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AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED BOUNDED REGION IN R^2

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ABSTRACT. The basic problem is to determine the geometry of an arbitrary multiply connected bounded region in R^2 together with the mixed boundary conditions, from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the Laplace operator, using the asymptotic expansion of the spectral function $\theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j)$ as $t \to 0$.

KEY WORDS AND PHRASES. Inverse problem, Laplace's operator, eigenvalue problem, spectral function.

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1. INTRODUCTION.

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the Laplace operator $\Delta_2 = \sum_{i=1}^{2} \left(\frac{\partial}{\partial x^i} \right)^2$ in the x^1x^2-plane.

Let $\Omega \subseteq R^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the Neumann/Dirichlet problem

$$(\Delta_2 + \lambda)u = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_j \le \ldots \to \infty \quad \text{as} \quad j \to \infty. \tag{1.3}$$

The problem of determining the geometry of Ω has been investigated by Pleijel [1], Kac [2], McKean and Singer [3], Stewartson and Waechter [4], Smith [5], Sleeman and Zayed [6,7], Gottlieb [8], Greiner [9], Zayed [10-13] and the references given there, using the asymptotic expansion of the trace function

$$\theta(t) = tr[\exp(-t\Delta_2)] = \sum_{j=1}^{\infty} \exp(-t\lambda_j) \quad \text{as} \quad t \to 0. \tag{1.4}$$

It has been shown that, in the case of Neumann boundary conditions (N.b.c.):

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} k^2(\sigma)d\sigma + 0(t) \quad \text{as} \quad t \to 0, \tag{1.5}$$

while, in the case of Dirichlet boundary conditions (D.b.c.):

$$\theta(t) = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} k^2(\sigma)d\sigma + 0(t) \quad \text{as} \quad t \to 0, \tag{1.6}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the total length of $\partial\Omega$ and $k(\sigma)$ is the curvature of $\partial\Omega$. The constant term a_0 has geometric significance, e.g., if Ω is smooth and convex, then $a_0 = \frac{1}{6}$ and if Ω is permitted to have a finite number of smooth convex holes "H", then $a_0 = \frac{1}{6}(1 - H)$.

The object of this paper is to discuss the following more general inverse problem: Let Ω be an arbitrary multiply connected bounded region in R^2 which is surrounded internally by simply connected bounded domains Ω_i with smooth boundaries $\partial\Omega_i$, $i = 1,\dots,m-1$ and externally by a simply connected bounded domain Ω_m with a smooth boundary $\partial\Omega_m$. Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

$$(\Delta_2 + \lambda)u = 0 \quad \text{in} \quad \Omega, \tag{1.7}$$

together with one of the following mixed boundary conditions:

$$\frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = 1,\dots,k \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = k+1,\dots,m, \tag{1.8}$$

$$u = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = 1,\dots,k \quad \text{and} \quad \frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = k+1,\dots,m, \tag{1.9}$$

where $\frac{\partial}{\partial n_i}$ denote differentiations along the inward pointing normals to the boundaries $\partial\Omega_i$, $i = 1,\dots,m$, respectively.

The basic problem is to determine the geometry of Ω from the asymptotic expansion of the spectral function (1.4) for small positive t.

Note that problems (1.7)-(1.9) have been investigated recently by Zayed [11] in the special case where Ω is an arbitrary doubly connected bounded region (i.e., m=2).

2. STATEMENT OF OUR RESULTS.

Suppose that the boundaries $\partial\Omega_i$, $i = 1,\dots,m$ are given locally by the equations $x^n = y^n(\sigma_i)$, $n = 1,2$ in which σ_i, $i = 1,\dots,m$ are the arc-lengths of the counterclockwise oriented boundaries $\partial\Omega_i$ and $y^n(\sigma_i) \in C^\infty(\partial\Omega_i)$. Let L_i and $k_i(\sigma_i)$ be the lengths and the curvatures of $\partial\Omega_i$, $i = 1,\dots,m$ respectively. Then, the results of our main problem (1.7)-(1.9) can be summarized in the following cases:

CASE 1. (N.b.c. on $\partial\Omega_i$, $i = 1,\dots,k$ and D.b.c. on $\partial\Omega_i$, $i = k+1,\dots,m$)

$$\begin{aligned}\theta(t) = {} & \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\left\{\sum_{i=1}^{k} L_i - \sum_{i=k+1}^{m} L_i\right\} + \frac{1}{6}(2-m) \\ & + \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2}\left\{7\sum_{i=1}^{k}\int_{\partial\Omega_i} k_i^2(\sigma_i)d\sigma_i + \sum_{i=k+1}^{m}\int_{\partial\Omega_i} k_i^2(\sigma_i)d\sigma_i\right\} \\ & + 0(t) \quad \text{as} \quad t \to 0.\end{aligned} \tag{2.1}$$

CASE 2. (D.b.c. on $\partial\Omega_i$, $i = 1,\dots,k$ and N.b.c. on $\partial\Omega_i$ $i = k+1,\dots,m$)

In this case the asymptotic expansion of $\theta(t)$ as $t \to 0$ has the same form (2.1) with the interchanges $\partial\Omega_i$, $i = 1,\dots,k \leftrightarrow \partial\Omega_i$, $i = k+1,\dots,m$.

With reference to formulae (1.4), (1.5) and to articles [6], [11], [12] the asymptotic expansion (2.1) may be interpreted as follows:

(i) Ω is an arbitrary multiply connected bounded region in R^2 and we have the mixed boundary conditions (1.8) or (1.9) as indicated in the specifications of the two respective cases.

(ii) For the first four terms, Ω is an arbitrary multiply connected bounded region in R^2 of area $|\Omega|$.

In case 1, it has $H = (m-1)$ holes, the boundaries $\partial\Omega_i$, $i = 1,\dots,k$ are of lengths $\sum_{i=1}^{k} L_i$ and of curvatures $k_i(\sigma_i)$, $i = 1,\dots,k$ together with Neumann boundary conditions, while the boundaries $\partial\Omega_i$, $i = k+1,\dots,m$ are of lengths $\sum_{i=k+1}^{m} L_i$ and of curvatures $k_i(\sigma_i)$, $i = k+1,\dots,m$ together with Dirichlet boundary conditions, provided H is an integer.

We close this section with the following remarks:

REMARK 2.1. On setting $k = 0$ in formula (2.1) with the usual definition that $\sum_{i=1}^{0}$ is zero, we obtain the results of Dirichlet boundary conditions on $\partial\Omega_i$, $i = 1,\dots,m$.

REMARK 2.2. On setting $k = m$ in formula (2.1) with the usual definition that $\sum_{i=m+1}^{m}$ is zero, we obtain the results of Neumann boundary conditions on $\partial\Omega_i$, $i = 1,\dots,m$.

3. FORMULATION OF THE MATHEMATICAL PROBLEM

It is easy to show that the spectral function (1.4) associated with problems (1.7)-(1.9) is given by

$$\theta(t) = \iint_{\Omega} G\left(\underset{\sim}{x}, \underset{\sim}{x}; t\right) d\underset{\sim}{x}, \tag{3.1}$$

where $G\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right)$ is Green's function for the heat equation

$$\left(\Delta_2 - \frac{\partial}{\partial t}\right) u = 0, \tag{3.2}$$

subject to the mixed boundary conditions (1.8) or (1.9) and the initial condition

$$\lim_{t \to 0} G\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right) = \delta\left(\underset{\sim}{x}_1 - \underset{\sim}{x}_2\right), \tag{3.3}$$

where $\delta\left(\underset{\sim}{x}_1 - \underset{\sim}{x}_2\right)$ is the Dirac delta function located at the source point $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$. Let us write

$$G\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right) = G_0\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right) + \chi\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right), \tag{3.4}$$

where

$$G_0\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right) = (4\pi t)^{-1} \exp\left\{-\frac{\left|\underset{\sim}{x}_1 - \underset{\sim}{x}_2\right|^2}{4t}\right\}, \tag{3.5}$$

is the "fundamental solution" of the heat equation (3.2), while $\chi\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right)$ is the "regular solution" chosen so that $G\left(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t\right)$ satisfies the mixed boundary conditions (1.8) or (1.9).

On setting $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{x}$ we find that

$$\theta(t) = \frac{|\Omega|}{4\pi t} + K(t), \tag{3.6}$$

where

$$K(t) = \iint_{\Omega} \chi\left(\underset{\sim}{x}, \underset{\sim}{x}; t\right) d\underset{\sim}{x}. \tag{3.7}$$

The problem now is to determine the asymptotic expansion of *K(t)* for small positive *t*. In what follows we shall use Laplace transforms with respect to *t*, and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right) = \int_0^\infty e^{-s^2 t} G\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; t\right) dt. \tag{3.8}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right)$ satisfies the membrane equation

$$(\Delta_2 - s^2)\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right) = -\delta\left(\underset{\sim}{x_1} - \underset{\sim}{x_2}\right) \quad \text{in} \quad \Omega, \tag{3.9}$$

together with the mixed boundary conditions (1.8) or (1.9).

The asymptotic expansion of *K(t)* for small positive *t*, may then be deduced directly from the asymptotic expansion of $\overline{K}(s^2)$ for large positive *s*, where

$$\overline{K}(s^2) = \iint_{\Omega} \overline{\chi}\left(\underset{\sim}{x}, \underset{\sim}{x}; s^2\right) d\underset{\sim}{x}. \tag{3.10}$$

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [6] that the membrane equation (3.9) has the fundamental solution

$$\overline{G}_0(x_1, x_2; s^2) = \frac{1}{2\pi} K_0(s r_{x_1 x_2}) \tag{4.1}$$

where $r_{x_1 x_2} = \left|\underset{\sim}{x_1} - \underset{\sim}{x_2}\right|$ is the distance between the points $\underset{\sim}{x_1} = (x_1^1, x_1^2)$ and $\underset{\sim}{x_2} = (x_2^1, x_2^2)$ of the region Ω while K_0 is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for $\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right)$ satisfying the mixed boundary conditions (1.8) or (1.9). Therefore, Green's theorem gives:

CASE 1. (N.b.c. on $\partial\Omega_i$, $i = 1, \ldots, k$ and D.b.c. on $\partial\Omega_i$, $i = k+1, \ldots, m$)

$$\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right) = \frac{1}{2\pi} K_0\left(s r_{\underset{\sim}{x_1}\underset{\sim}{x_2}}\right) + \frac{1}{\pi}\sum_{i=1}^{k} \int_{\partial\Omega_i} \overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{y}; s^2\right) \frac{\partial}{\partial n_{\underset{\sim}{i y}}} K_0\left(s r_{\underset{\sim}{y}\underset{\sim}{x_2}}\right) d\underset{\sim}{y}$$

$$+ \frac{1}{\pi}\sum_{i=k+1}^{m} \int_{\partial\Omega_i} \frac{\partial}{\partial n_{i\underset{\sim}{y}}} \overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{y}; s^2\right) K_0\left(s r_{\underset{\sim}{y}\underset{\sim}{x_2}}\right) d\underset{\sim}{y}. \tag{4.2}$$

CASE 2. (D.b.c. on $\partial\Omega_i$, $i = 1, \ldots, k$ and N.b.c. on $\partial\Omega_i$, $i = k+1, \ldots, m$)

In this case Green's function $\overline{G}\left(\underset{\sim}{x_1}, \underset{\sim}{x_2}; s^2\right)$ has the same form (4.2) with the interchanges $\partial\Omega_i$, $i = 1, \ldots, k \leftrightarrow \partial\Omega_i$, $i = k+1, \ldots, m$.

On applying the iteration method (see [11], [12]) to the integral equation (4.2), we obtain Green's function $\overline{G}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2\right)$ which has the regular part:

$$\overline{\chi}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2\right)=\frac{1}{2\pi^2}\sum_{i=1}^{k}\int_{\partial\Omega_i}K_0\left(sr_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)\frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)d\underset{\sim}{y}$$

$$+\frac{1}{2\pi^2}\sum_{i=k+1}^{m}\int_{\partial\Omega_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0\left(sr_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)d\underset{\sim}{y}$$

$$+\frac{1}{2\pi^2}\sum_{i=1}^{k}\int_{\partial\Omega_i}\int_{\partial\Omega_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0\left(sr_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)M_i\left(\underset{\sim}{y},\underset{\sim}{y}'\right)\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}K_0\left(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)d\underset{\sim}{y}\,d\underset{\sim}{y}'$$

$$+\frac{1}{2\pi^2}\sum_{i=k+1}^{m}\int_{\partial\Omega_i}\int_{\partial\Omega_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})M_i^*\left(\underset{\sim}{y},\underset{\sim}{y}'\right)K_0\left(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)d\underset{\sim}{y}\,d\underset{\sim}{y}'$$

$$+\frac{1}{2\pi^2}\sum_{i=1}^{k}\int_{\partial\Omega_i}\left\{\sum_{i=k+1}^{m}\int_{\partial\Omega_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0\left(sr_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)L_i\left(\underset{\sim}{y},\underset{\sim}{y}'\right)d\underset{\sim}{y}\right\}\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}K_0\left(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)d\underset{\sim}{y}'$$

$$+\frac{1}{2\pi^2}\sum_{i=k+1}^{m}\int_{\partial\Omega_i}\left\{\sum_{i=1}^{k}\int_{\partial\Omega_i}K_0\left(sr_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)L_i^*\left(\underset{\sim}{y},\underset{\sim}{y}'\right)d\underset{\sim}{y}\right\}K_0\left(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)d\underset{\sim}{y}', \qquad (4.3)$$

where

$$M_i\left(\underset{\sim}{y},\underset{\sim}{y}'\right)=\sum_{\nu=0}^{\infty}K_i^{(\nu)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \qquad (4.4)$$

$$M_i^*\left(\underset{\sim}{y},\underset{\sim}{y}'\right)=\sum_{\nu=0}^{\infty}{}^*K_i^{(\nu)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \qquad (4.5)$$

$$L_i\left(\underset{\sim}{y},\underset{\sim}{y}'\right)=\sum_{\nu=0}^{\infty}\underline{K}_i^{(\nu)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \qquad (4.6)$$

$$L_i^*\left(\underset{\sim}{y},\underset{\sim}{y}'\right)=\sum_{\nu=0}^{\infty}{}^*\underline{K}_i^{(\nu)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \qquad (4.7)$$

$$K_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right)=\frac{1}{\pi}\frac{\partial}{\partial n_{iy}}K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right), \qquad (4.8)$$

$${}^*K_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right)=\frac{1}{\pi}\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right), \qquad (4.9)$$

$$\underline{K}_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right)=\frac{1}{\pi}K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right), \qquad (4.10)$$

and

$${}^*\underline{K}_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right)=\frac{1}{\pi}\frac{\partial^2}{\partial n_{i\underset{\sim}{y}}\partial n_{i\underset{\sim}{y}'}}K_0\left(sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right). \qquad (4.11)$$

In the same way, we can show that in case 2 Green's function $\overline{G}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2\right)$ has a regular part of the same form (4.3) with the interchanges $\partial\Omega_i,\ i=1,\ldots,k \leftrightarrow \partial\Omega_i,\ i=k+1,\ldots,m$.

On the basis of (4.3) the function $\overline{\chi}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2\right)$ will be estimated for large values of s. The case when $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ lie in the neighborhoods of $\partial\Omega_i$, $i = 1,\ldots,m$ is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOODS OF $\partial\Omega_i$, $i = 1,\ldots,m$.

Let n_i, $i = 1,\ldots,m$ be the minimum distances from a point $\underset{\sim}{x} = \left(\underset{\sim}{x}^1,\underset{\sim}{x}^2\right)$ of the region Ω to the boundaries $\partial\Omega_i$, $i = 1,\ldots,m$ respectively. Let $\underset{\sim}{n}_i(\sigma_i)$, $i = 1,\ldots,m$ denote the inward drawn unit normals to $\partial\Omega_i$, $i = 1,\ldots,m$ respectively. We note that the coordinates in the neighborhood of $\partial\Omega_i$, $i = k+1,\ldots,m$ and its diagrams (see [11]) are in the same form as in section 5.1 of [11] with the interchanges $\sigma_2 \leftrightarrow \sigma_i$, $n_2 \leftrightarrow n_i$, $h_2 \leftrightarrow h_i$, $I_2 \leftrightarrow I_i$, $\mathcal{D}(I_2) \leftrightarrow \mathcal{D}(I_i)$ and $\delta_2 \leftrightarrow \delta_i$, $i = k+1,\ldots,m$. Thus, we have the same formulae (5.1.1)-(5.1.5) of section 5.1 in [11] with the interchanges $n_2 \leftrightarrow n_i$, $\underset{\sim}{n}_2(\sigma_2) \leftrightarrow \underset{\sim}{n}_i(\sigma_i)$, $\underset{\sim}{t}_2(\sigma_2) \leftrightarrow \underset{\sim}{t}_i(\sigma_i)$, $k_2(\sigma_2) \leftrightarrow k_i(\sigma_i)$, $i = k+1,\ldots,m$.

Similarly, the coordinates in the neighborhood of $\partial\Omega_i$, $i = 1,\ldots,k$ and its diagrams (see [11]) are similar to those obtained in section 5.2 of [11] with the interchanges $\sigma_1 \leftrightarrow \sigma_i$, $n_1 \leftrightarrow n_i$, $h_1 \leftrightarrow h_i$, $I_1 \leftrightarrow I_i$, $\mathcal{D}(I_1) \leftrightarrow \mathcal{D}(I_i)$ and $\delta_1 \leftrightarrow \delta_i$, $i = 1,\ldots,k$. Thus, we have the same formulae (5.2.1)-(5.2.5) of section 5.2 in [11] with the interchanges $n_1 \leftrightarrow n_i$, $\underset{\sim}{n}_1(\sigma_1) \leftrightarrow \underset{\sim}{n}_i(\sigma_i)$, $\underset{\sim}{t}_1(\sigma_1) \leftrightarrow \underset{\sim}{t}_i(\sigma_i)$ and $k_1(\sigma_1) \leftrightarrow k_i(\sigma_i)$, $i = 1,\ldots,k$.

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions

$$K_0\left(sr_{\underset{\sim}{x}\underset{\sim}{y}}\right),\quad \frac{\partial}{\partial n_{i\underset{\sim}{y}}}K_0\left(sr_{\underset{\sim}{x}\underset{\sim}{y}}\right),\quad i = 1,\ldots,m \tag{6.1}$$

when the distance between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is small, are very similar to those obtained in section 6 of [11]. Consequently, for $i = 1,\ldots,k,k+1,\ldots,m$, the local behavior of the following kernels:

$$K_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\quad {}^{*}\underline{K}_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \tag{6.2}$$

$${}^{*}K_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\quad \underline{K}_i\left(\underset{\sim}{y}',\underset{\sim}{y}\right), \tag{6.3}$$

when the distance between $\underset{\sim}{y}$ and $\underset{\sim}{y}'$ is small, follows directly from the knowledge of the local expansions of (6.1).

DEFINITION 1. Let $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ be points in the upper half-plane $\xi^2 > 0$, then we define

$$\hat{\rho}_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 + \xi_2^2)^2}. \tag{6.4}$$

An $e^{\lambda}\left(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s\right)$-function is defined for points $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ belong to sufficiently small domains $\mathcal{D}(I_i)$ except when $\underset{\sim}{\xi}_1 = \underset{\sim}{\xi}_2 \in I_i$, $i = 1,\ldots,m$ and λ is called the degree of this function. For every positive integer Λ it has the local expansion (see [11]):

$$e^{\lambda}\left(\underset{\sim}{\xi_1},\underset{\sim}{\xi_2};s\right)=\sum{}^{*} f(\xi_1^1)(\xi_1^2)^{P_1}(\xi_2^2)^{P_2}\left(\frac{\partial}{\partial\xi_1^1}\right)^{l}\left(\frac{\partial}{\partial\xi_1^2}\right)^{m} K_0(s\hat{\rho}_{12})+R^{\Lambda}\left(\underset{\sim}{\xi_1},\underset{\sim}{\xi_2},s\right), \tag{6.5}$$

where $\sum^{*}$ denotes a sum of a finite number of terms in which $f(\xi_1^1)$ is an infinitely differentiable function. In this expansion, P_1, P_2, l, m are integers, where $P_1 \geq 0, P_2 \geq 0, l \geq 0, \lambda = \min(P_1+P_2-q), q = l+m$ and the minimum is taken over all terms which occur in the summation $\sum^{*}$. The remainder $R^{\Lambda}\left(\underset{\sim}{\xi_1},\underset{\sim}{\xi_2};s\right)$ has continuous derivatives of order $d \leq \Lambda$ satisfying

$$D^{d}R^{\Lambda}\left(\underset{\sim}{\xi_1},\underset{\sim}{\xi_2};s\right)=0\left(s^{-\Lambda}e^{-As\hat{\rho}_{12}}\right) \quad \text{as} \quad s\to\infty, \tag{6.6}$$

where A is a positive constant.

Thus, using methods similar to those obtained in section 7 of [11], we can show that the functions (6.1) are e^{λ}-functions with degrees $\lambda = 0, -1$ respectively. Consequently, the functions (6.2) are e^{λ}-functions with degrees $\lambda = 0, -1$, while the functions (6.3) are e^{λ}-functions with degrees $\lambda = 0, 1$ respectively.

DEFINITION 2. If x_1 and x_2 are points in large domains $\Omega + \partial\Omega_i$, $i = 1,\ldots,k,k+1,\ldots,m$, then we define

$$\hat{r}_{12}=\min_{\underset{\sim}{y}}\left(r_{\underset{\sim}{x_1}\underset{\sim}{y}}+r_{\underset{\sim}{x_2}\underset{\sim}{y}}\right) \quad \text{if} \quad \underset{\sim}{y}\in\partial\Omega_i, \quad i=1,\ldots,k,$$

and

$$\hat{R}_{12}=\min_{\underset{\sim}{y}}\left(r_{\underset{\sim}{x_1}\underset{\sim}{y}}+r_{\underset{\sim}{x_2}\underset{\sim}{y}}\right) \quad \text{if} \quad \underset{\sim}{y}\in\partial\Omega_i, \quad i=k+1,\ldots,m.$$

An $E^{\lambda}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s\right)$-function is defined and infinitely differentiable with respect to $\underset{\sim}{x_1}$ and $\underset{\sim}{x_2}$ when these points belong to large domains $\Omega+\partial\Omega_i$ except when $\underset{\sim}{x_1}=\underset{\sim}{x_2}\in\partial\Omega_i$, $i=1,\ldots,m$. Thus, the E^{λ}-function has a similar local expansion of the e^{λ}-function (see [6], [11]).

By the help of section 8 in [11], it is easily seen that formula (4.3) is an $E^{0}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s\right)$-function and consequently

$$\overline{G}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right)=\sum_{i=1}^{k} O\left\{[1+|\log s\hat{r}_{12}|]e^{-A_i s\hat{r}_{12}}\right\}$$

$$+\sum_{i=k+1}^{m} O\left\{[1+|\log s\hat{R}_{12}|]e^{-A_i s\hat{R}_{12}}\right\}, \tag{6.7}$$

which is valid for $s\to\infty$, where A_i, $i=1,\ldots,m$ are positive constants.

Formula (6.7) shows $\overline{G}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right)$ is exponentially small for $s\to\infty$.

7. THE ASYMPTOTIC BEHAVIOR OF $\overline{\chi}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right)$.

With reference to sections 7 and 9 in [11], if the e^{λ}-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of section 7 in [11], we obtain the following local behavior of $\overline{\chi}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right)$ as $s\to\infty$ which is valid when $\hat{r}_{12}$ and $\hat{R}_{12}$ are small:

$$\overline{\chi}\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right)=\sum_{i=1}^{m}\overline{\chi}_i\left(\underset{\sim}{x_1},\underset{\sim}{x_2};s^2\right), \tag{7.1}$$

where, if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_i)$, $i = 1,\dots,k,k+1,\dots,m$, then

$$\bar{\chi}_i\left(x_1,x_2;s^2\right) = -\frac{1}{2\pi}K_0(s\hat{\rho}_{12}) + O\{s^{-1}\exp(-A_i s\hat{\rho}_{12})\}. \tag{7.2}$$

When $\hat{r}_{12} \geq \delta_i > 0$, $i = 1,\dots,k$ and $\hat{R}_{12} \geq \delta_i > 0$, $i = k+1,\dots,m$ the function $\bar{\chi}\left(x_1,x_2;s^2\right)$ is of order $O\{\exp(-cs)\}$ as $s \to \infty$, $c > 0$. Thus, since $\lim_{\hat{r}_{12}\to 0}\frac{\hat{r}_{12}}{\hat{\rho}_{12}} = \lim_{\hat{R}_{12}\to 0}\frac{\hat{R}_{12}}{\hat{\rho}_{12}} = 1$, then if x_1 and x_2 belong to large domains $\Omega + \partial\Omega_i$, $i = 1,\dots,k$, we deduce for $s \to \infty$ that

$$\bar{\chi}_i\left(x_1,x_2;s^2\right) = -\frac{1}{2\pi}K_0(s\hat{r}_{12}) + O\{s^{-1}\exp(-A_i s\hat{r}_{12})\}, \tag{7.3}$$

while, if x_1 and x_2 belong to large domains $\Omega + \partial\Omega_i$, $i = k+1,\dots,m$, we deduce for $s \to \infty$ that

$$\bar{\chi}_i\left(x_1,x_2;s^2\right) = -\frac{1}{2\pi}K_0(s\hat{R}_{12}) + O\{s^{-1}\exp(-A_i s\hat{R}_{12})\}. \tag{7.4}$$

8. CONSTRUCTION OF OUR RESULTS.

Since for $\xi^2 \geq h_i > 0$, $i = 1,\dots,k,k+1,\dots,m$, the functions $\bar{\chi}_i\left(x,x;s^2\right)$ are of order $O\{\exp(-2sA_i h_i)\}$, the integral of the function $\bar{\chi}\left(x,x;s^2\right)$ over the region Ω can be approximated in the following way (see (3.10)):

$$\begin{aligned}\bar{K}(s^2) = {} & \sum_{i=k+1}^{m}\int_{\xi^2=0}^{h_i}\int_{\xi^1=0}^{L_i}\bar{\chi}_i\left(x,x;s^2\right)\{1-k_i(\xi^1)\xi^2\}d\xi^1 d\xi^2 \\ & -\sum_{i=1}^{k}\int_{\xi^2=0}^{h_i}\int_{\xi^1=0}^{L_i}\bar{\chi}_i\left(x,x;s^2\right)\{1+k_i(\xi^1)\xi^2\}d\xi^1 d\xi^2 \\ & +\sum_{i=1}^{m}O\{\exp(-2sA_i h_i)\} \qquad \text{as} \qquad s\to\infty. \end{aligned} \tag{8.1}$$

If the e^{λ}-expansions of $\bar{\chi}_i\left(x,x;s^2\right)$, $i = 1,\dots,k,k+1,\dots,m$, are introduced into (8.1), one obtains an asymptotic series of the form:

$$\bar{K}(s^2) = \sum_{n=1}^{j} a_n s^{-n} + O(s^{-j-1}) \quad \text{as} \quad s\to\infty, \tag{8.2}$$

where the coefficients a_n are calculated from the e^{λ}-expansions by the help of formula (10.3) of section 10 in [11].

Now, the first three coefficients a_1, a_2, a_3 take the forms:

$$a_1 = \frac{1}{8}\left(\sum_{i=1}^{k} L_i - \sum_{i=k+1}^{m} L_i\right),$$

$$a_2 = \frac{1}{6}(2-m), \tag{8.3}$$

$$a_3 = \frac{1}{512}\left\{7\sum_{i=1}^{k} \int_{\partial\Omega_i} k_i^2(\sigma_i)d\sigma_i + \sum_{i=k+1}^{m} \int_{\partial\Omega_i} k_i^2(\sigma_i)d\sigma_i\right\}.$$

On inserting (8.3) int (8.2) and inverting Laplace transforms and using (3.6) we arrive at our result (2.1).

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