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ON THE INTEGRABILITY OF A K-CONFORMAL KILLING EQUATION IN A KAEHLERIAN MANIFOLD

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Abstract. We show that a necessary and sufficient condition in order that K- conformal Killing equation is completely integrable is that the Kaehlerian manifold K^{2m} $(m > 2)$ is of constant holomorphic sectional curvature.

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§1. Introduction. Let M^n be an n-dimensional Riemannian manifold. Denote respectively by g_{ji}, $R_{kji}{}^r$, $R_{ji} = R_{rji}{}^r$ and $R = g^{ji}R_{ji}$ the metric, the curvature tensor, the Ricci tensor, and the scalar curvature of Reimannian manifold in terms of local coordinates $\{x^i\}$, where Latin indices run over the range $\{1, 2, \ldots, n\}$.

In M^n, a p-form u is said to be Killing, if it satisfies the Killing-Yano's equation:

$$\nabla_{i_0} u_{i_1 i_2 \ldots i_p} + \nabla_{i_1} u_{i_0 i_2 \ldots i_p} = 0,$$

where ∇ denotes the operator of the Riemannian covarient derivative.

The following theorem is well known:

Theorem A ([1], [5]). *A necessary and sufficient condition in order that the Killing-Yano's equation is completely integrable is that the Riemannian manifold M^n $(n > 2)$ is space of constant curvature.*

A Riemannian manifold M^n is called a Sasakian manifold if it admits a unit special Killing 1-form η with constant 1 such that

$$\nabla_i \phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \quad \phi_{kj} = \nabla_k \eta_j.$$

In a Sasakian manifold M^n, a 1-form u is called D-Killing if it satisfies the D-Killing equation of type α:

$$\nabla_j u_i + \nabla_i u_j = -2\alpha u_r (\phi_j{}^r \eta_i + \phi_i{}^r \eta_j),$$

where α is constant.

Then it is known that

Theorem B ([8]). *A necessary and sufficient condition in order that the D-Killing equation of type α is completely integrable is that the Sasakian manifold M^n $(n > 3)$ is a space of constant ϕ-holomorphic sectional curvature with $H = 1 - 4\alpha$.*

We consider the analogy of Theorem A and B in a Kaehlerian manifold, namely, the purpose of this paper is to prove the followings.

Theorem 3.1. *If there exists (locally) a K-conformal Killing 2-form u_{ji} satisfying $u_{ji}(p) = C_{ji}$ for any point p of a Kaehlerian manifold K^{2m} $(m > 2)$ and any constants $C_{ji}(= -C_{ij})$, then K^{2m} is of constant holomorphic sectional curvature.*

Theorem 4.1. *A necessary and sufficient condition in order that K-conformal Killing equation is completely integrable is that the Kaehlerian manifold K^{2m} $(m > 2)$ is of constant holomorphic sectional curvature.*

We shall recall a K-conformal Killing 2-form and an HP-Killing 1-form in §2. In §3 we shall give the proof of Theorem 3.1. Moreover §4 will be devoted to the integrability condition of the K-conformal Killing equation, that is, the proof of Theorem 4.1 will be given.

§2. Pleriminaries. We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential p-form

$$u = \frac{1}{p!} u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with skew symmetric coefficients $u_{i_1 \dots i_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given by

$$(du)_{i_1 \dots i_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_{i_a} u_{i_1 \dots \hat{i}_a \dots i_{p+1}},$$

$$(\delta u)_{i_2 \dots i_p} = -\nabla^r u_{r i_2 \dots i_p},$$

where $\nabla^h = g^{hj} \nabla_j$ and $\hat{i}_a$ means i_a to be deleted.

A Kaehlerian manifold K^{2m} with metric g of real dimension $n = 2m$ is M^n admitting a parallel tensor field $\phi_i{}^h$ such that

$$\phi_i{}^r \phi_r{}^j = -\delta_i^j, \quad \phi_{ji} = -\phi_{ij},$$

where we put $\phi_{ji} = \phi_j{}^r g_{ri}$. A Kaehlerian manifold is called of constant holomorphic sectional curvature if the curvature tensor satisfies the following equation:

$$R_{kji}{}^r = \frac{R}{n(n+2)} (g_{ji} \delta_k^r - g_{ki} \delta_j^r + \phi_{ji} \phi_k{}^r - \phi_{ki} \phi_j{}^r - 2\phi_{kj} \phi_i{}^r).$$

In the sequel, we shall consider a Kaehlerian manifold K^{2m} and assume that $m > 1$.

Now, we want to recall the operator for differential forms in K^{2m}. Denote by $\mathcal{F}^p$ the set of all p-forms on K^{2m}. The operators $\Gamma : \mathcal{F}^p \to \mathcal{F}^{p+1}$ and $\Phi : \mathcal{F}^p \to \mathcal{F}^p$ are defined by

$$(\Gamma u)_{i_0 \dots i_p} = \sum_{a=0}^{p} (-1)^a \phi_{i_a}{}^r \nabla_r u_{i_0 \dots \hat{i}_a \dots i_p},$$

$$(\Phi u)_{i_1 \dots i_p} = \sum_{a=1}^{p} \phi_{i_a}{}^r u_{i_1 \dots r \dots i_p}$$

for any p-form u. For 0-form u_0, we defined $\Phi u_0 = 0$. In K^{2m}, the curvature tensor and the Ricci tensor satisfy ([2])

$$\phi_h{}^r R_{rijk} = \phi_i{}^r R_{rhjk}, \quad R_{hijk} = \phi_h{}^r \phi_i{}^s R_{rsjk}, \tag{2.1}$$

$$\frac{1}{2} \phi^{rs} R_{rsji} = \phi^{rs} R_{rjsi} = -S_{ji} = S_{ij}. \tag{2.2}$$

where we put $S_{ji} = \phi_j{}^r R_{ri}$.

A 2-form u will be called K-conformal Killing, if it satisfies

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\tilde{\rho}_k \phi_{ji} + \tilde{\rho}_j \phi_{ki}), \tag{2.3}$$

where we put

$$\rho_i = -\frac{1}{n+2} (\delta u)_i, \quad \tilde{\rho}_i = (\Phi \rho)_i.$$

It is easy to see from (2.3) that

$$\phi_k{}^r \nabla_r u_{ji} + \nabla_k (\Phi u)_{ji} = (\Gamma u)_{kji} - 2\rho_k \phi_{ji} - \rho_j \phi_{ki} + \rho_i \phi_{kj} - \tilde{\rho}_j g_{ki} + \tilde{\rho}_i g_{kj}. \tag{2.4}$$

By interchanging alternatively indices as $k \to j \to i$ at (2.4) and adding all together, we have

$$d\Phi u = 2\Gamma u. \tag{2.5}$$

Let $v^{\#}$ be the vector field obtained from 1-form $v = v_i dx^i$ by virtue of the metric tensor field g. Then we have $v^{\#} = (v^i)$. We denote $\mathcal{L}(v^{\#})$ by the Lie derivative with respect to the vector field $v^{\#}$. Then we have

$$\mathcal{L}(v^{\#})\left\{ {i \atop kj} \right\} = \nabla_k \nabla_j v^i + v^r R_{rkj}{}^i, \tag{2.6}$$

where $\left\{ {i \atop kj} \right\}$ denotes the Christoffel's symbols. Let $v^{\#}$ be an HP-Killing vector field. Then we have ([10])

$$\mathcal{L}(v^{\#})\left\{ {i \atop kj} \right\} = -\frac{1}{n+2}[(d\delta v)_k \delta_j{}^i + (d\delta v)_j \delta_k{}^i - (\Phi d\delta v)_k \phi_j{}^i - (\Phi d\delta v)_j \phi_k{}^i],$$

which together with (2.6) satisfies that

$$\nabla_k \nabla_j v_i + v_r R^r{}_{kji} = -\frac{1}{n+2}[(d\delta v)_k g_{ji} + (d\delta v)_j g_{ki} - (\Phi d\delta v)_k \phi_{ji} - (\Phi d\delta v)_j \phi_{ki}].$$

In the following, any 1-form v satisfying the above equation is said to be an HP-Killing 1-form.

For an HP-Killing 1-form and a K-conformal Killing 2-form, it is known that the following holds:

Theorem C ([9]). *In a Kaehlerian manifold K^{2m}, for an HP-Killing 1-form v, $d\Phi v$ is a closed K-conformal Killing 2-form.*

§3. Proof of Theorem 3.1. At first, for a K-conformal Killing 2-form u_{ji}, the following equations hold ([7]):

$$\begin{aligned} & R_{jkl}{}^r u_{ir} + R_{ilk}{}^r u_{jr} + R_{lij}{}^r u_{kr} + R_{kji}{}^r u_{lr} \\ &\quad = (\rho_{ij} + \rho_{ji}) g_{kl} - (\rho_{ik} + \rho_{ki}) g_{jl} + (\rho_{kl} + \rho_{lk}) g_{ij} \\ &\quad - (\rho_{jl} + \rho_{lj}) g_{ik} + (\tilde\rho_{ij} - \tilde\rho_{ji})\phi_{kl} - (\tilde\rho_{ik} - \tilde\rho_{ki})\phi_{jl} \\ &\quad + (\tilde\rho_{kl} - \tilde\rho_{lk})\phi_{ij} - (\tilde\rho_{jl} - \tilde\rho_{lj})\phi_{ik} + 2(\tilde\rho_{kj} - \tilde\rho_{jk})\phi_{il} \\ &\quad + 2(\tilde\rho_{il} - \tilde\rho_{li})\phi_{kj}, \end{aligned} \tag{3.1}$$

$$\rho_{ij} + \rho_{ji} = \frac{1}{(n-2)(n+4)}[(n+1)(R_i{}^r u_{jr} + R_j{}^r u_{ir}) - 3(\phi_i{}^e S_j{}^r + \phi_j{}^e S_i{}^r) u_{er}], \tag{3.2}$$

$$\tilde\rho_{ij} - \tilde\rho_{ji} = \frac{1}{(n+2)(n+4)}[(n+3)(S_i{}^r u_{rj} + S_j{}^r u_{ir}) + (\phi_i{}^e R_j{}^r - \phi_j{}^e R_i{}^r) u_{er}], \tag{3.3}$$

where we put

$$\rho_{ji} = \nabla_j \rho_i, \qquad \tilde\rho_{ji} = \nabla_j \tilde\rho_i.$$

Because of (3.2) and (3.3), equation (3.1) is rewritten as follows:

$$R_{jkl}{}^r u_{ir} + R_{ilk}{}^r u_{jr} + R_{lij}{}^r u_{kr} + R_{kji}{}^r u_{lr} = A_{lkji} + B_{lkji}, \tag{3.4}$$

where we put

$$\begin{aligned} A_{lkji} = & \frac{1}{(n-2)(n+4)}[(n+1)(R_i{}^r u_{jr} + R_j{}^r u_{ir}) g_{kl} - 3(\phi_i{}^e S_j{}^r + \phi_j{}^e S_i{}^r) g_{kl} u_{er} \\ & - (n+1)(R_i{}^r u_{kr} + R_k{}^r u_{ir}) g_{jl} + 3(\phi_i{}^e S_k{}^r + \phi_k{}^e S_i{}^r) g_{jl} u_{er} \\ & + (n+1)(R_k{}^r u_{lr} + R_l{}^r u_{kr}) g_{ij} - 3(\phi_k{}^e S_l{}^r + \phi_l{}^e S_k{}^r) g_{ij} u_{er} \\ & - (n+1)(R_j{}^r u_{lr} + R_l{}^r u_{jr}) g_{ik} + 3(\phi_j{}^e S_l{}^r + \phi_l{}^e S_j{}^r) g_{ik} u_{er}], \end{aligned}$$

$$
\begin{aligned}
B_{lkji} =& \frac{1}{(n+2)(n+4)}[(n+3)(S_i{}^r u_{rj} + S_j{}^r u_{ir})\phi_{kl} + (\phi_i{}^e R_j{}^r - \phi_j{}^e R_i{}^r)\phi_{kl} u_{er} \\
& - (n+3)(S_i{}^r u_{rk} + S_k{}^r u_{ir})\phi_{jl} - (\phi_i{}^e R_k{}^r - \phi_k{}^e R_i{}^r)\phi_{jl} u_{er} \\
& + (n+3)(S_k{}^r u_{rl} + S_l{}^r u_{kr})\phi_{ij} + (\phi_k{}^e R_l{}^r - \phi_l{}^e R_k{}^r)\phi_{ij} u_{er} \\
& - (n+3)(S_j{}^r u_{rl} + S_l{}^r u_{jr})\phi_{ik} - (\phi_j{}^e R_l{}^r - \phi_l{}^e R_j{}^r)\phi_{ik} u_{er} \\
& + 2(n+3)(S_k{}^r u_{rj} + S_j{}^r u_{kr})\phi_{il} + 2(\phi_k{}^e R_j{}^r - \phi_j{}^e R_k{}^r)\phi_{il} u_{er} \\
& + 2(n+3)(S_i{}^r u_{rl} + S_l{}^r u_{ir})\phi_{kj} + 2(\phi_i{}^e R_l{}^r - \phi_l{}^e R_i{}^r)\phi_{kj} u_{er}].
\end{aligned}
$$

By our assumption, we have from (3.4)

$$
\begin{aligned}
& R_{jkl}{}^r \delta_i{}^e + R_{ilk}{}^r \delta_j{}^e + R_{lij}{}^r \delta_k{}^e + R_{kji}{}^r \delta_l{}^e \\
& \quad - R_{jkl}{}^e \delta_i{}^r - R_{ilk}{}^e \delta_j{}^r - R_{lij}{}^e \delta_k{}^r - R_{kji}{}^e \delta_l{}^r \\
& \quad = P_{lkji}{}^{re} + Q_{lkji}{}^{re},
\end{aligned}
$$

where we put

$$
\begin{aligned}
P_{lkji}{}^{re} =& \frac{1}{(n-2)(n+4)}[(n+1)(R_i{}^r \delta_j{}^e + R_j{}^r \delta_i{}^e - R_i{}^e \delta_j{}^r - R_j{}^e \delta_i{}^r) g_{kl} \\
& - 3(\phi_i{}^e S_j{}^r + \phi_j{}^e S_i{}^r - \phi_i{}^r S_j{}^e - \phi_j{}^r S_i{}^e) g_{kl} \\
& - (n+1)(R_i{}^r \delta_k{}^e + R_k{}^r \delta_i{}^e - R_i{}^e \delta_k{}^r - R_k{}^e \delta_i{}^r) g_{jl} \\
& + 3(\phi_i{}^e S_k{}^r + \phi_k{}^e S_i{}^r - \phi_i{}^r S_k{}^e - \phi_k{}^r S_i{}^e) g_{jl} \\
& + (n+1)(R_k{}^r \delta_l{}^e + R_l{}^r \delta_k{}^e - R_k{}^e \delta_l{}^r - R_l{}^e \delta_k{}^r) g_{ij} \\
& - 3(\phi_k{}^e S_l{}^r + \phi_l{}^e S_k{}^r - \phi_k{}^r S_l{}^e - \phi_l{}^r S_k{}^e) g_{ij} \\
& - (n+1)(R_j{}^r \delta_l{}^e + R_l{}^r \delta_j{}^e - R_j{}^e \delta_l{}^r - R_l{}^e \delta_j{}^r) g_{ik} \\
& + 3(\phi_j{}^e S_l{}^r + \phi_l{}^e S_j{}^r - \phi_j{}^r S_l{}^e - \phi_l{}^r S_j{}^e) g_{ik}],
\end{aligned}
$$

$$
\begin{aligned}
Q_{lkji}{}^{re} =& \frac{1}{(n+2)(n+4)}[(n+3)(\delta_i{}^e S_j{}^r - \delta_j{}^e S_i{}^r - \delta_i{}^r S_j{}^e + \delta_j{}^r S_i{}^e)\phi_{kl} \\
& + (\phi_i{}^e R_j{}^r - \phi_j{}^e R_i{}^r - \phi_i{}^r R_j{}^e + \phi_j{}^r R_i{}^e)\phi_{kl} \\
& - (n+3)(\delta_i{}^e S_k{}^r - \delta_k{}^e S_i{}^r - \delta_i{}^r S_k{}^e + \delta_k{}^r S_i{}^e)\phi_{jl} \\
& - (\phi_i{}^e R_k{}^r - \phi_k{}^e R_i{}^r - \phi_i{}^r R_k{}^e + \phi_k{}^r R_i{}^e)\phi_{jl} \\
& + (n+3)(\delta_k{}^e S_l{}^r - \delta_l{}^e S_k{}^r - \delta_k{}^r S_l{}^e + \delta_l{}^r S_k{}^e)\phi_{ij} \\
& + (\phi_k{}^e R_l{}^r - \phi_l{}^e R_k{}^r - \phi_k{}^r R_l{}^e + \phi_l{}^r R_k{}^e)\phi_{ij} \\
& - (n+3)(\delta_j{}^e S_l{}^r - \delta_l{}^e S_j{}^r - \delta_j{}^r S_l{}^e + \delta_l{}^r S_j{}^e)\phi_{ik} \\
& - (\phi_j{}^e R_l{}^r - \phi_l{}^e R_j{}^r - \phi_j{}^r R_l{}^e + \phi_l{}^r R_j{}^e)\phi_{ik} \\
& + 2(n+3)(\delta_k{}^e S_j{}^r - \delta_j{}^e S_k{}^r - \delta_k{}^r S_j{}^e + \delta_j{}^r S_k{}^e)\phi_{il} \\
& + 2(\phi_k{}^e R_j{}^r - \phi_j{}^e R_k{}^r - \phi_k{}^r R_j{}^e + \phi_j{}^r R_k{}^e)\phi_{il} \\
& + 2(n+3)(\delta_i{}^e S_l{}^r - \delta_l{}^e S_i{}^r - \delta_i{}^r S_l{}^e + \delta_l{}^r S_i{}^e)\phi_{kj} \\
& + 2(\phi_i{}^e R_l{}^r - \phi_l{}^e R_i{}^r - \phi_i{}^r R_l{}^e + \phi_l{}^r R_i{}^e)\phi_{kj}].
\end{aligned}
$$

Transvecting the above equation with $\delta_e{}^l$, by virtue of the Bianchi's identity we have

$$
\begin{aligned}
(n-1)R_{kji}{}^r =& \frac{1}{(n^2-4)(n+4)}[(n^3+2n^2-4n+4) \\
& \times (R_{ij}\delta_k{}^r - R_{ik}\delta_j{}^r + g_{ij}R_k{}^r - g_{ik}R_j{}^r) \\
& - (n^3-12n+4)(\phi_{ij}S_k{}^r - \phi_{ik}S_j{}^r + 2\phi_{kj}S_i{}^r) \\
& - 4(n+1)(S_{ij}\phi_k{}^r - S_{ik}\phi_j{}^r + 2S_{kj}\phi_i{}^r) \\
& - (n+1)(n+2)R(g_{ij}\delta_k{}^r - g_{ik}\delta_j{}^r) \\
& - (n-2)R(\phi_{ij}\phi_k{}^r - \phi_{ik}\phi_j{}^r + 2\phi_{kj}\phi_i{}^r)].
\end{aligned}
\tag{3.5}
$$

If we transvect (3.5) with $\phi_h{}^k \phi_l{}^j$ and regard to (2.1) and (2.2), then we get

$$\begin{aligned}(n-1)R_{hli}{}^r = \frac{1}{(n^2-4)(n+4)}[&(n^3+2n^2-4n+4)\\ &\times (S_{li}\phi_h{}^r - S_{hi}\phi_l{}^r + \phi_{li}S_h{}^r - \phi_{hi}S_l{}^r)\\ &-(n^3-12n+4)(g_{ih}R_l{}^r - g_{il}R_h{}^r + 2\phi_{hl}S_i{}^r)\\ &-4(n+1)(R_{ih}\delta_l{}^r - R_{il}\delta_h{}^r + 2S_{hl}\phi_i{}^r)\\ &-(n+1)(n+2)R(\phi_{li}\phi_h{}^r - \phi_{hi}\phi_l{}^r)\\ &-(n-2)R(g_{ih}\delta_l{}^r - g_{il}\delta_h{}^r + 2\phi_{hl}\phi_i{}^r)].\end{aligned}$$

Furthermore, contracting the above equation with $\delta_r{}^h$, if $m > 2$ we obtain

$$R_{li} = \frac{R}{n}g_{li},$$

that is, K^{2m} is an Einstein manifold. Substituting this into (3.5), we can easily get

$$R_{kji}{}^r = \frac{R}{n(n+2)}(g_{ji}\delta_k{}^r - g_{ki}\delta_j{}^r + \phi_{ji}\phi_k{}^r - \phi_{ki}\phi_j{}^r - 2\phi_{kj}\phi_i{}^r),$$

which means that K^{2m} is of constant holomorphic sectional curvature. Consequently, we complete the proof of Theorem 3.1.

From Theorem 3.1 and C we can get

Cororally. *If there exists (locally) a 2-form $(d\Phi v)_{ji}$ obtained from an HP-Killing 1-form v satisfying $(d\Phi v)_{ji}(p) = C_{ji}$ for any point p of a Kaehlerian manifold K^{2m} $(m > 2)$ and any constants $C_{ji}(= -C_{ij})$, then K^{2m} is of constant holomorphic sectional curvature.*

§4. **Proof of Theorem 4.1.** The purpose of this section is to prove Theorem 4.1 stated in §1. Namely we shall show the converse of Theorem 3.1 is true.

In the first place, it is known that

Theorem D ([5]). *If a Kaehlerian manifold K^{2m} is an Einstein manifold, the associated 1-form ρ_i of K-conformal Killing 2-form is Killing, namely,*

$$\nabla_j\rho_i + \nabla_i\rho_j = 0.$$

Moreover, by virtue of (3.3) we find the following.

Lemma 4.1. *If a Kaehlerian manifold K^{2m} is an Einstein manifold, then we have for the associated 1-form ρ of K-conformal Killing 2-form u*

$$d\Phi\rho = \frac{R}{n(n+2)}\Phi u,$$

hence, Φu is closed.

In a Kaehlerian manifold K^{2m} we consider the K-conformal Killing equation as a system of partial differential equations of unknown function u_{ji}. This system is equivalent to the following system of partial differential equations with unknown functions $u_{ji}(= -u_{ij})$ and $u_{kji}(= -u_{kij})$:

$$u_{kji} + u_{jki} = 2\rho_i g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\tilde{\rho}_k\phi_{ji} + \tilde{\rho}_j\phi_{ki}), \tag{4.1}$$

$$\nabla_k u_{ji} = u_{kji}, \tag{4.2}$$

$$\begin{aligned}\nabla_l u_{kji} =& \frac{1}{2}(R_{jkl}{}^r u_{ir} + R_{kil}{}^r u_{jr} + R_{ijl}{}^r u_{kr}) \\ &+ \frac{1}{2}(\rho_{ki} - \rho_{ik})g_{jl} + \frac{1}{2}(\rho_{jk} - \rho_{kj})g_{li} \\ &+ \frac{1}{2}(\rho_{ij} - \rho_{ji})g_{kl} - \rho_{lj}g_{ki} + \rho_{li}g_{kj} \\ &+ \frac{1}{2}(\tilde{\rho}_{kj} - \tilde{\rho}_{jk})\phi_{li} + \frac{1}{2}(\tilde{\rho}_{ji} - \tilde{\rho}_{ij})\phi_{lk} \\ &+ \frac{1}{2}(\tilde{\rho}_{ik} - \tilde{\rho}_{ki})\phi_{lj} + (\tilde{\rho}_{il} - \tilde{\rho}_{li})\phi_{kj} \\ &+ (\tilde{\rho}_{lj} - \tilde{\rho}_{jl})\phi_{ki} + (2\tilde{\rho}_{lk} + \tilde{\rho}_{kl})\phi_{ji}.\end{aligned} \tag{4.3}$$

We shall show that the system is completely integrable if K^{2m} is a space of constant holomorphic sectional curvature.

From our assumption, we can replace (4.3) by the following equation:

$$\begin{aligned}\nabla_l u_{kji} = \frac{R}{n(n+2)}[& g_{lk}u_{ij} + g_{lj}u_{ki} + g_{li}u_{jk} \\ &+ \phi_{lk}(\Phi u)_{ji} + \phi_{lj}(\Phi u)_{ik} + \phi_{li}(\Phi u)_{kj} \\ &- \phi_{ji}(\Phi u)_{lk} - \phi_{ik}(\Phi u)_{lj} - \phi_{kj}(\Phi u)_{li} \\ &- \phi_{jk}\phi_l{}^r u_{ir} - \phi_{ki}\phi_l{}^r u_{jr} - \phi_{ij}\phi_l{}^r u_{kr}] \\ + g_{lk}\rho_{ij} + g_{lj}\rho_{ki} &+ g_{li}\rho_{jk} + g_{ki}\rho_{jl} + g_{kj}\rho_{li} + 3\tilde{\rho}_{lk}\phi_{ji},\end{aligned} \tag{4.4}$$

where we have used Lemma 4.1 and Theorem D.

The equation obtained from (4.1) by differentiation:

$$\partial_l u_{kji} + \partial_l u_{jki} = \partial_l[2\rho_i g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\tilde{\rho}_k\phi_{ji} + \tilde{\rho}_j\phi_{ki})]$$

is satisfied identically by (4.1), (4.2) and (4.3).

Next, we discuss the integrability condition (4.2):

$$\nabla_l\nabla_k u_{ji} - \nabla_k\nabla_l u_{ji} = -R_{lkj}{}^r u_{ri} - R_{lki}{}^r u_{jr}. \tag{4.5}$$

Taking account that K^{2m} is a space of constant holomorphic sectional curvature, we have

$$\begin{aligned}&- R_{lkj}{}^r u_{ri} - R_{lki}{}^r u_{jr} \\ &= \frac{R}{n(n+2)}[g_{lj}u_{ki} + g_{li}u_{jk} - g_{kj}u_{li} - g_{ki}u_{jl} + \phi_{lj}\phi_k{}^r u_{ri} \\ &\qquad + \phi_{li}\phi_k{}^r u_{jr} - \phi_{kj}\phi_l{}^r u_{ri} - \phi_{ki}\phi_l{}^r u_{jr} + 2\phi_{lk}(\Phi u)_{ji}].\end{aligned}$$

On the other hand, by virtue of (4.2) and (4.4), $\nabla_l\nabla_k u_{ji} - \nabla_k\nabla_l u_{ji}$ becomes the right hand side of the above equation. Thus (4.5) holds.

The integrability condition of (4.4) is

$$\nabla_m\nabla_l u_{kji} - \nabla_l\nabla_m u_{kji} = -R_{mlk}{}^r u_{rji} - R_{mlj}{}^r u_{kri} - R_{mli}{}^r u_{kjr}. \tag{4.6}$$

Since K^{2m} is a space of constant holomorphic sectional curvature, we can get

$$\begin{aligned}&- R_{mlk}{}^r u_{rji} - R_{mlj}{}^r u_{kri} - R_{mli}{}^r u_{kjr} \\ &= \frac{R}{n(n+2)}[g_{mk}u_{lji} + g_{mj}u_{kli} + g_{mi}u_{kjl} \\ &\qquad - g_{lk}u_{mji} - g_{lj}u_{kmi} - g_{li}u_{kjm} \\ &\qquad + \phi_l{}^r(\phi_{mk}u_{rji} + \phi_{mj}u_{kri} + \phi_{mi}u_{kjr}) \\ &\qquad - \phi_m{}^r(\phi_{lk}u_{rji} + \phi_{lj}u_{kri} + \phi_{li}u_{kjr}) \\ &\qquad - 2\phi_{ml}(2\rho_k\phi_{ji} + \rho_j\phi_{ki} - \rho_i\phi_{kj} + \tilde{\rho}_j g_{ki} - \tilde{\rho}_i g_{kj})],\end{aligned} \tag{4.7}$$

where we have used (2.4), (2.5), (4.2) and Lemma 4.1.

On the other hand, operating ∇_m to (4.4) and owing to (4.1) and (4.2), we obtain

$$
\begin{aligned}
\nabla_m \nabla_l u_{kji} = \frac{R}{n(n+2)}[& -g_{lk}u_{mji} - g_{lj}u_{kmi} - g_{li}u_{kjm} \\
& + \phi_{lk}\nabla_m(\Phi u)_{ji} + \phi_{lj}\nabla_m(\Phi u)_{ik} + \phi_{li}\nabla_m(\Phi u)_{kj} \\
& - \phi_{ji}\nabla_m(\Phi u)_{lk} - \phi_{ik}\nabla_m(\Phi u)_{lj} - \phi_{kj}\nabla_m(\Phi u)_{li} \\
& + \phi_{kj}\phi_l{}^r u_{rmi} + \phi_{ik}\phi_l{}^r u_{rmj} + \phi_{ji}\phi_l{}^r u_{rmk} \\
& + (2\rho_i g_{mk} - \rho_m g_{ki} - \rho_k g_{mi} + 3\tilde{\rho}_k \phi_{mi})g_{lj} \\
& - (2\rho_j g_{mk} - \rho_m g_{kj} - \rho_k g_{mj} + 3\tilde{\rho}_k \phi_{mj})g_{li} \\
& + (3\rho_l \phi_{mi} + 2\rho_i \phi_{ml} + \rho_m \phi_{li} + \tilde{\rho}_l g_{mi})\phi_{kj} \\
& + (3\rho_l \phi_{mj} + 2\rho_j \phi_{ml} + \rho_m \phi_{lj} + \tilde{\rho}_l g_{mj})\phi_{ik} \\
& + (3\rho_l \phi_{mk} + 2\rho_k \phi_{ml} + \rho_m \phi_{lk} + \tilde{\rho}_l g_{mk} + 3\tilde{\rho}_m g_{lk})\phi_{ji}] \\
& + g_{lk}\nabla_m \rho_{ij} + g_{lj}\nabla_m \rho_{ki} + g_{li}\nabla_m \rho_{jk} + g_{ki}\nabla_m \rho_{jl} \\
& + g_{kj}\nabla_m \rho_{li} + 3\phi_{ji}\nabla_m \tilde{\rho}_{lk}.
\end{aligned}
$$

By interchanging the indices m and l in the above equation, subtracting from the original and owing to (2.4), (2.5), (4.1), (4.2), Theorem D, Lemma 4.1 and the Ricci's identity, we find that $\nabla_m \nabla_l u_{kji} - \nabla_l \nabla_m u_{kji}$ reduces to the right hand side of (4.7). Therefore (4.6) holds. Consequently, we complete the proof of Theorem 4.1.

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