\mathbf{\Gamma DGA}_*(\mathbf{flat})$ of $\mathbf{DGA}_*(\mathbf{flat})$ whose morphisms are the dga morphisms α satisfying the condition (3.3) and thereby $\mathbf{F}$ becomes a functor $\mathbf{DGA}_*(\mathbf{flat})/_{\simeq} \to \mathbf{\Gamma}$ which satisfies the properties of a "detecting functor", a notion introduced by H.J. Baues in [**5**], which implies that the functor F induces the following results:

Theorem 1. *Two objects A and B in $\mathbf{DGA}_*(\mathbf{flat})$ have the same homotopy type if their Γ-systems are isomorphic.*

Theorem 2. *Homotopy types in the category* $\mathbf{\Gamma DGA_*(flat)}/_{\simeq}$ *are in bijection with the classes of isomorphisms of objects in the category* $\mathbf{\Gamma}$.

Moreover, we have:

Theorem 3. *Two objects* A *and* B *in* $\mathbf{\Gamma DGA_*(flat)}$ *have the same homotopy type if and only if their Whitehead exact sequences are isomorphic.*

If the condition (4.1) is satisfied, we can identify the category $\mathbf{DGA_*(flat)}$ with its subcategory $\mathbf{\Gamma DGA_*(flat)}$. Therefore, under this condition, we conclude that theorems 2 and 3 are also true in $\mathbf{DGA_*(flat)}$.

The functor F is surjective on objects then for every Γ-system there exists a free dga $(T(V), \partial)$ such that $F((T(V), \partial)) = \Gamma$-system. When R is a field of any characteristic we show that the dga $(T(V), \partial)$ coincides with the minimal model defined in [**10**]. So the notion of Γ-system is probably the best substitute for the minimal model in the case of differential algebras over a P.I.D R, rather than over a field.

In the last section we treat a particular case where the definition of the category $\mathbf{\Gamma}$ may be simplified. Indeed when the dga A is an object of $\mathbf{\Gamma DGA_n^{3n+2}(flat)}$ (the subcategory of $\mathbf{\Gamma DGA_*(flat)}$ of which the objects are those satisfying the relations $Tor_i^A(R, R) = 0$ for $i \leqslant n$ and $i \geqslant 3n + 3$) we can denote the graded module Γ_* simply by the graded module $Tor_*^A(R, R)$. Then we define an algebraic category $\mathbf{\Gamma_n^{3n+2}}$ and a functor:

$$F_n^{3n+2} : \mathbf{\Gamma DGA_n^{3n+2}(flat)}/_{\simeq} \to \mathbf{\Gamma_n^{3n+2}}$$

and we show that F_n^{3n+2} is also a "detecting functor". Hence we derive the following homotopy classification theorem in $\mathbf{\Gamma DGA_n^{3n+2}(flat)}$:

Theorem 4. *Homotopy types of objects in* $\mathbf{\Gamma DGA_n^{3n+2}(flat)}$ *are in bijection with the proper equivalence classes (see definition 15) of tuples* $(b_{3n+2}, \pi_{3n}, ..., b_{2n+2}, \pi_{2n})$ *where* $b_{k+2} \in Hom(H_{k+2}, \Gamma_k)$ *and where* $\pi_k \in \mathrm{Ext}(H_{k+1}, \mathrm{coker}\, b_{k+2})$ *for each* $k \leqslant 3n$.

This article is organized as follows. In section 2, the Whitehead exact sequence associated with a dga is defined and its essential properties are given. Section 3 is devoted to the Γ-homotopy systems of order n, a notion needed to define the category Γ and to introduce the functor F and therefore to announce the main results in section 4. We conclude with some geometric applications and examples in section 5.

2. Whitehead exact sequence associated with a dga

In this section we give the definition and the essential properties of the Whitehead exact sequence associated to an associative differential graded algebra. Recall that Baues has constructed this sequence for dgas in [**5**] and he proved that it is an exact sequence.

Let A be a dga and let $(T(V),\partial)$ be a model of A. Form the following long exact sequence:

$$\cdots \to H_n(T(V_{\leqslant n})) \xrightarrow{j_n} V_n \xrightarrow{\beta_n} H_{n-1}(T(V_{\leqslant n-1})) \to \cdots$$

where the connecting β_n is defined by:

$$\beta_n(v_n) = \overline{\partial(v_n)}, \tag{2.1}$$

where $\overline{\partial(v_n)} \in H_{n-1}(T(V_{\leqslant n-1}))$ is the homology class of the $(n-1)$-cycle $\partial(v_n) \in T_{n-1}(V_{\leqslant n-1})$, we define the graded module $(\Gamma_n)_{n\geqslant 2}$ by setting:

$$\Gamma_n = \ker(H_n(T(V_{\leqslant n})) \xrightarrow{j_n} V_n). \tag{2.2}$$

Recall that the linear part d of the differential ∂ is given by:

$$d_{n+1} = j_{n-1}\circ\beta_n \quad \forall n \geqslant 2. \tag{2.3}$$

The Whitehead exact sequence associated with the dga A is by definition (see [**5**]) the following long exact sequence:

$$\cdots \to Tor^A_{n+2}(R,R) \xrightarrow{b_{n+2}} \Gamma_n \longrightarrow H_n(A) \longrightarrow Tor^A_{n+1}(R,R) \xrightarrow{b_{n+1}} \cdots$$

where $b_{n+2}(\overline{z}) = \beta_{n+1}(z)$.

Remark 1. *Since V_n is free, for each $n \geqslant 2$, from the short exact sequence:*

$$\Gamma_n \rightarrowtail H_n(T(V_{\leqslant n})) \twoheadrightarrow \ker\beta_n \subset V_n,$$

we deduce that:

$$H_n(T(V_{\leqslant n})) \cong \Gamma_n \oplus \ker\beta_n, \tag{2.4}$$

and in terms of the differential $d_{n+2} : V_{n+1} \to V_n$ we deduce the following decomposition:

$$V_{n+1} \cong (\operatorname{Im} d_{n+2})' \oplus \ker d_{n+2}, \tag{2.5}$$

where $(\operatorname{Im} d_{n+2})' \subset V_{n+1}$ is a copy of $\operatorname{Im} d_{n+2} \subset V_n$. Therefore the short exact sequence:

$$(\operatorname{Im} d_{n+2})' \overset{d_{n+1}}{\rightarrowtail} \ker d_{n+1} \twoheadrightarrow Tor^A_{n+1}(R,R),$$

is a free resolution of the module $Tor^A_{n+1}(R,R)$.

- Since $d_{n+2}\left((\operatorname{Im} d_{n+2})'\right) \subset \ker\beta_n$ then the short exact sequence:

$$(\operatorname{Im} d_{n+2})' \overset{d_{n+2}}{\rightarrowtail} \ker\beta_n \twoheadrightarrow \ker b_{n+1}, \tag{2.6}$$

is a free resolution of the sub-module $\ker b_{n+1} \subset Tor^A_{n+1}(R,R)$.

- According to the relations (2.5) and (2.4), if $(z_{n+2,\sigma})_{\sigma\in\sum}$ and $(l_{n+2,\sigma'})_{\sigma'\in\sum'}$ denote respectively the basis of the free modules $\ker d_{n+2}$ and $(\operatorname{Im} d_{n+2})'$, the formula (2.1) can be written :

$$\beta_{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) = b_{n+2}(\overline{z_{n+2,\sigma}}) + \varphi_n(l_{n+2,\sigma'}) + d_{n+2}(l_{n+2,\sigma'}), \tag{2.7}$$

where $\varphi_n : (\operatorname{Im} d_{n+2})' \to \Gamma_n$ is a homomorphism given by the differential in ∂.

Proposition 1. *Let A be a dga, then for each $n \geqslant 2$, we have:*

$$H_n(A) \cong \frac{\operatorname{coker} b_{n+2} \oplus \ker \beta_n}{\operatorname{Im}\overline{\varphi_n} \oplus \operatorname{Im} d_{n+2}}, \tag{2.8}$$

where:

$$\overline{\varphi_n} : (\operatorname{Im} d_{n+2})' \xrightarrow{\varphi_n} \Gamma_n \twoheadrightarrow \operatorname{coker} b_{n+2}.$$

Proof. Let $(T(V), \partial)$ be a model of A. From the following exact sequence:

$$V_{n+1} \xrightarrow{\beta_{n+1}} H_n(T(V_{\leqslant n})) \to H_n(T(V_{\leqslant n+1})) \to 0,$$

and the relation (2.4) we get:

$$H_n(T(V_{\leqslant n+1})) \cong \frac{\Gamma_n \oplus \ker \beta_n}{\operatorname{Im} \beta_{n+1}}. \tag{2.9}$$

Substituting the relation (2.7) in the formula (2.9) we get:

$$H_n(T(V_{\leqslant n+1})) \cong \frac{\Gamma_n \oplus \ker \beta_n}{\operatorname{Im} \varphi_n + \operatorname{Im} b_{n+2} + \operatorname{Im} d_{n+2}}, \tag{2.10}$$

but $T(V)$ is a model of A, so:

$$H_n(T(V_{\leqslant n+1})) = H_n(T(V)) = H_n(A),$$

thus the relation (2.10) can be written as:

$$H_n(A) \cong H_n(T(V)) \cong \frac{\operatorname{coker} b_{n+2} \oplus \ker \beta_n}{\operatorname{Im} \overline{\varphi_n} + \operatorname{Im} d_{n+2}},$$

as desired. □

It's well know that the Whitehead exact sequence is natural with respect to dga morphisms; namely, a dga morphism $f : A \to B$ induces the following useful commutative diagram:

$$(A) \qquad \begin{array}{ccccccccc}
\ldots \to Tor^A_{n+2}(R,R) & \xrightarrow{b_{n+2}} & \Gamma^A_n & \longrightarrow & H_n(A) & \longrightarrow & Tor^A_{n+1}(R,R) & \xrightarrow{b_{n+1}} & \ldots \\
\downarrow{\scriptstyle Tor^f_{n+2}(R,R)} & & \downarrow{\scriptstyle \gamma^f_n} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle Tor^f_{n+1}(R,R)} & & \\
\ldots \to Tor^B_{n+2}(R,R) & \xrightarrow{b'_{n+2}} & \Gamma^B_n & \longrightarrow & H_n(B) & \longrightarrow & Tor^B_{n+1}(R,R) & \xrightarrow{b'_{n+1}} & \ldots
\end{array}$$

where $\gamma^f_* : \Gamma^A_* \to \Gamma^B_*$ is the graded homomorphism induced by the dga morphism f. The commutativity of the above diagram induces the formula:

$$\left(Tor^f_{n+1}(R,R)\right)^* ([H_n(B)]) = (\overline{\gamma^f_n})_*([H_n(A)]), \tag{2.11}$$

where $[H_n(A)] \in \operatorname{Ext}(\ker b_{n+1}, \operatorname{coker} b_{n+2})$, $[H_n(B)] \in \operatorname{Ext}(\ker b'_{n+1}, \operatorname{coker} b'_{n+2})$ and where:

$$\left(Tor^{f}_{n+1}(R,R)\right)^{*} : \mathrm{Ext}(\ker b'_{n+1}, \mathrm{coker}\, b'_{n+2}) \to \mathrm{Ext}(\ker b_{n+1}, \mathrm{coker}\, b'_{n+2})$$
$$(\overline{\gamma^{f}_{n}})_{*} : \mathrm{Ext}(\ker b_{n+1}, \mathrm{coker}\, b_{n+2}) \to \mathrm{Ext}(\ker b_{n+1}, \mathrm{coker}\, b'_{n+2}).$$

Remark 2. *Formula (2.11) means the following:*
consider:

$$(\mathrm{Im}\, d_{n+2})' \overset{d_{n+2}}{\to} \ker \beta_n \twoheadrightarrow \ker b_{n+1}$$
$$(\mathrm{Im}\, d'_{n+2})' \overset{d'_{n+2}}{\to} \ker \beta'_n \twoheadrightarrow \ker b'_{n+1},$$

as two free resolutions of $\ker b_{n+1}$ *and* $\ker b'_{n+1}$ *respectively. To the given extensions* $[H_n(A)]$, $[H_n(B)]$ *and homomorphisms* γ^{f}_{n}, $Tor^{f}_{n+1}(R,R)$ *there correspond the following diagrams:*

$$\begin{tikzcd}
(\mathrm{Im}\, d_{n+2})' \arrow[r, tail, "d_{n+2}"] \arrow[d, "\overline{\varphi_n}"] & \ker \beta_n \arrow[r, two heads] & \ker b_{n+1} \\
\mathrm{coker}\, b_{n+1} \arrow[d, "\overline{\gamma^{f}_{n}}"] & & \\
\mathrm{coker}\, b'_{n+1} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{Im}\, d_{n+2})' \arrow[r, tail, "d_{n+2}"] \arrow[d, "\xi_{n+2}"] & \ker \beta_n \arrow[r, two heads] & \ker b_{n+1} \\
(\mathrm{Im}\, d'_{n+2})' \arrow[r, tail, "d'_{n+2}"] \arrow[d, "\overline{\varphi'_n}"] & \ker \beta'_n \arrow[r, two heads] & \ker b'_{n+1} \\
\mathrm{coker}\, b'_{n+1} & &
\end{tikzcd}$$

where $[\varphi_n] = [H_n(A)]$, $[\varphi'_n] = [H_n(B)]$ *and where* $\overline{\gamma^{f}_{n}}$ *(respectively* ξ_{n+2}*) is the homomorphism induced by* γ^{f}_{n} *(respect. by* $Tor^{f}_{n+1}(R,R)$*) on the quotient module* $\mathrm{coker}\, b_{n+2}$ *(respectively on the sub-module* $(\mathrm{Im}\, d_{n+2})'$*).*
The homomorphisms$(\overline{\gamma^{f}_{n}})_{*}$ *and* $\left(Tor^{f}_{n+1}(R,R)\right)^{*}$ *satisfy the following relations:*

$$(\overline{\gamma^{f}_{n}})_{*}([H_n(A)]) = [\overline{\gamma^{f}_{n}} \circ \overline{\varphi_n}]$$
$$(Tor^{f}_{n+1}(R,R))^{*}([H_n(B)]) = [\overline{\varphi'_n} \circ \xi_{n+2}],$$

so the formula (2.11) is equivalent to the relation:

$$[\overline{\gamma^{f}_{n}} \circ \overline{\varphi_n}] = [\overline{\varphi'_n} \circ \xi_{n+2}] \;\text{ in }\; \mathrm{Ext}(\ker b_{n+1}, \mathrm{coker}\, b'_{n+2}),$$

or that there exists a homomorphism $g_n : \ker \beta_n \longrightarrow \mathrm{coker}\, b'_{n+2}$ *satisfying the relation:*

$$\overline{\gamma^{f}_{n}} \circ \overline{\varphi_n} - \overline{\varphi'_n} \circ \xi_{n+2} = g_n \circ d_{n+2}. \tag{2.12}$$

3. Γ-Homotopy systems of order n and their category

The notion of homotopy systems of order n was defined by Baues in [**5**] for CW-complexes. For him a homotopy system of order n is a triple constituting with a

CW-complex X^n of dimension n, a chain complex (C_*, d) which coincides with the cellular chain complex $C_*(X^n)$ in dimension below $n+1$ and a homomorphism of abelian groups $f_{n+1} : C_{n+1} \to \pi_n(X^n)$ which satisfies the *cocycle* condition $f_{n+1} \circ d_{n+2} = 0$.

In this section we introduce the notion of homotopy systems of order n for dgas and their morphisms. Although this definition is completely different from Baues one but they express the same ideas.

Definition 1. *Let $n \geqslant 2$. A Γ-homotopy system of order n is a triple $(T(V, \partial^n), b_{n+2}, \pi_n)$ where:*
1- $T(V_{\leqslant n}, \partial^n)$ is an R-free graded algebra.
2- $(V_{\geqslant 1}, d_*)$ is a positive chain complex.
3- b_{n+2} is a homomorphism of modules:

$$b_{n+2} : H_{n+2}(s^{-1}V_*) \to \Gamma_n \subset H_n(T(V_{\leqslant n})).$$

4- π_n is an extension such that:

$$\pi_n \in \mathrm{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b_{n+2}).$$

Recall that the R-module Γ_n is given by the formula (2.2) and recall that s^{-1} is the desuspension graded homomorphism.

Definition 2. *A morphism between two Γ-homotopy systems $(T(V, \partial^n), b_{n+2}, \pi_n)$ and $(T(W, \delta^n), b'_{n+2}, \pi'_n)$ of order n is a pair (ξ_*, α^n) such that:*
$\alpha^n : T((V_{\leqslant n}, \partial^n)) \to T(W_{\leqslant n}, \delta^n))$ is a dga-morphism,
$\xi_* : s^{-1}V_* \to s^{-1}W_*$ is a chain map such that:

$$H_i(\xi_*) = Tor_i^{\alpha^n}(R, R), \forall i \leqslant n+1,$$

satisfying the following two conditions:

$$1 - b'_{n+2} \circ H_{n+2}(\xi_*) = \gamma_n^{\alpha^n} \circ b_{n+2}$$
$$2 - [H_{n+1}(\xi_*)]^*(\pi'_n) = (\gamma_n^{\alpha^n})_*(\pi_n),$$

where the homomorphism $\gamma_n^{\alpha^n}$ is induced by α^n and where:

$$[H_{n+1}(\xi_*)]^* : \mathrm{Ext}(H_{n+1}(s^{-1}W_*), \operatorname{coker} b'_{n+2}) \to \mathrm{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b'_{n+2})$$
$$(\gamma_n^{\alpha^n})_* : \mathrm{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b_{n+2}) \to \mathrm{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b'_{n+2}).$$

Remark 3. *As in the remark (2), the last condition means the following:*
Put $H_{n+1} = H_{n+1}(s^{-1}V_*)$ and $H'_{n+1} = H_{n+1}(s^{-1}W_*)$ and consider:

$$(\operatorname{Im} d_{n+2})' \overset{d_{n+2}}{\to} \ker d_{n+1} \twoheadrightarrow H_{n+1}$$
$$(\operatorname{Im} d'_{n+2})' \overset{d'_{n+2}}{\to} \ker d'_{n+1} \twoheadrightarrow H'_{n+1},$$

as two free resolutions of H_{n+1} and H'_{n+1} respectively. To the given extensions π_n and π'_n and homomorphisms $\gamma_n^{\alpha^n}$, $H_{n+1}(\xi_*)$ there corresponds the following diagram:

$$\begin{tikzcd}
(\operatorname{Im} d_{n+2})' \arrow[r, tail, "d_{n+2}"] \arrow[d, "\overline{\varphi_n}"] & \ker d_{n+1} \arrow[r, two heads] & H_{n+1} \\
\operatorname{coker} b_{n+1} \arrow[d, "\overline{\gamma_n^{\eta}}"] & & \\
\operatorname{coker} b'_{n+1} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\operatorname{Im} d_{n+2})' \arrow[r, tail, "d_{n+2}"] \arrow[d, "\xi_{n+2}"] & \ker d_{n+1} \arrow[r, two heads] & H_{n+1} \\
(\operatorname{Im} d'_{n+2})' \arrow[r, tail, "d'_{n+2}"] \arrow[d, "\overline{\varphi'_n}"] & \ker d'_{n+1} \arrow[r, two heads] & H'_{n+1} \\
\operatorname{coker} b'_{n+1} & &
\end{tikzcd}$$

where $[\overline{\varphi_n}] = \pi_n$, $[\overline{\varphi'_n}] = \pi'_n$ *and where* $\overline{\gamma_n^{\alpha^n}}$ *is the homomorphism induced by* $\gamma_n^{\alpha^n}$ *on the quotient module* $\operatorname{coker} b_{n+2}$.
The homomorphisms $(\overline{\gamma_n^{\alpha^n}})_*$ *and* $[H_{n+1}(\xi_*)]^*$ *satisfy the relation:*

$$(\overline{\gamma_n^{\alpha^n}})_*(\pi_n) = [\overline{\gamma_n^{\alpha^n}} \circ \overline{\varphi_n}]$$
$$[H_{n+1}(\xi_*)]^* \, (\pi'_n) = [\overline{\varphi'_n} \circ \xi_{n+2}].$$

So the second condition is equivalent to the existence of a homomorphism h_n : $\ker d_{n+1} \longrightarrow \operatorname{coker} b'_{n+2}$ *satisfying the relation:*

$$\overline{\gamma_n^f} \circ \overline{\varphi_n} - \overline{\varphi'_n} \circ \xi_{n+2} = h_n \circ d_{n+2}. \tag{3.1}$$

Denote by $\mathbf{H_n}$ the category of Γ-homotopy systems of order n and their morphisms.

Definition 3 (see [5]). *Let* (ξ_*, α^n) *and* (ξ'_*, α'^n) *be two morphisms in the category* $\mathbf{H}_n$. *We say that they are homotopic in* $\mathbf{H}_n$ *and we write* $(\xi_*, \alpha^n) \simeq (\xi'_*, \alpha'^n)$, *if* α^n *and* α'^n *are homotopic as a dga morphisms and if there exist homomorphisms* $\sigma_{k+1} : (s^{-1}V)_k \to (s^{-1}W)_{k+1}$, $k \geqslant n$, *such that:*

$$\xi_k - \xi'_k = \sigma_k \circ d_k + d'_{k+1} \circ \sigma_{k+1}, k \geqslant n+1.$$

Denote by $\mathbf{H_n}\diagup_{\simeq}$ the category whose objects are those of $\mathbf{H}_n$ and whose morphisms are the homotopy classes $\{(\xi_*, \alpha^n)\}$ of morphisms (ξ_*, α^n) of $\mathbf{H}_n$. We denote also by $\mathbf{H_n^{n+1}}$ the subcategory of $\mathbf{H_n}$ whose objects are the Γ-homotopy systems $(T(V, \partial^n), b_{n+2}, \pi_n)$ of order n such that $V_i = 0$ for all $i \geqslant n+2$.

3.1. Definition of the functor F_n^{n+1}

The Whitehead exact sequence introduced in the previous section allows us to define a function:

$$F_n : \mathbf{OBDGA}_*(\mathbf{flat})\diagup_{\simeq} \to \mathbf{OBH_n}\diagup_{\simeq},$$

as follows :
let $(T(V, \partial) \in \mathbf{OBDGA}_*(\mathbf{flat})\diagup_{\simeq}$ and let:

$$\cdots \to \operatorname{Tor}_{n+2}^{T(V)}(R, R) \xrightarrow{b_{n+2}} \Gamma_n \to H_n(T(V)) \to \operatorname{Tor}_{n+1}^{T(V)}(R, R) \xrightarrow{b_{n+1}} \cdots$$

be the Whitehead exact sequence associated with $T(V)$.

This sequence gives us a homomorphism $b_{n+2} : \mathrm{Tor}^{T(V)}_{n+2}(R,R) \to \Gamma_n$ and a short exact sequence $\mathrm{coker}\, b_{n+1} \rightarrowtail H_n(T(V)) \twoheadrightarrow \ker b_{n+1}$. But according to relation (2.8) we have:

$$H_n(A) \cong \frac{\mathrm{coker}\, b_{n+2} \oplus \ker \beta_n}{\mathrm{Im}\, \overline{\varphi_n} + \mathrm{Im}\, d_{n+2}},$$

where $\bar{\varphi_n} : (\mathrm{Im}\, d_{n+1})' \overset{\varphi_n}{\rightarrowtail} \Gamma_n \overset{pr}{\to} \mathrm{coker}\, b_{n+1}$.
Since the relation (2.3) implies $\ker \beta_n \subset \ker d_{n+1}$ we set:

$$\pi_n = \left[\frac{\mathrm{coker}\, b_{n+2} \oplus \ker d_{n+1}}{\mathrm{Im}\, \overline{\varphi_n} + \mathrm{Im}\, d_{n+2}} \right]. \tag{3.2}$$

Therefore, $\pi_n \in \mathrm{Ext}(Tor^{T(V)}_{n+1}(R,R), \mathrm{coker}\, b_{n+2})$. We define F_n by:

$$F_n(T(V)) = (T(V, \partial^n), b_{n+2}, \pi_n).$$

We now wish to define F_n on dga morphisms by setting $F_n(\alpha) = Tor^{\alpha}_*(R,R)$, but unfortunately this definition is not natural with respect to dga morphisms since we have seen that if α is a dga morphism then the induced graded homomorphism $Tor^{\alpha}_*(R,R)$ implies the formula (2.12). But as we know, in order to be a morphism in $\mathbf{H}_n$ a graded homomorphism f_* should satisfy the formula (3.1).

Therefore to make this correspondence natural with respect to dga morphisms, we define $\mathbf{\Gamma DGA}_*(\mathbf{flat})$ as the subcategory of $\mathbf{DGA}_*(\mathbf{flat})$ whose morphisms are the dga morphisms α satisfying the following condition: for all $n \geqslant 2$:

$$[\overline{\gamma^{\alpha}_n} \circ \overline{\varphi_n}] = [\overline{\varphi'_n} \circ \xi_{n+2}] \text{ in } \mathrm{Ext}(Tor^{T(V)}_{n+1}(R,R), \mathrm{coker}\, b'_{n+2}). \tag{3.3}$$

Note that this condition is trivial when we work over a field of any characteristic.

Thus we define the functor:

$$F_n : \mathbf{\Gamma DGA}_*(\mathbf{flat})/_{\simeq} \to \mathbf{H_n}/_{\simeq},$$

by setting:

$$\begin{aligned} F_n(T(V)) &= (T(V, \partial^n), b_{n+2}, \pi_n) \\ F_n(\{\alpha\}) &= \{(\xi_*, \alpha^n)\}, \end{aligned}$$

where α^n denote the restriction of the dga-morphism α on $T(V_{\leqslant n}, \partial^n)$ and where the ξ_* is the chain transformation induces by the dga-morphism α on the indecomposables.

Corollary 1. *The condition (3.3) implies that the morphism (ξ_*, α^n) satisfies the second condition which define the morphisms in $\mathbf{H}_n$.*

Let $\mathbf{DGA}^{\mathbf{n+2}}_*(\mathbf{flat})$ be the subcategory of $\mathbf{DGA}_*(\mathbf{flat})$ on which the objects A are those satisfying the relation $Tor^A_i(R,R) = 0$, $\forall i \geqslant n+3$.

Remark 4. *If A is an object of the category $\mathbf{DGA}^{\mathbf{n+2}}_*(\mathbf{flat})$ then a model $T(V_{\leqslant n+1}, \partial)$ of A can be chosen such that $V_i = 0$ for each $i \geqslant n+2$. Therefore $Tor^A_{n+2}(R,R) = \ker d_{n+2} \subset V_{n+1}$. So $Tor^A_{n+2}(R,R)$ is free.*

We can also consider the functor:

$$F_n^{n+1} : \mathbf{\Gamma DGA_*^{n+2}(flat)}\diagup_{\simeq} \to \mathbf{H_n^{n+1}}\diagup_{\simeq},$$

given by the formula:

$$\begin{aligned} F_n^{n+1}(T(V_{\leqslant n+1}, \partial)) &= (T(V_{\leqslant n+1}, \partial^n), b_{n+2}, \pi_n) \\ F_n^{n+1}(\{\alpha^{n+1}\}) &= \{(\xi_*, \alpha^n)\}, \end{aligned}$$

which is the restriction of the functor F_n to the subcategory $\mathbf{\Gamma DGA_*^{n+2}(flat)}\diagup_{\simeq}$ of $\mathbf{DGA_*(flat)}\diagup_{\simeq}$.

3.2. Properties of the functor F_n^{n+1}

Now we approach the study of the functor F_n^{n+1}.

Theorem 5. *A morphism $\alpha : T(V_{\leqslant n+1}, \partial) \to T(W_{\leqslant n+1}, \delta)$ in $\mathbf{\Gamma DGA_*^{n+2}(flat)}$ is a quasi-isomorphism if and only if $F_n^{n+1}(\alpha)$ is an isomorphism in $\mathbf{H_n^{n+1}}\diagup_{\simeq}$.*

Proof. Recall that by definition of the functor F_n^{n+1} we have the formulas:

$$\begin{aligned} F_n^{n+1}(T(V_{\leqslant n+1}, \partial)) &= (T(V_{\leqslant n+1}, \partial^n), b_{n+2}, \pi_n) \\ F_n^{n+1}(T(W_{\leqslant n+1}, \delta)) &= (T(W_{\leqslant n+1}, \delta^n), b'_{n+2}, \pi'_n), \end{aligned}$$

and the relation:

$$\begin{aligned} Tor_*^{T(V_{\leqslant n+1})}(R, R) &= H_*(s^{-1}V_{\leqslant n+1}, d_*) \\ Tor_*^{T(W_{\leqslant n+1})}(R, R) &= H_*(s^{-1}W_{\leqslant n+1}, d'_*). \end{aligned}$$

So if $F_n^{n+1}(\alpha)$ is an isomorphism in $\mathbf{H}_n^{n+1}\diagup_{\simeq}$ then we deduce that the graded homomorphism:

$$Tor_*^{\alpha}(R, R) : Tor_*^{T(V_{\leqslant n+1})}(R, R) \to Tor_*^{T(W_{\leqslant n+1})}(R, R),$$

is an isomorphism and then we apply the classical theorem of Moore [**13, 15**] which asserts that a dga morphism f in $\mathbf{DGA_*(flat)}$ is a quasi-isomorphism if and only if $Tor_*^f(R, R)$ is a quasi-isomorphism as a chain morphism. □

Theorem 6. *For each Γ-homotopy system $(T(V_{\leqslant n+1}, \partial^n), b_{n+2}, \pi_n)$ in $\mathbf{H}_n^{n+1}$, there exists a free dga $T(V_{\leqslant n+1}, \partial)$ in $\mathbf{DGA_*^{n+2}(flat)}$ such that:*

$$\begin{aligned} F_n^{n+1}(T(V_{\leqslant n+1}, \partial)) &= (T(V_{\leqslant n+1}, \partial^{n+1}), b_{n+2}, \pi_n) \\ Tor_{n+2}^{T(V_{\leqslant n+1})}(R, R) &= H_{n+2}(s^{-1}V_{\leqslant n+1}). \end{aligned}$$

Proof. Consider:

$$(\operatorname{Im} d_{n+2})' \overset{d_{n+2}}{\rightarrowtail} \ker \beta_n \twoheadrightarrow \ker b_{n+1},$$

as a free resolution of $\ker b_{n+1}$. The inclusion $\ker b_{n+1} \overset{i}{\hookrightarrow} H_{n+1}(s^{-1}V_*)$ implies the homomorphism:

$$\operatorname{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b_{n+2}) \xrightarrow{i_*} \operatorname{Ext}(\ker b_{n+1}, \operatorname{coker} b_{n+2}).$$

Therefore for the given extension $\pi_n \in \operatorname{Ext}(H_{n+1}(s^{-1}V_*), \operatorname{coker} b_{n+2})$ and the homomorphism b_{n+2}, there exist homomorphisms μ_{n+2}, φ_n which make the following diagram commutative:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & & (\mathrm{Im}\, d_{n+2})' \arrow[dll, "\varphi_n"'] \arrow[ddl, "\overline{\varphi_n}"] \arrow[d] \\
\ker d_{n+2} \arrow[r, "\mu_{n+2}"] \arrow[dr, "b_{n+2}"'] & Z_n(T(V_{\leqslant n})) \arrow[d, "pr"] & & \ker \beta_n \arrow[d] \\
 & \Gamma_n \arrow[r, two heads] & \operatorname{coker} b_{n+2} & \ker b_{n+1}
\end{tikzcd}
$$

where $[\overline{\varphi_n}] = i_*(\pi_n)$.
Using the decomposition of the module $V_{n+1} = (\mathrm{Im}\, d_{n+2})' \oplus \ker d_{n+2}$ in the relation (2.5), we define ∂ on $T(V_{\leqslant n+1})$ by the formulas:

$$
\begin{aligned}
\partial_{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) &= d_{n+2}(l_{n+2,\sigma'}) + \varphi_n(l_{n+2,\sigma'}) + \mu_{n+2}(z_{n+2,\sigma}) \\
\partial_{\leqslant n} &= \partial^n_{\leqslant n},
\end{aligned}
$$

where $(z_{n+2,\sigma})_{\sigma\in\sum}$ and $(l_{n+2,\sigma'})_{\sigma'\in\sum'}$ denote respectively bases of the free submodules $\ker d_{n+2}$ and $(\mathrm{Im}\, d_{n+2})'$.
Note that by the above diagram the element:

$$\varphi_n(l_{n+2,\sigma'}) + \mu_{n+2}(z_{n+2,\sigma}) \in Z_n(T(V_{\leqslant n})),$$

hence $\partial^n_{\leqslant n}(\varphi_n(l_{n+2,\sigma'}) + \mu_{n+2}(z_{n+2,\sigma})) = 0$.
Since $\partial^n_{\leqslant n}(d_{n+2}(l_{n+2,\sigma'})) = d_{n+1}\,(d_{n+2}(l_{n+2,\sigma'})) = 0$ we deduce that ∂ is a differential on $T(V_{\leqslant n+1})$.
It's easy to see that the Whitehead exact sequence associated with the dga $A = (T(V_{\leqslant n+1}), \partial)$ can be written:

$$\ker d_{n+2} = Tor^A_{n+2}(R,R) \xrightarrow{b_{n+2}} \Gamma_n \to H_n(A) \to Tor^A_{n+1}(R,R) \to \cdots$$

then according to the definition of the functor F^{n+1}_n, it easy to check that:

$$
\begin{aligned}
F^{n+1}_n(A) &= (T(V_{\leqslant n+1}, \partial^n), b_{n+2}, \pi_n) \\
Tor^A_{\leqslant n+2}(R,R) &= H_{\leqslant n+2}(s^{-1}V_{\leqslant n+1}),
\end{aligned}
$$

and the proof is completed. □

Theorem 6 allows us to describe an action of the group $Hom(Tor^{T(V)}_{n+2}(R,R), \Gamma_n)$ on the set $\mathbf{OBDGA}_*(\mathbf{flat})$ as follows:

Corollary 2. *Given a free dga $(T(V), \partial)$, let $b_{n+2} \in \operatorname{Hom}(Tor^{T(V)}_{n+2}(R,R), \Gamma_n)$. Perturb the differential ∂^{n+1} in the dga $(T(V_{\leqslant n+1}, \partial^{n+1})$ by setting:*

$$\partial'^{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) = \partial^{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) + \mu_{n+2}(z_{n+2,\sigma}). \tag{3.4}$$

For a given extension $\pi_n \in \mathrm{Ext}(Tor_{n+1}^{T(V)}(R,R), \mathrm{coker}\, b'_{n+2})$*, also perturb* ∂'^{n+1} *to obtain a new differential* δ^{n+1} *of* $T(V_{\leqslant n+1})$ *by setting:*

$$\delta^{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) = \partial'^{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) + d_{n+2}(l_{n+2,\sigma'}) \tag{3.5}$$
$$+\varphi_n(l_{n+2,\sigma'}), \tag{3.6}$$

so that the differential δ *is defined by the formulas:*

$$\delta^i = \partial'^i = \partial^i \qquad \forall i \neq n+1$$
$$\delta^{n+1} = \partial'^{n+1}_{n+1} + d_{n+2} + \varphi_n = \partial^{n+1} + d_{n+2} + \mu_{n+2}.$$

Definition 4. *The dga* $(T(V_{\leqslant n+1}), \partial)$ *constructed below is called the dga associated with the* Γ*-homotopy system* $(T(V_{\leqslant n+1}, \partial^n), b_{n+2}, \pi_n)$ *of order n.*

Definition 5. *We call the pair* (b_{n+2}, π_n) *an adapted couple for the dga* $(T(V_{\leqslant n}), \partial^n)$*.*

Definition 6. *A dga morphism* $\alpha^n : (T(V_{\leqslant n}), \partial^n) \to (T(W_{\leqslant n}), \delta^n)$ *is said to be n-diagonal if* α^n *maps* $T(V_{\leqslant n-1})$ *onto* $T(W_{\leqslant n-1})$ *and the direct factor* $\ker d_{n+1}$ *of* V_n *onto the module* W_n*.*

Lemma 1. *If* $\alpha^n : (T(V_{\leqslant n}), \partial^n) \to (T(W_{\leqslant n}), \delta^n)$ *is n-diagonal then according to the splitting* $H_n(T(V_{\leqslant n})) \cong \Gamma_n \oplus \ker b_{n+1}$ *(respectively.* $H_n(T(W_{\leqslant n})) \cong \Gamma'_n \oplus \ker b'_{n+1}$*), the homomorphism:*

$$H_n(\alpha^n) : H_n(T(V_{\leqslant n})) \to H_n(T(W_{\leqslant n})),$$

splits into:

$$H_n(\alpha^n) = \gamma_n^{\alpha^n} \oplus \xi_{n+1}, \tag{3.7}$$

where $\gamma_n^{\alpha^n}$ *is the homomorphism induced by* α^n *on the sub-module* Γ_n*, and where* $\xi_{n+1} : \ker \beta_n \to \ker \beta'_n$ *is the chain transformation induced by* α^n *on the indecomposables.*

Theorem 7. *Let* $(T(V_{\leqslant n+1}, \partial)$ *and* $(T(W_{\leqslant n+1}, \delta)$ *be two free dgas in* $\mathbf{DGA}_*^{n+2}(\mathbf{flat})$ *and let:*

$$(\xi_*, \alpha^n) : F_n^{n+1}((T(V_{\leqslant n+1}, \partial)) \to F_n^{n+1}((T(W_{\leqslant n+1}, \delta)),$$

be a morphism in $\mathbf{H_n^{n+1}}$ *such that* α^n *is n-diagonal. Then there exists a dga-morphism* $\Lambda : (T(V_{\leqslant n+1}), \partial) \to (T(W_{\leqslant n+1}), \delta)$ *such that* Λ *is* $(n+1)$*-diagonal and satisfies:*

$$F_n^{n+1}(\Lambda) = H_{n+2}(\xi_*). \tag{3.8}$$

In order to prove the theorem we consider the following diagram:

$$\begin{tikzcd}
& V_{n+1} \arrow[dl, "\beta_{n+1}"'] \arrow[dd, "d_{n+2}"] \arrow[rr, "\xi_{n+2}"] & & W_{n+1} \arrow[dr, "\beta'_{n+1}"] \arrow[dd, "d'_{n+2}"] & \\
H_n(T(V_{\leqslant n})) \arrow[rrrr, "H_n(\alpha^n)"] \arrow[dr, "j_n"'] & & & & H_n(T(W_{\leqslant n})) \arrow[dl, "j'_n"] \\
& V_n \arrow[rr, "\xi_{n+1}"] \arrow[d] & & W_n \arrow[d] & \\
& \vdots & & \vdots &
\end{tikzcd}$$

- where the left and right triangles commute by the definition of the linear differentials,
- where the lower trapezoid commutes by the definition of the given morphism (ξ_*, α^n),
- where the central square commutes by the definition of ξ_*.

Since $j'_n(H_k(\alpha^n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2}) = 0$, then:

$$\operatorname{Im}(H_n(\alpha^n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2}) \subset \Gamma'_n.$$

We begin by giving the following lemma:

Lemma 2. *There exists a homomorphism $g_n : V_n \to \Gamma'_n$ such that*

$$H_n(\alpha^n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2} = g_n \circ d_{n+1} \pmod{\operatorname{Im} b'_{n+2}}.$$

Proof. Since the dga morphism α^n is n-diagonal, the relations (2.7) and (3.7) give us an explicit expression of the homomorphism $H_n(\alpha^n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2}$ so it's easy to show that:

$$H_n(\alpha^n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2} = \gamma_n^{\alpha^n} \circ \varphi_n - \varphi'_n \circ \xi_{n+2}.$$

Since $pr \circ \left(\gamma_n^{\alpha^n} \circ \varphi_n - \varphi'_n \circ \xi_{n+2}\right) = \overline{\gamma_n^{\alpha^n}} \circ \overline{\varphi_n} - \overline{\varphi'_n} \circ \xi_{n+2}$ and according to the condition (2) defining a morphism in the category $\mathbf{H}_n$ and remark 3, there exists a homomorphism g_n which makes the following diagram commutative after composition with the projection pr:

$$\begin{tikzcd}
0 \arrow[r] & (\operatorname{Im} d_{n+2})' \arrow[r, "d_{n+2}"] \arrow[d, "\gamma_n^{\alpha^n}\varphi_n - \varphi'_n\xi_{n+2}"'] & \ker d_{n+1} \arrow[dl, "g_n"] \\
& \Gamma'_n \arrow[r, two heads, "pr"] & \operatorname{coker} b'_{n+2}
\end{tikzcd}$$

Since $V_n = (\operatorname{Im} d'_{n+1}) \oplus \ker d_{n+1}$ then we can extend g_n to V_n and the proof is completed. $\square$

Proof. (of theorem (7))
According to lemma 2 we have:

$$H_n(\alpha) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2} - g_n \circ d_{n+2} : (\operatorname{Im} d'_{n+2}) \to \operatorname{Im} b'_{n+2},$$

or:

$$(H_n(\alpha) - g_n \circ j_n) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+2} : (\operatorname{Im} d'_{n+2}) \to \operatorname{Im} b'_{n+2}.$$

As a result there exists a homomorphism λ_{n+1} which makes the upper triangle in the following diagram commutative:

$$\begin{tikzcd}
 & \ker d'_{n+2} \arrow[d] \\
V_{n+1} \arrow[ur, "\lambda_{n+1}"] \arrow[r, "(H_n(\alpha^n) - g_n j_n)\beta_{n+1} - \beta'_{n+1}\xi_{n+2}"] \arrow[dr, "(\alpha^n - h_n)\partial - \delta\xi_{n+2}"'] & \operatorname{Im} b'_{n+2} \subset \Gamma'_n \\
 & Z_n(T(W_{\leqslant n})) \arrow[u, "pr"']
\end{tikzcd}$$

where the homomorphism $h_n : V_n \to Z_n(T(W_{\leqslant n}))$ satisfies the relation $pr \circ h_n = g_n$ and where $Z_n(T(W_{\leqslant n}))$ is the module of n-cycles of the dga $T(W)$.
Choose $(z_{n+2,\sigma})_{\sigma\in\sum}$ and $(l_{n+2,\sigma'})_{\sigma'\in\sum'}$ respectively as bases of the free sub-modules $(\operatorname{Im} d_{n+2})'$ and $\ker d_{n+2}$. Recall that:

$$V_{n+1} \cong (\operatorname{Im} d_{n+2})' \oplus (\ker d_{n+2}).$$

By the commutativity of the above diagram, for each $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$, there exists an element $t_{n+2,\sigma,\sigma'} \in T_{n+1}(W_{\leqslant n})$ such that:

$$\begin{aligned}((\alpha_n - h_n)\partial - \delta\xi_{n+2})(z_{n+2,\sigma} + l_{n+2,\sigma'}) = \delta \circ \lambda_{n+1}(l_{n+2,\sigma'}) \\ +\delta(t_{n+2,\sigma,\sigma'}).\end{aligned} \tag{3.9}$$

If $z_{n+2,\sigma} + l_{n+2,\sigma'} \in V_{n+1}$, then we define $\Lambda : (T(V_{\leqslant n+1}), \partial) \longrightarrow (T(W_{\leqslant n+1}), \delta)$ by setting:

$$\begin{aligned}
\Lambda_{n+1}(z_{n+2,\sigma} + l_{n+2,\sigma'}) &= \lambda_{n+1}(l_{n+2,\sigma,\sigma'}) + t_{n+2,\sigma,\sigma'} + \xi_{n+2}(z_{n+2,\sigma} + l_{n+2,\sigma'}) \\
\Lambda_n &= \alpha_n - h_n \\
\Lambda_i &= \alpha_i \qquad i \neq n, n+1.
\end{aligned}$$

Λ is a dga morphism i.e. $\Lambda \circ \partial = \delta \circ \Lambda$. Indeed, since for each $z_{n+2,\sigma} + l_{n+2,\sigma'} \in V_{n+1}$ we have $\partial(z_{n+2,\sigma} + l_{n+2,\sigma'}) \in Z_n\left(T(W_{\leqslant n})\right)$, then according to the definition of Λ we have

$$\Lambda \circ \partial(z_{n+2,\sigma} + l_{n+2,\sigma'}) = (\alpha_n - h_n) \circ \partial(z_{n+2,\sigma} + l_{n+2,\sigma'}),$$

and, on the other hand, according to the relation (3.9) and the definition of Λ, we have

$$\begin{aligned}\delta\circ\Lambda_{n+1}(z_{n+2,\sigma}+l_{n+2,\sigma'}) &= \delta\circ\lambda_{n+1}(l_{n+2,\sigma'})+\delta\circ\xi_{n+2}(z_{n+2,\sigma}+l_{n+2,\sigma'})\\ &\quad+\delta(t_{n+2,\sigma'})\\ &=((\alpha_n-h_n)\circ\partial-\delta\circ\xi_{n+2})((z_{n+2,\sigma}+l_{n+2,\sigma'}))\\ &\quad+\delta\circ\xi_{n+2}(z_{n+2,\sigma}+l_{n+2,\sigma'})\\ &=(\alpha_n-h_n)\circ\partial((z_{n+2,\sigma}+l_{n+2,\sigma'})).\end{aligned}$$

Therefore Λ is a dga morphism. Finally by the definition, it's clear that α is $(n+1)$-diagonal and satisfies the relation (3.8). ☐

Following H.J. Baues in [**5**], let **A** and **B** be two categories and let $F : \mathbf{A}\to\mathbf{B}$ be a functor. We say that F is a "detecting functor " if the following conditions are satisfied:
1 -A morphism $\alpha : A_1 \to A_2$ in the category **A** is an isomorphism if and only if $F(\alpha) : F(A_1)\to F(A_2)$ is an isomorphism.
2 - For each object B in the category **B**, there exists an object A in **A** such that $F(A) = B$.
3 - Let A_1 and A_2 be two objects in **A** and let $\beta : F(A_1)\to F(A_2)$ be a morphism in **B**, there exists a morphism $\alpha : A_1 \to A_2$ verifying $F(\alpha) = \beta$.
We summarize the properties of the functor F_n^{n+1} in the following result:

Theorem 8. *The functor:*

$$F_n^{n+1} : \mathbf{\Gamma DGA_*^{n+2}(flat)}/_{\simeq} \to \mathbf{H_n^{n+1}}/_{\simeq},$$

is a detecting functor.

4. The category Γ

4.1. Notion of Γ-system and Γ-morphism

Roughly speaking, each object of the category Γ gives a long exact sequence which is the Whitehead exact sequence associated with a certain object in $\mathbf{DGA}_*(\mathbf{flat})$. Given a graded R-module H_*, choose a free chain complex $(s^{-1}V_*, d)$ such that $H_* = H_*(s^{-1}V_*, d)$.

Definition 7. *A Γ-system is a triple $(H_n, b_{n+2},\pi_n)_{n\geqslant 2}$ where, for each n, (b_{n+2},π_n) is an adapted couple (see definition 5) for the dga $(T(V_{\leqslant n}),\partial^n)$ which is constructed from the dga $(T(V_{\leqslant 2}), d)$ and the pairs $(b_{k+2},\pi_k)_{2\leqslant k\leqslant n-1}$ according to theorem 6.*

Thus if we iterate this process we find a free dga $(T(V),\partial)$ having the following property:
for each $n\geqslant 2$ we have:

$$F_n((T(V),\partial)) = (T(V,\partial^n), b_{n+2},\pi_n).$$

Definition 8. *The dga $(T(V),\partial)$ constructed below is called the dga associated with the Γ-system $(H_n, b_{n+2},\pi_n)_{\geqslant 2}$.*

Remark 5. *The $(T(V),\partial)$ is not unique but we shall prove later that $(T(V),\partial)$ is unique up to homotopy.*

Let $(H_n, b_{n+2}, \pi_n)_{n\geqslant 2}$ and $(H'_n, b'_{n+2}, \pi'_n)_{n\geqslant 2}$ be two Γ-systems and let $f_* : H_* \to H'_*$ be a graded homomorphism of degree 0. We say that f_* is a Γ-morphism if f_* satisfies the following inductive construction:
let $(T(V),\partial)$ and $(T(W),\delta)$ be two free dgas associated respectively with the given Γ-systems. By virtue of the homotopy extension theorem [**14**], there exists a chain transformation $\xi_* : (s^{-1}V, d) \to (s^{-1}W, d')$ such that $H_*(\xi_*) = f_*$. We begin by considering the dga morphism $\alpha^2 : (T(V_{\leqslant 2}), d) \to (T(W_{\leqslant 2}), d')$ given by:

$$\alpha^2 = \xi_3 \text{ on } V_2$$
$$\alpha^2 = \xi_2 \text{ on } V_1.$$

Since $\ker d_3 = H_3$ we deduce that (ξ_*, α^2) is a morphism in the category $\mathbf{H}_2^3$ and it's clear that α^2 is 2-diagonal by the definition, so theorem 7 gives a dga morphism $\alpha^3 : (T(V_{\leqslant 3}), \partial^3) \to (T(W_{\leqslant 3}), \delta^3)$ which is 3-diagonal and such that $Tor_i^{\alpha^3}(R,R) = f_i$ for all $i \leqslant 4$.
Suppose now that we have built a dga morphism:

$$\alpha^n : (T(V_{\leqslant n}), \partial^n) \to (T(W_{\leqslant n}), \delta^n),$$

by using the above process, such that α^n is n-diagonal and such that:

$$Tor_i^{\alpha^n}(R,R) = f_i \text{ for all } i \leqslant n+1.$$

If we assume that (ξ_*, α^n) is a morphism in $\mathbf{H}_n^{n+1}$, then theorem 7 allows us to construct a dga morphism $\alpha^{n+1} : (T(V_{\leqslant n+1}), \partial^{n+1}) \to (T(W_{\leqslant n+1}), \delta^{n+1})$ which is $(n+1)$-diagonal and such that:

$$Tor_i^{\alpha^{n+1}}(R,R) = f_i \text{ for all } i \leqslant n+2.$$

Thus if we iterate this process for all $n \geqslant 2$ we find a dga morphism $\alpha : (T(V),\partial) \to (T(W),\delta)$ satisfying the following properties:
1- $Tor_*^{\alpha}(R,R) = f_*$.
2- For each $n \geqslant 2$, (ξ_*, α^n) is a morphism in the category $\mathbf{H}_n$, where $\alpha^n : T(V_{\leqslant n}), \partial) \to (T(W_{\leqslant n}), \delta)$ is the restriction of α.

Definition 9. *A graded homomorphism $f_* : H_* \to H'_*$ satisfying the above inductive condition for each $n \geqslant 2$ is called a Γ-morphism.*

Definition 10. *The dga morphism $\alpha : (T(V),\partial) \to (T(W),\delta)$ constructed by the above process is called the dga morphism associated with the Γ-morphism f_*.*

Remark 6. *Since $Tor_*^{\alpha}(R,R) = f_*$ we deduce that the dga morphism α constructed below satisfies condition (3.3). So it is a morphism in the category $\mathbf{\Gamma DGA}_*(\mathbf{flat})$.*

4.2. Definition of the category Γ and the functor F

Now we are able to give the following definition:

Definition 11. *The category Γ is defined as follows:*
Objects: *the Γ-systems defined in the definition (7)*
Morphisms: *the Γ-morphisms defined in the definition (9)*

Thus we define the functor $F : \mathbf{\Gamma DGA_*(flat)}/_{\simeq} \to \mathbf{\Gamma}$ by setting:

$$F((T(V),\partial)) = (Tor_k^{T(V)}(R,R), b_{k+2}, \pi_k)_{k\geqslant 2}$$
$$F(\{\alpha\}) = Tor_*^{\alpha}(R,R).$$

We now give some properties of the functor F and we show that the functor satisfies the properties of a "detecting functor". We begin with the following propositions:

Proposition 2. *Let $(T(V),\partial)$ and $(T(W),\delta)$ be two free dgas. If the Γ-systems $(Tor_k^{T(V)}(R,R), b_{k+2}, \pi_k)_{k\geqslant 2}$ and $(Tor_k^{T(W)}(R,R), b'_{k+2}, \pi'_k)_{k\geqslant 2}$ are isomorphic in the category Γ, then the given dgas have the same homotopy type.*

Proof. Let $f_* : (Tor_k^{T(V)}(R,R), b_{k+2}, \pi_k)_{k\geqslant 2} \to (Tor_k^{T(W)}(R,R), b'_{k+2}, \pi'_k)_{k\geqslant 2}$ be the isomorphism between the two Γ-*systems* and let $\alpha : (T(V),\partial) \to (T(W),\delta)$ be the dga morphism associated with the Γ-morphism f_* (see definition 10). Since $Tor_*^{\alpha}(R,R) = f_*$ and again we apply the classical Moore's theorem to get the result. □

Proposition 3. *A dga-morphism α is a quasi-isomorphism if and only if $F(\alpha)$ is an isomorphism.*

Proof. Obvious since $F(\alpha) = Tor_*^{\alpha}(R,R)$. □

Remark 7. *¿From the proposition 3, we conclude that the dga morphism α associated to the Γ-morphism f_* is unique up to homotopy.*

Now we are able to announce the main theorem in this paper:

Theorem 9. *The functor:*

$$F : \mathbf{\Gamma DGA_*(flat)}/_{\simeq} \to \mathbf{\Gamma},$$

is a detecting functor.

Proof. Definition 7 implies that to each Γ-system is associated a dga $(T(V),\partial)$ such that $F(T(V))$ is the given Γ-system. Definition 10 implies that to each Γ-morphism is associated a morphism in $\mathbf{\Gamma DGA_*(flat)}$ which has for image, by the functor F, the given Γ-morphism. We conclude the proof by proposition 3. □

As a consequence of theorem 9 we deduce the following result:

Theorem 10. *Homotopy types in the category $\mathbf{\Gamma DGA_*(flat)}/_{\simeq}$ are in bijection with the classes of isomorphisms of objects in the category Γ.*

Moreover, we derive the following theorem:

Theorem 11. *Two objects of the category $\mathbf{\Gamma DGA_*(flat)}$ have the same homotopy type if and only if their Whitehead exact sequences are isomorphic.*

Now we give a relation identifying the subcategory $\mathbf{\Gamma DGA_*(flat)}$ with $\mathbf{DGA_*(flat)}$.

Proposition 4. *Let:*

$$\cdots \to Tor_{n+2}^{T(V)}(R,R) \xrightarrow{b_{n+2}} \Gamma_n \longrightarrow H_n(T(V)) \longrightarrow Tor_{n+1}^{T(V)}(R,R) \xrightarrow{b_{n+1}} \cdots$$

$$\cdots \to Tor_{n+2}^{T(W)}(R,R) \xrightarrow{b'_{n+2}} \Gamma'_n \longrightarrow H_n(T(W)) \longrightarrow Tor_{n+1}^{T(W)}(R,R) \xrightarrow{b'_{n+1}} \cdots$$

be the Whitehead exact sequences associated respectively with two free dgas $(T(V),\partial)$ and $(T(W),\delta)$. If

$$\operatorname{Ext}\left(\frac{Tor_{n+1}^{T(V)}(R,R)}{\ker b_{n+1}}, \operatorname{coker} b'_{n+1}\right) = 0 \text{ for all } n \geqslant 2, \tag{4.1}$$

then the condition (3.3) is satisfied.

Proof. ¿From the following short exact sequence:

$$\ker\beta_n \overset{i}{\hookrightarrow} \ker d_{n+1} \twoheadrightarrow \frac{\ker d_{n+1}}{\ker\beta_n},$$

where i is the inclusion, and by the second isomorphism theorem which asserts that:

$$\frac{\ker d_{n+1}}{\ker\beta_n} \cong \frac{Tor_{n+1}^{T(V)}(R,R)}{\ker b_{n+1}},$$

we deduce that:

$$\operatorname{Ext}\left(\frac{Tor_{n+1}^{T(V)}(R,R)}{\ker b_{n+1}}, \operatorname{coker} b'_{n+1}\right) = \frac{Hom(\ker\beta_n, \operatorname{coker} b'_{n+1})}{i^*(Hom(\ker d_{n+1}, \operatorname{coker} b'_{n+1}))} = 0. \tag{4.2}$$

¿From the relation (2.12) we know that the homomorphism $\overline{\gamma_n^f} \circ \overline{\varphi_n} - \overline{\varphi'_n} \circ \xi_{n+1}$ satisfies the following relation:

$$\overline{\gamma_n^f} \circ \overline{\varphi_n} - \overline{\varphi'_n} \circ \xi_{n+1} = g_n \circ d_{n+1}$$

where $g_n : \ker\beta_n \longrightarrow \operatorname{coker} b'_{n+2}$. So the relation (4.2) implies that $g_n \in i^*(Hom(\ker d_{n+1}, \operatorname{coker} b'_{n+1}))$. Hence there exists a homomorphism $h_n : \ker d_{n+1} \to \operatorname{coker} b'_{n+1}$ such that $g_n = h_n \circ i$. Therefore the relation (4.1) implies the condition (3.3). □

As a consequence of proposition 4, we derive the following result, which is the main result in this paper

Theorem 12. *If the relation (4.1) is satisfied, then two free dgas $(T(V),\partial)$ and $(T(W),\delta)$ have the same homotopy type if and only if their Whitehead exact sequences are isomorphic.*

Corollary 3. *If the relation (4.1) is satisfied for all dga $(T(V),\partial)$, then we have:*

$$\mathbf{DGA}_*(\mathbf{flat}) = \mathbf{\Gamma DGA}_*(\mathbf{flat}).$$

Let k be a field of any characteristic. In [[**10**], thm 2.3] Baues and Lemaire establish the following result: for every dga A over k there exists a quasi-isomorphism

$\alpha : (T(V), \partial) \to A$, where the differential ∂ is decomposable, i.e: $\partial : V \to T_{\geqslant 2}(V)$. This tensor algebra is called the minimal model of A. It is unique up to isomorphism. Over a field k, following the construction in definition (7), the tensor algebra $(Ts^{-1}(Tor_*^A(R,R)), \partial)$ can be taken as the dga associated with the Γ-system $\Gamma(A)$. Therefore by theorem 9 there exists a quasi-isomorphism $\alpha : (T(s^{-1}Tor_*^A(R,R)), \partial) \to A$, where the differential ∂ is decomposable. The dga $(T(s^{-1}Tor_*^A(R,R)), \partial)$ is not unique, but the Γ-system $\Gamma(A) = (Tor_n^A(R,R)), b_{n+2})_{n\geqslant 2}$ is unique.

5. The category Γ_n^{3n+2}

In this section we treat a partial special case where the notion of the Γ-system may be simplified, even over a P.I.D. Indeed we will show when the dga A is an object of $\mathbf{\Gamma DGA_n^{3n+2}(flat)}$ (the subcategory of $\mathbf{\Gamma DGA_*(flat)}$ of which the objects are those satisfying the relations $Tor_i^A(R,R) = 0$ for $i \leqslant n$ and $i \geqslant 3n+3$, we can denote the graded module Γ_* simply by the graded module $Tor_*^A(R,R)$ and the homomorphism b_{2n+2}.
This section is motivated by the following theorem which gives us explicitly the graded module Γ_*^A which appear in the Whitehead sequence associated with A.

Theorem 13. *Let A be an object in $\mathbf{DGA_n^{3n+2}(flat)}$. If we put $H_k = Tor_k^A(R,R)$, then we have:*

$$\Gamma_k^A = \bigoplus_{i=n+1}^{k-n+1} H_i \otimes H_{k-i+2} \oplus \bigoplus_{i=n+1}^{k-n} Tor_R(H_i, H_{k+1-i}) \quad , 2n \leqslant k \leqslant 3n-1$$

$$\Gamma_{3n}^A = \bigoplus_{i=n+1}^{2n+1} H_i \otimes H_{3n-i+2} \oplus \frac{H_{n+1} \otimes H_{n+1} \otimes H_{n+1}}{(\operatorname{Im} b_{2n+2}) \otimes H_{n+1} + H_{n+1} \otimes (\operatorname{Im} b_{2n+2})}$$
$$\oplus \bigoplus_{i=n+1}^{2n} Tor_R(H_i, H_{3n+1-i})$$

$$\Gamma_k^A = 0, \qquad k \leqslant 2n-1.$$

Proof. Recall that Γ_* is defined, for each $i \geqslant 2$, by the formula:

$$\Gamma_i = \ker(H_i(T(V_{\leqslant i})) \to V_i), \tag{5.1}$$

where $(T(V), \delta)$ is a free model of A. Since A is an object of $\mathbf{DGA_n^{3n+2}(flat)}$, we can choose $(T(V), \delta)$ such that:

$$V_i = 0 \text{ if } i \leqslant n-1 \text{ and } i \geqslant 3n+1. \tag{5.2}$$

Now we filter the dga $(T(V), \delta)$ by setting :

$$F^p = \bigoplus_{i \geqslant p} V^{\otimes i},$$

which induces a spectral sequence $(E_{*,*}^{-r}, d_{*,*}^{-r})$ of the first quadrant converging to $H_*(T(V), \delta)$.
According to the relation (5.1), we have

$$\Gamma_k^A = \bigoplus_{i \geqslant 2} E_{i,k-i}^{\infty} \ , \ k \leqslant 3n,$$

from (5.2), we have

$$E^{\infty}_{i,k-i} = E^{-1}_{i,k-i} = H_k(V^{\otimes i}, d) \ , \ k \leqslant 3n-1,$$

so for $k \leqslant 3n-1$ we deduce

$$E^{\infty}_{i,k-i} = 0 \text{ for } i \geqslant 2,$$

and we get

$$\Gamma_k = E^{\infty}_{2,k-2} = H_k(V^{\otimes 2}, d).$$

Finally by the Künneth formula we get

$$\Gamma_k = \overset{k}{\underset{i=0}{\oplus}} H_i \otimes H_{k-i+2} \oplus \overset{k-n}{\underset{i=n+1}{\oplus}} Tor_k(H_i, H_{k+1-i}), k \leqslant 3n-1.$$

Since $H_i = 0$, $i \leqslant n$ then we deduce

$$\Gamma_k = 0 \ , \ i \leqslant 2n-1.$$

For $k = 3n$, we remark that $E^{\infty}_{i,3n-i} = 0 \ , \ i \geqslant 4$ because $E^{-1}_{i,3n-i} = H_{3n}(V^{\otimes i}, d)$, so we get

$$\Gamma_{3n} = E^{\infty}_{2,3n-2} \oplus E^{\infty}_{3,3n-3}.$$

Observe that $E^{\infty}_{2,3n-2} = E^{-1}_{2,3n-2} = H_{3n}(V^{\otimes 2})$ so by the Künneth formula we get

$$E^{\infty}_{2,3n-2} = \overset{2n+1}{\underset{i=n+1}{\oplus}} H_i \otimes H_{3n-i+2} \oplus \overset{2n}{\underset{i=n+1}{\oplus}} Tor_R(H_i, H_{3n-i+1}).$$

To compute $E^{\infty}_{3,3n-3}$, note that we have

$$E^{-1}_{2,2n-1} \overset{d^{-1}_{2,3n-1}}{\longrightarrow} E^{-1}_{3,3n-3} \to E^{-1}_{4,3n-5} = 0. \tag{5.3}$$

Applying the Künneth formula to (5.3), we get

$$\overset{2n+2}{\underset{i=n+1}{\oplus}} H_i \otimes H_{3n-i+2} \oplus \overset{2n+1}{\underset{i=n+1}{\oplus}} Tor_R(H_i, H_{3n-i+1}) \overset{d^{-1}_{i,3n-1}}{\longrightarrow} H_{n+1} \otimes H_{n+1} \otimes H_{n+1} \to 0$$

and it's easy to see that the differential $d^{-1}_{i,3n-1}$ is identified with the homomorphism:

$$d^{-1}_{2,3n-1} = b_{2n+2} \otimes id_{H_{n+1}} + (-1)^n id_{H_{n+1}} \otimes b_{2n+2},$$

so

$$\mathrm{Im} d^{-1}_{2,3n-1} = \mathrm{Im}\, b_{2n+2} \otimes H_{n+1} + H_{n+1} \otimes \mathrm{Im}\, b_{2n+2}.$$

Finally we get

$$E^{\infty}_{3,3n-3} = \frac{H_{n+1} \otimes H_{n+1} \otimes H_{n+1}}{\mathrm{Im}\, b_{2n+2} \otimes H_{n+1} + H_{n+1} \otimes \mathrm{Im}\, b_{2n+2}},$$

and the theorem is proved. □

As a consequence of this result, we derive the following proposition, which is the version of the Hurewicz theorem in the category $\mathbf{DGA}^{3n+2}_n(\mathbf{flat})$.

Proposition 5. *Let R be a P.I.D and A an object of* $\mathbf{DGA_n^{3n+2}(flat)}$. *The Hurewicz homomorphism:*

$$h_k : H_k(A) \longrightarrow Tor^A_{k+1}(R, R),$$

is an isomorphism for $k \leqslant 2n-1$ and surjective for $k = 2n$.

Proof. One only need apply theorem 13 to the Whitehead exact sequence associated with the dga A. □

That's the same result obtained in ([**10**]) where R is a field of any characteristic. The next corollary is a known geometrical version of proposition 5.

Corollary 4. *Let X be a $(n-1)$-connected $(3n+1)$-dimensional CW-complex. Then the homomorphism:*

$$h_k : H_k(\Omega X, R) \longrightarrow H_{k+1}(X, R),$$

is an isomorphism for $k \leqslant 2n-1$ and surjective for $k = 2n$.

Proof. It suffices to apply proposition 5 to the free dga $(T_X(V), \partial^X)$ which is the Adams-Hilton model of X in [**1**]. Note that since X is $(n-1)$-connected $(3n+1)$-dimensional then the dga $(T_X(V), \partial^X)$ is an object in $\mathbf{DGA_n^{3n+2}(flat)}$. □

5.1. Definition of the category $\Gamma_{\mathrm{n}}^{3\mathrm{n}+2}$

We recall that H_* is said to be n-connected, $(3n+2)$-dimensional graded module if $H_i = 0$ for $i \leqslant n$ and $i \geqslant 3n+3$.

Definition 12. *For each n-connected, $(3n+2)$-dimensional graded module H_* and for each $b_{2n+2} \in Hom(H_{2n+2}, H_{n+1} \otimes H_{n+1})$, we define the graded module $\Gamma_{\leqslant 3n}$ by setting:*

$$\begin{aligned}
\Gamma_k &= 0 \qquad\qquad k \leqslant 2n-1\\
\Gamma_k &= \bigoplus_{i=n+1}^{k-n+1} H_i \otimes H_{k-i+2} \oplus \bigoplus_{i=n+1}^{k-n} Tor_R(H_i, H_{k+1-i}) \qquad 2n \leqslant k \leqslant 3n-1\\
\Gamma_{3n} &= \bigoplus_{i=n+1}^{2n+1} H_i \otimes H_{3n-i+2} \oplus \frac{H_{n+1} \otimes H_{n+1} \otimes H_{n+1}}{\operatorname{Im} b_{2n+2} \otimes H_{n+1} + H_{n+1} \otimes \operatorname{Im} b_{2n+2}} \oplus\\
&\quad \bigoplus_{i=n+1}^{2n} Tor_R(H_i, H_{3n+1-i}).
\end{aligned}$$

$\Gamma_{\leqslant 3n}$ is called the graded module associated with H_* and the homomorphism b_{2n+2}.

Note that by theorem 13, $\Gamma^A_{\leqslant 3n}$ is the graded module associated with $Tor^A_*(R, R)$ and b_{2n+2} given by the Whitehead sequence associated with A.

Definition 13. *For each graded homomorphism $f_* : H_* \to H'_*$ which makes the following diagram commutative:*

$$\begin{tikzcd} H_{2n+2} \arrow[r, "b_{2n+2}"] \arrow[d, "f_{2n+2}"] & H_{n+1} \otimes H_{n+1} \arrow[d, "f_{n+1} \otimes f_{n+1}"] \\ H'_{2n+2} \arrow[r, "b'_{2n+2}"] & H'_{n+1} \otimes H'_{n+1} \end{tikzcd}$$

we define the homomorphisms $\gamma_{\leqslant 3n}$ by setting:

$$\begin{aligned} \gamma_k &= 0 && k \leqslant 2n-1 \\ \gamma_k &= \overset{k-n+1}{\underset{i=n+1}{\oplus}} f_i \otimes f_{k-i+2} \oplus \overset{k-n}{\underset{i=n+1}{\oplus}} Tor_R(f_i, f_{k+1-i}) && 2n \leqslant k \leqslant 3n-1 \\ \gamma_{3n} &= \overset{2n+1}{\underset{i=n+1}{\oplus}} f_i \otimes f_{3n-i+2} \oplus \overline{f_{n+1}}^{\otimes 3} \oplus \overset{2n}{\underset{i=n+1}{\oplus}} Tor_R(f_i, f_{3n+1-i}), \end{aligned}$$

where:

$$\overline{f_{n+1}}^{\otimes 3} : \frac{H_{n+1} \otimes H_{n+1} \otimes H_{n+1}}{Imb_{2n+2} \otimes H_{n+1} + H_{n+1} \otimes \operatorname{Im} b_{2n+2}} \to \frac{H'_{n+1} \otimes H'_{n+1} \otimes H'_{n+1}}{\operatorname{Im} b'_{2n+2} \otimes H'_{n+1} + H'_{n+1} \otimes \operatorname{Im} b'_{2n+2}}.$$

γ_* is called the graded homomorphism associated with f_*.

Proposition 6. *Let $\alpha : A \to B$ be a morphism in $\mathbf{DGA_n^{3n+2}(flat)}$, then $\gamma_*^\alpha : \Gamma_*^A \to \Gamma_*^B$ is the graded homomorphism associated with $Tor_*^\alpha(R, R)$.*

Proof. The proof is very simple but it's very long. It is just a simple computation and verification (see [**11**]) □

Definition 14. *The category $\mathbf{\Gamma_n^{3n+2}}$ is defined as follows:*
***Object:** a collection $(H_*, (b_{i+2}, \pi_i)_{2n \leqslant i \leqslant 3n})$ such that:*
H_* is an n-connected, $(3n+2)$-dimensional graded module.
For each $2n \leqslant i \leqslant 3n$, b_{i+2} is a homomorphism of modules $H_{i+2} \to \Gamma_i$ where $\Gamma_{\leqslant 3n}$ is the graded module associated with H_*.
For each $2n \leqslant i \leqslant 3n$, $\pi_i \in \operatorname{Ext}(H_{i+1}, \operatorname{coker} b_{i+2})$.

***Morphism:** a morphism between two objects $(H_*, (b_{i+2}, \pi_i))_{2n \leqslant i \leqslant 3n}$ and $(H'_*, (b'_{i+2}, \pi'_i))_{2n \leqslant i \leqslant 3n}$ is a graded homomorphism $f_* : H_* \to H'_*$ satisfying the conditions:*

$$f_{i+2} \circ b'_{i+2} = \gamma_i \circ b_{i+2}, \qquad (f_{i+1})^*(\pi'_i) = (\overline{\gamma_i})_*(\pi_i) \qquad i \leqslant 3n,$$

where $\gamma_{\leqslant 3n}$ is the homomorphism associated with f_* and where the homomorphisms $(f_{i+1})^*$, $(\overline{\gamma_i})_*$ are given by:

$$\begin{aligned} (\overline{\gamma_i})_* &: \operatorname{Ext}(H_{i+1}, \operatorname{coker} b_{i+2}) \to \operatorname{Ext}(H'_{i+1}, \operatorname{coker} b'_{i+2}) \\ (f_{i+1})^* &: \operatorname{Ext}(H'_{i+1}, \operatorname{coker} b'_{i+2}) \to \operatorname{Ext}(H_{i+1}, \operatorname{coker} b'_{i+2}) \end{aligned}$$

and where $\overline{\gamma_i}$ is induced by γ_i on $\operatorname{coker} b_{i+2}$.

5.2. The functor F_n^{3n+2}

As in the previous section we define the functor:

$$F_n^{3n+2} : \mathbf{\Gamma DGA_n^{3n+2}(flat)}/_{\simeq} \to \mathbf{\Gamma_n^{3n+2}},$$

by setting:

$$F_n^{3n+2}(A) = (Tor_*^A(R,R), (b_{i+2}, \pi_i)_{2n \leqslant i \leqslant 3n})$$
$$F_n^{3n+2}(\{\alpha\}) = Tor_*^{\alpha}(R,R).$$

The main result in this section is the following theorem:

Corollary 5. *Under the condition (4.1), the functor*

$$F_n^{3n+2} : \mathbf{DGA_n^{3n+2}(flat)}/_{\simeq} \to \mathbf{\Gamma_n^{3n+2}},$$

is a "detecting functor "

Proof. The proof is the same as in theorem 9. □

As a corollary we have the result which we can use to compute the number of the homotopy types in the category $\mathbf{DGA_n^{3n+2}(flat)}$.

Theorem 14 (Homotopy classification theorem). *Homotopy types of objects in* $\Gamma\mathbf{DGA_n^{3n+2}(flat)}$ *are in bijection with the proper equivalence classes of tuples* $(b_{3n+2}, \pi_{3n}, ..., b_{2n+2}, \pi_{2n})$ *where* $b_{i+2} \in Hom(H_{i+2}, \Gamma_i)$, $\pi_i \in \mathrm{Ext}(H_{i+1}, \mathrm{coker}\, b'_{i+2})$, $2n \leqslant i \leqslant 3n$ *and where* H_* *is* n*-connected,* $(3n+2)$*-dimensional graded module.*

Recall that $(\Gamma_i)_{i \leqslant 3n}$ is the graded module associated with H_* and b_{2n+2}.

Definition 15. *Two tuples* $(b_{3n+2}, \pi_{3n}, ..., b_{2n+2}, \pi_{2n})$ *and* $(b'_{3n+2}, \pi'_{3n}, ..., b'_{2n+2}, \pi'_{2n})$ *are called proper equivalent if there exists a graded automorphism* $f_{\leqslant 3n} : H_{\leqslant 3n} \cong H_{\leqslant 3n}$ *such that for every* $i \leqslant 3n$*:*

$$(f_{i+1})^*(\pi'_i) = (\overline{\gamma_i})_*(\pi_i), \text{ and } \gamma_i \circ b_{i+2} = b'_{i+2} \circ f_{i+2}.$$

5.3. Examples

We conclude this work by giving some geometric applications to the above theorems.

Proposition 7. *Let* A *an object of* $\mathbf{DGA_n^{3n+2}(flat)}$*. Denote by* $\mathbf{S}_A$ *the set consisting of the dgas* B *satisfying:*

$$Tor_i^B(R,R) = Tor_i^A(R,R) \text{ for every } n \leqslant i \leqslant 3n.$$

Assume that $Tor_i^A(R,R) = 0$ *for each* i *satisfying:*

$$\begin{aligned} i &\leqslant \frac{3n+2}{2} && \text{if } n \text{ is even} \\ i &\leqslant \frac{3n+1}{2} && \text{if } n \text{ is odd} \end{aligned} \tag{5.4}$$

then all objects of the set $\mathbf{S}_A$ *have the same homotopy type.*

Proof. If B is an element of $\mathbf{S}_A$ then from the hypothesis (5.4) and theorem 13 we deduce that the module Γ_i is trivial for all $i \leqslant 3n$. Hence we get:

$$\mathbf{F}_n^{3n+2}(B) = (Tor_*^B(R,R),0,0) = (Tor_*^A(R,R),0,0) = \mathbf{F}_n^{3n+2}(A).$$

The relation (4.1) is trivially satisfied . So by corollary 5, $\mathbf{F}_n^{3n+2}$ is a detecting functor, thus B and A have the same homotopy type. □

Proposition 8. *Let $n \in \mathbf{N}$ and X be a CW-complex such that $H_k(X,R) = 0$ for each k satisfying:*

$$k \leqslant \frac{3n+2}{2} \quad \text{if } n \text{ is even}$$
$$k \leqslant \frac{3n+1}{2} \quad \text{if } n \text{ is odd}$$
$$k \geqslant 3n+3$$

If we put $H_k = H_k(X,R)$ then we have :

$$H_i(\Omega X,R) = \underset{k}{\oplus} H_k^{\otimes q} \qquad \text{if } i = q(k-1)$$
$$H_i(\Omega X,R) = \underset{k}{\oplus}(\underset{s+t=q}{\oplus} Tor_R(H_k^{\otimes s}, H_k^{\otimes t})) \qquad \text{if } i = q(k-1)+1$$
$$H_i(\Omega X,R) = 0 \qquad \text{otherwise.}$$

Proof. Let $M(H_k,k)$ be a Moore space and let :

$$Y = \underset{k}{\vee} M(H_k,k)$$

be the wedge of the spaces $M(H_k,k)$ and let $A(X)$ and $A(Y)$ be the Adams-Hilton models associated respectively to X and Y. We recall that we have:

$$Tor_*^{A(X)}(R,R) = H_*(X,R) \text{ and } Tor_*^{A(Y)}(R,R) = H_*(Y,R).$$

Since $H_*(X,R) = H_*(Y,R)$, the relation implies that $A(Y)$ is an object of the set $\Sigma_{A(X)}$ in proposition 7, so that $A(X)$ and $A(Y)$ have the same homotopy type. We deduce that:

$$H_*(A(X)) \cong H_*(A(Y)), \tag{5.5}$$

since $A(X)$ and $A(Y)$ are respectively the Adams-Hilton models of X and Y. Then:

$$H_*(A(X)) = H_*(\Omega X,R) \text{ and } H_*(A(Y)) = H_*(\Omega Y,R),$$

so according to (5.5) we get:

$$H_*(\Omega X,R) \cong H_*(\Omega \underset{k}{\vee} M(H_k,k),R) \cong \underset{k}{\oplus} H_*(M(H_k,k),R).$$

Now the next lemma allows us to achieve the proof. □

Lemma 3. *The graded algebra $H_*(\Omega M(H_k,k),R)$ is given by:*

$$H_i(\Omega M(H_k,k),R) = H_k^{\otimes q} \qquad \text{if } i = q(k-1)$$
$$H_i(\Omega M(H_k,k),R) = \underset{s+t=q}{\oplus} Tor_R(H_k^{\otimes s}, H_k^{\otimes t}) \qquad \text{if } i = q(k-1)+1$$
$$H_i(\Omega M(H_k,k),R) = 0 \qquad \text{otherwise.}$$

Proof. Let $A(M(H_k, k))$ be an Adams-Hilton model of the Moore space $M(H_k, k)$. The Whitehead sequence associated with $A(M(H_k, k))$ is:

$$\to H_{i+2}(M(H_k,k),R) \to \Gamma_i \to H_i(\Omega M(H_k,k),R) \to H_{i+1}(M(H_k,k),R) \to$$

Since we have:

$$H_{i+2}(M(H_k,k),R) = 0 \text{ unless } i = k-2$$

it follows that

$$H_i(\Omega M(H_k,k),R) \cong \Gamma_i.$$

According to section (3), we can realize the object $\Gamma(\{A(M(H_k,k)\})$ by the free dga $T(V_k, V_{k-1}, d)$ where $V_k \overset{d}{\rightarrowtail} V_{k-1} \twoheadrightarrow H_k$ is a free resolution of H_k.
Since the module $\Gamma_i = \ker\ (H_i(T(V_k, V_{k-1}, d)) \longrightarrow V_i)$ and since the differential d is linear, the Künneth formula allows us to write:

$$\begin{array}{ll} \Gamma_i = H_k^{\otimes q} & \text{if } \ i = q(k-1) \\ \Gamma_i = \bigoplus\limits_{s+t=q} Tor_R(H_k^{\otimes s}, H_k^{\otimes t}) & \text{if } \ i = q(k-1)+1 \\ \Gamma_i = 0 & \text{otherwise} \end{array}$$

and the proof is achieved. □

Example 1. *Let X be a CW-complex 1-connected having the following homology groups:*

$$\begin{array}{l} H_2(X,\mathbb{Z}) = \mathbb{Z}_2,\ H_3(X,\mathbb{Z}) = \mathbb{Z}_3,\ H_4(X,\mathbb{Z}) = \mathbb{Z}_3, \\ H_5(X,\mathbb{Z}) = \mathbb{Z}, H_i(X,\mathbb{Z}) = 0 \ \textit{otherwise}. \end{array}$$

How many homotopy type of the dga $C_*(\Omega X)$ exist?

According to theorem 13 we have:

$$\Gamma_2 = H_2 \otimes H_2 = \mathbb{Z}_2$$

$$\Gamma_3 = H_2 \otimes H_3 \oplus H_3 \otimes H_2 \oplus \frac{H_2 \otimes H_2 \otimes H_2}{\operatorname{Im} b_4 \otimes H_2 + H_2 \otimes \operatorname{Im} b_4} = \mathbb{Z}_2.$$

We begin by computing the homomorphisms b_4. We have:

$$Hom(H_4(X), \Gamma_2) = Hom(\mathbb{Z}_3, \mathbb{Z}_2) = 0.$$

So we deduce that $b_4 = 0$ and:

$$\operatorname{Ext}(H_3, \operatorname{coker} b_4) = \operatorname{Ext}(\mathbb{Z}_3, \mathbb{Z}_2) = 0.$$

Then we only find the trivial extension $\pi_2 = 0$. Now we have:

$$b_5 \in Hom(H_5, \Gamma_3) = Hom(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2,$$

so we find two homomorphisms $b_5^{(0)} = 0$ and $b_5^{(1)} = 1$.
For $b_5^{(0)} = 0$ we get $\operatorname{Ext}(H_4, \operatorname{coker} b_5^{(0)}) = \operatorname{Ext}(\mathbb{Z}_3,\mathbb{Z}_2) = 0$ so in this case we also find the trivial extension $\pi_3^{(0)} = 0$. For $b_5^{(1)} = 1$ then $b_5^{(1)}$ is onto and we deduce that $\operatorname{Ext}(H_4, \operatorname{coker} b_5^{(1)}) = 0$ and also we get the trivial extension $\pi_3^{(1)} = 0$.

Then we find two Γ-systems $(0,0,0,0)$, $(1,0,0,0)$ which are obviously not isomorphic in their category.

Now since by computation we have:

$$\mathrm{Ext}\left(\frac{H_4}{\ker b_4}, \operatorname{coker} b_5^{(0)}\right) = \mathrm{Ext}(\mathbb{Z}_3, \mathbb{Z}_2) = 0$$

$$\mathrm{Ext}\left(\frac{H_4}{\ker b_4}, \operatorname{coker} b_5^{(1)}\right) = \mathrm{Ext}(\mathbb{Z}_3, 0) = 0$$

$$\mathrm{Ext}\left(\frac{H_5}{\ker b_5^{(0)}}, \operatorname{coker} b_6\right) = \mathrm{Ext}(0, \mathbb{Z}_2) = 0$$

$$\mathrm{Ext}\left(\frac{H_5}{\ker b_5^{(1)}}, \operatorname{coker} b_6\right) = \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_2) = 0$$

$$\mathrm{Ext}\left(\frac{H_{n+1}}{\ker b_{n+1}}, \operatorname{coker} b_{n+2}\right) = \mathrm{Ext}(0, \mathbb{Z}_2) = 0 \quad \text{for } n \geqslant 5,$$

then condition (4.1) is satisfied. Therefore, according to the theorem 14 of classification of homotopy types, we have two homotopy types having the above homology groups.

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