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SOME OBSERVATIONS ON KHOVANSKII'S MATRIX METHODS FOR EXTRACTING ROOTS OF POLYNOMIALS

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Abstract

In this article we apply a formula for the n-th power of a 3×3 matrix (found previously by the authors) to investigate a procedure of Khovanskii's for finding the cube root of a positive integer.

We show, for each positive integer α, how to construct certain families of integer sequences such that a certain rational expression, involving the ratio of successive terms in each family, tends to $\alpha^{1/3}$. We also show how to choose the optimal value of a free parameter to get maximum speed of convergence.

We apply a similar method, also due to Khovanskii, to a more general class of cubic equations, and, for each such cubic, obtain a sequence of rationals that converge to the real root of the cubic.

We prove that Khovanskii's method for finding the m-th ($m \geq 4$) root of a positive integer works, provided a free parameter is chosen to satisfy a very simple condition.

Finally, we briefly consider another procedure of Khovanskii's, which also involves $m \times m$ matrices, for approximating the root of an arbitrary polynomial of degree m.

1. Introduction

In [1] Khovanskii described a method which uses powers of 3×3 matrices to approximate cube roots of integers. More precisely, let α be a positive integer whose cube root is desired and let a be an arbitrary integer. Define the matrix A by

$$A = \begin{pmatrix} a & \alpha & \alpha \\ 1 & a & \alpha \\ 1 & 1 & a \end{pmatrix}, \tag{1}$$

and let $A_{n,i,j}$ denote the (i,j)-th entry of A^n. Suppose

$$\lim_{n\to\infty}\frac{A_{n,1,1}}{A_{n,3,1}}=x, \qquad \lim_{n\to\infty}\frac{A_{n,2,1}}{A_{n,3,1}}=y, \tag{2}$$

where x and y are finite and $x+y+1\neq 0$. Then $x=\sqrt[3]{\alpha^2}$ and $y=\sqrt[3]{\alpha}$.

Khovanskii did not give conditions which insure the convergence of the sequences above. Also, he did not investigate the speed of convergence or the question of the optimal choice of the integer a to ensure the most rapid convergence. Further, there is the difficulty that is necessary to compute the powers of the matrix A.

In this present paper we show that the sequences $\{A_{n,1,1}/A_{n,3,1}\}_{n=1}^{\infty}$, $\{A_{n,2,1}/A_{n,3,1}\}_{n=1}^{\infty}$ converge for all integers a greater than a certain explicit lower bound. We also determine, for a given α, the choice of a which insures the most rapid convergence. We also give precise estimates for $|A_{n,2,1}/A_{n,3,1}-\alpha^{1/3}|$, for this optimal choice of a. Finally, we employ a closed formula for the n-th power of a 3×3 matrix from our paper [2], which actually makes it unnecessary to perform the matrix multiplications. We have the following theorems.

Theorem 1. *Let $\alpha>1$ be an integer and a be any integer such that $a>-\dfrac{\alpha^{2/3}}{1+\alpha^{1/3}}$. Set*

$$a_n = \sum_{i,j}(-1)^i\binom{i+j}{j}\binom{n-i-2j}{i+j}(3a)^{n-2i-3j}(3a^2-3\alpha)^i(a^3+\alpha-3a\alpha+\alpha^2)^j.$$

Then

$$\lim_{n\to\infty}1+\frac{\alpha-1}{\dfrac{a_n}{a_{n-1}}-a+1}=\alpha^{1/3}.$$

Note that the limit is independent of the choice of the parameter a.

Theorem 2. *Let α and a be as described in Theorem 1. Let the matrix A be as described in (1). Then the choice of a which gives the most rapid convergence is one of the two integers closest to $\bar{a}=\dfrac{\alpha^{1/3}+\alpha}{1+\alpha^{1/3}}$. For this choice of a and $n\geq 3$,*

$$\frac{A_{n,2,1}}{A_{n,3,1}}-\alpha^{1/3}=\left((\omega-1)\,\omega\left(\left(\frac{-\omega}{2}\right)^n-\left(\frac{-\omega^2}{2}\right)^n\right)+\frac{n\delta_3}{2^n\alpha^{1/3}}+\frac{\delta_4}{2^{2n}}\right)\alpha^{1/3},$$

where $\omega=\exp(2\pi\imath/3)$, $|\delta_3|\leq 8$ and $|\delta_4|\leq 48$.

We also investigate two other procedures due to Khovanskii. One is a method for finding a root of $x^3-p\,x-q$ and the other is a method for finding $\alpha^{1/m}$, where α and m are arbitrary positive integers. Again, Khovanskii's methods involve sequences of powers of matrices and rely on the ratios of certain matrix entries converging, and he did not give any conditions which guarantee convergence. We give criteria which insure convergence. In the case of $x^3-p\,x-q$, we again prove a result which makes the actual matrix multiplications unnecessary. We have the following theorems.

Theorem 3. *Let $p > 0$, $q > 0$ be integers such that $27q^2 - 4p^3 > 0$. Define*

$$a_n = \sum_{2i+3j \le n} \binom{i+j}{j} \binom{n-i-2j}{i+j} 3^{n-2i-3j} (3-p)^i (q-p+1)^j.$$

Then

$$-1 + \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{(2/3)^{1/3}\, p}{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}} + \frac{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}}{2^{1/3}\, 3^{2/3}}, \tag{3}$$

is the real root of $f(x) = x^3 - px - q$.

Theorem 4. *Let A be the $m \times m$ matrix*

$$A = \begin{pmatrix} a & \alpha & \alpha & \alpha & \dots & \alpha \\ 1 & a & \alpha & \alpha & \dots & \alpha \\ 1 & 1 & a & \alpha & \dots & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & \alpha \\ 1 & 1 & 1 & \dots & 1 & a \end{pmatrix}.$$

Let $A_{n,\,i,\,j}$ denote the $(i,\,j)$ entry of A^n and suppose $a > 0$. Then $\lim\limits_{n\to\infty} \dfrac{A_{n,\,i,\,j}}{A_{n,\,u,\,v}} = \alpha^{(j+u-i-v)/m}$.

Some of the work in this paper relies heavily on results proved in our paper [2]:

Theorem 5. *Suppose $A \in M_k(K)$ and let $T^k - s_1 T^{k-1} + s_2 T^{k-2} + \cdots + (-1)^k s_k\, I$ denote its characteristic polynomial. Then, for all $n \ge k$, one has $A^n = b_{k-1}A^{k-1} + b_{k-2}A^{k-2} + \cdots + b_0\, I$, where*

$$\begin{aligned} b_{k-1} &= a(n-k+1), \\ b_{k-2} &= a(n-k+2) - s_1 a(n-k+1), \\ &\vdots \\ b_1 &= a(n-1) - s_1 a(n-2) + \cdots + (-1)^{k-2} s_{k-2} a(n-k+1), \\ b_0 &= a(n) - s_1 a(n-1) + \cdots + (-1)^{k-1} s_{k-1} a(n-k+1) \\ &= (-1)^{k-1} s_k a(n-k), \end{aligned}$$

and

$$a(n) = c(i_2, \cdots, i_k, n) s_1^{n-i_2-2i_3-\cdots-(k-1)i_k} (-s_2)^{i_2} s_3^{i_3} \cdots ((-1)^{k-1} s_k)^{i_k},$$

with

$$c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k-1)i_k)!}{i_2! \cdots i_k! (n - 2i_2 - 3i_3 - \cdots - (ki_k)!}.$$

For the case $k = 3$ we get the following corollary.

Corollary 1. *(i) Let $A \in M_3(K)$ and let $X^3 = tX^2 - sX + d$ denote the characteristic polynomial of A. Then, for all $n \geq 3$,*

$$A^n = a_{n-1}A + a_{n-2}Adj(A) + (a_n - ta_{n-1})\, I, \tag{4}$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} t^{n-2i-3j} s^i d^j$$

for $n > 0$ and $a_0 = 1$.

We use this corollary in conjunction with Khovanskii's ideas to determine sequences of rational approximations to the real root of certain types of polynomials.

2. Approximating Cube Roots of Positive Integers

We start with a theorem.

Theorem 6. *Let $\alpha > 1$ be an integer and a be any integer such that*

$$a > -\frac{\alpha^{2/3}}{1+\alpha^{1/3}}. \tag{5}$$

Set

$$a_n = \sum_{i,j} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (3a)^{n-2i-3j} (3a^2 - 3\alpha)^i (a^3 + \alpha - 3a\alpha + \alpha^2)^j.$$

Then

$$\lim_{n\to\infty} 1 + \frac{\alpha - 1}{\dfrac{a_n}{a_{n-1}} - a + 1} = \alpha^{1/3}. \tag{6}$$

Proof. Let $\omega := \exp(2\pi\imath/3)$ and set

$$A = \begin{pmatrix} a & \alpha & \alpha \\ 1 & a & \alpha \\ 1 & 1 & a \end{pmatrix}.$$

The eigenvalues of A are

$$\begin{aligned} \beta_1 &= a + \alpha^{1/3} + \alpha^{2/3}, \\ \beta_2 &= a + \alpha^{1/3}\,\omega + \alpha^{2/3}\,\omega^2, \\ \beta_3 &= a + \alpha^{2/3}\,\omega + \alpha^{1/3}\,\omega^2. \end{aligned} \tag{7}$$

Note that β_1 is positive for any a satisfying (5). Further, for such a,

$$\left|\frac{\beta_2}{\beta_1}\right|^2 = \left|\frac{\beta_3}{\beta_1}\right|^2 = \frac{\beta_2\beta_3}{\beta_1^2} = \frac{a^2 - a\,\alpha^{1/3} + \alpha^{2/3} - a\,\alpha^{2/3} - \alpha + \alpha^{4/3}}{\left(a + \alpha^{1/3} + \alpha^{2/3}\right)^2} < 1,$$

so that $\beta_1 > |\beta_2| = |\beta_3|$. Let

$$M = \begin{pmatrix} \alpha^{2/3} & \alpha^{2/3}\,\omega^2 & \alpha^{2/3}\,\omega \\ \alpha^{1/3} & \alpha^{1/3}\,\omega & \alpha^{1/3}\,\omega^2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

Then $A = M\,D\,M^{-1}$ and so

$$A^n \;=\; M\,D^n\,M^{-1} \;=\; \begin{pmatrix} \frac{{\beta_1}^n+{\beta_2}^n+{\beta_3}^n}{3} & \frac{\alpha^{1/3}\left({\beta_1}^n+\omega\,{\beta_2}^n+\omega^2\,{\beta_3}^n\right)}{3} & \frac{\alpha^{2/3}\left({\beta_1}^n+\omega^2\,{\beta_2}^n+\omega\,{\beta_3}^n\right)}{3} \\ \frac{{\beta_1}^n+\omega^2\,{\beta_2}^n+\omega\,{\beta_3}^n}{3\,\alpha^{1/3}} & \frac{{\beta_1}^n+{\beta_2}^n+{\beta_3}^n}{3} & \frac{\alpha^{1/3}\left({\beta_1}^n+\omega\,{\beta_2}^n+\omega^2\,{\beta_3}^n\right)}{3} \\ \frac{{\beta_1}^n+\omega\,{\beta_2}^n+\omega^2\,{\beta_3}^n}{3\,\alpha^{2/3}} & \frac{{\beta_1}^n+\omega^2\,{\beta_2}^n+\omega\,{\beta_3}^n}{3\,\alpha^{1/3}} & \frac{{\beta_1}^n+{\beta_2}^n+{\beta_3}^n}{3} \end{pmatrix}.$$

Let $A_{n,\,i,\,j}$ denote the (i,j)-th entry of A^n. It is now easy to see (since $\beta_1 > |\beta_2| = |\beta_3|$) that

$$\lim_{n\to\infty} \frac{A_{n,\,2,\,1}}{A_{n,\,3,\,1}} = \alpha^{1/3} \lim_{n\to\infty} \frac{{\beta_1}^n + \omega^2\,{\beta_2}^n + \omega\,{\beta_3}^n}{{\beta_1}^n + \omega\,{\beta_2}^n + \omega^2\,{\beta_3}^n} = \alpha^{1/3}. \tag{8}$$

On the other hand, the characteristic polynomial of A is

$$X^3 = 3\,a\,X^2 - (3a^2 - 3\alpha)X + a^3 + \alpha - 3a\alpha + \alpha^2.$$

It follows from Corollary 1, that if $t = 3a$, $s = 3a^2 - 3\alpha$, $d = a^3 + \alpha - 3a\alpha + \alpha^2$ and

$$\begin{aligned} a_n &= \sum_{2i+3j\le n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} t^{n-2i-3j} s^i d^j, \\ \gamma_n &:= \left(a^3 + \alpha - 3\,a\,\alpha + \alpha^2\right)\, a_{n-3} - 2\,\left(a^2 - \alpha\right)\, a_{n-2} + a\,a_{n-1}, \\ \delta_n &:= (-a + \alpha)\, a_{n-2} + a_{n-1}, \\ \rho_n &:= (1 - a)\, a_{n-2} + a_{n-1}, \end{aligned}$$

then

$$A^n = \begin{pmatrix} \gamma_n & \alpha\rho_n & \alpha\delta_n \\ \delta_n & \gamma_n & \alpha\rho_n \\ \rho_n & \delta_n & \gamma_n \end{pmatrix}.$$

Thus (6) now follows by comparing $\lim_{n\to\infty} \delta_n/\rho_n$ with the limit found above. □

Remarks: (a) Note that the limit in (6) is independent of the choice of a, so that various corollaries can be obtained from particular choices of a.
(b) A similar method can be used to approximate square roots and roots of higher order (see Section 4).
(c) The pairs $(1,2)$ and $(1,3)$ in (8) can be replaced by other pairs to give limits of the form $\alpha^{j/3}$, $-4 \le j \le 4$.

Corollary 2. *Let α be a positive integer. Set*

$$a_n = \sum_{2i+3j\le n} \binom{i+j}{j} \binom{n-i-2j}{i+j} 3^{n-i-3j} (\alpha - 1)^{i+2j}.$$

Then $\lim_{n\to\infty} 1+(\alpha-1)\dfrac{a_{n-1}}{a_n} = \alpha^{1/3}$.

Proof. Let $a=1$ in Theorem 6. □

Corollary 3. *Let α be a positive integer. Set*

$$a_n = \sum_{i=0}^{n} \binom{2n+i}{2n-2i} 3^{3i}\alpha^{2n+i}(\alpha+1)^{2n-2i},$$

$$b_n = \sum_{i=0}^{n-1} \binom{2n+i}{2n-2i-1} 3^{3i+1}\alpha^{2n+i}(\alpha+1)^{2n-2i-1}.$$

Then

$$\lim_{n\to\infty} 1+\frac{\alpha-1}{\dfrac{a_n}{b_n}+1} = \alpha^{1/3}.$$

Proof. Let $a=0$ and replace n by $6n$ in Theorem 6. □

Corollary 4. *Let α be a positive integer. Set*

$$a_n = \sum_{i=0}^{\lfloor n/3\rfloor} \binom{n-2i}{i} 3^{n-3i}\alpha^{n-i}(\alpha-1)^{2i}.$$

Then

$$\lim_{n\to\infty} 1+\frac{\alpha^2-1}{\dfrac{a_n}{a_{n-1}}-\alpha+1} = \alpha^{2/3}.$$

Proof. Replace α by α^2 and then let $a=\alpha$ in Theorem 6. □

It is clear from (8) that the smaller the ratios $|\beta_2/\beta_1| = |\beta_3/\beta_1|$, the faster will be the rate of convergence in (6). It is also clear from (7) that these ratios can be made arbitrarily close to 1 by choosing a arbitrarily large. We are interested in how small this ratio can be made (to get fastest convergence) and what is the optimal choice of a for a given α to produce this smallest ratio.

Theorem 7. *Let α and a be as described in Theorem 6. Let the matrix A be as described at* (1). *Then the choice of a which gives the most rapid convergence is one of the two integers closest to*

$$\bar{a} = \frac{\alpha^{1/3}+\alpha}{1+\alpha^{1/3}}. \tag{9}$$

For this choice of a and $n \geq 3$,

$$\frac{A_{n,2,1}}{A_{n,3,1}} - \alpha^{1/3} = \left((\omega-1)\,\omega\left(\left(\frac{-\omega}{2}\right)^n - \left(\frac{-\omega^2}{2}\right)^n\right) + \frac{n\delta_3}{2^n\alpha^{1/3}} + \frac{\delta_4}{2^{2n}}\right)\alpha^{1/3}, \tag{10}$$

where $\omega = \exp(2\pi\imath/3)$, $|\delta_3| \leq 8$ and $|\delta_4| \leq 48$.

Proof. For the moment we consider a to be a real variable and define

$$h(a) = \left|\frac{\beta_2}{\beta_1}\right|^2 = \left|\frac{\beta_3}{\beta_1}\right|^2 = \frac{\beta_2\beta_3}{\beta_1^2} = \frac{a^2 - a\,\alpha^{1/3} + \alpha^{2/3} - a\,\alpha^{2/3} - \alpha + \alpha^{4/3}}{\left(a + \alpha^{1/3} + \alpha^{2/3}\right)^2}.$$

The function $h(a)$ achieves its minimum at

$$a = \bar{a} := \frac{\alpha^{1/3} + \alpha}{1 + \alpha^{1/3}} \text{ and } h(\bar{a}) = \frac{\left(-1 + \alpha^{1/3}\right)^2}{4\,\left(1 + \alpha^{1/3} + \alpha^{2/3}\right)}.$$

Hence for large α the best possible choice of a is one of the two integers closest to $\bar{a}$, say

$$a' = \frac{\alpha^{1/3} + \alpha}{1 + \alpha^{1/3}} + \eta,$$

with $|\eta| < 1$. With this choice,

$$\begin{aligned}\frac{\beta_2\beta_3}{\beta_1^2} &= \frac{1}{4} - \frac{3}{4}\frac{\left(1 + \alpha^{1/3} + \alpha^{2/3}\right) 4\,\alpha + \left(1 + \alpha^{1/3}\right)\,\eta\,\left(4\,\alpha^{2/3} - \eta - \alpha^{1/3}\,\eta\right)}{\left(2\,\left(1 + \alpha^{1/3} + \alpha^{2/3}\right)\,\alpha^{1/3} + \left(1 + \alpha^{1/3}\right)\,\eta\right)^2} \\ &=: \frac{1}{4} + \frac{3}{4}g(\eta).\end{aligned}$$

Next, considering $g(\eta)$ as a function of η,

$$g'(\eta) = \frac{4\left(1 + \alpha^{1/3}\right)^4 \alpha^{1/3}\,\eta}{\left(2\,\alpha^{1/3} + 2\,\alpha^{2/3} + 2\,\alpha + \eta + \alpha^{1/3}\,\eta\right)^3}$$

Thus, since $g'(0) = 0$ and $g(1) < g(-1)$,

$$\frac{1}{4} + \frac{3}{4}g(0) \le \frac{\beta_2\beta_3}{\beta_1^2} \le \frac{1}{4} + \frac{3}{4}g(-1),$$

or

$$\frac{1}{4} - \frac{3}{4}\frac{\alpha^{1/3}}{\left(1 + \alpha^{1/3} + \alpha^{2/3}\right)} \le \frac{\beta_2\beta_3}{\beta_1^2} \le \frac{1}{4} - \frac{3}{4}\frac{\left(-1 + \alpha^{1/3}\right)\,\left(1 + 3\,\alpha^{1/3} + 8\,\alpha^{2/3} + 8\,\alpha + 4\,\alpha^{4/3}\right)}{\left(-1 + \alpha^{1/3} + 2\,\alpha^{2/3} + 2\,\alpha\right)^2}.$$

Thus $\beta_2\beta_3/\beta_1^2 < 1/4$ or $|\beta_2/\beta_1| = |\beta_3/\beta_1| < 1/2$, for $\alpha > 1$. Also,

$$\begin{aligned}\frac{\beta_2}{\beta_1} &= \frac{\alpha^{1/3} - \alpha^{2/3} + \eta + \alpha^{1/3}\,\eta + \alpha^{1/3}\,\omega - \alpha\,\omega}{2\,\alpha^{1/3} + 2\,\alpha^{2/3} + 2\,\alpha + \eta + \alpha^{1/3}\,\eta} \\ &= -\frac{\omega}{2} + \frac{\delta_1}{\alpha^{1/3}}, \\ \frac{\beta_3}{\beta_1} &= \frac{-\alpha^{2/3} + \alpha + \eta + \alpha^{1/3}\,\eta - \alpha^{1/3}\,\omega + \alpha\,\omega}{2\,\alpha^{1/3} + 2\,\alpha^{2/3} + 2\,\alpha + \eta + \alpha^{1/3}\,\eta} \\ &= -\frac{\omega^2}{2} + \frac{\delta_2}{\alpha^{1/3}},\end{aligned} \tag{11}$$

where $|\delta_1|$, $|\delta_2| < 1$. (We omit the details of these calculations. The first equation is simply solved for δ_1, the solution is multiplied by its conjugate $\overline{\delta_1} = \delta_2$, the resulting real number is shown to be monotone decreasing as a function of η by differentiating with respect to η, and finally it is shown that $\delta_1\overline{\delta_1} < 1$ at $\eta = -1$.)

Note that these ratios $|\beta_2/\beta_1| = |\beta_3/\beta_1|$ increase quite slowly with α: $|\beta_2/\beta_1| < 0.45$, for $\alpha < 3000$, for example. Returning to large α,

$$\begin{aligned}(12)\qquad \frac{A_{n,2,1}}{A_{n,3,1}} - \alpha^{1/3} &= \left(\frac{{\beta_1}^n + \omega^2\,{\beta_2}^n + \omega\,{\beta_3}^n}{{\beta_1}^n + \omega\,{\beta_2}^n + \omega^2\,{\beta_3}^n} - 1\right)\alpha^{1/3}\\ &= \frac{(1-\omega)\;\omega\;(-{\beta_2}^n + {\beta_3}^n)}{{\beta_1}^n + \omega\,{\beta_2}^n + \omega^2\,{\beta_3}^n}\alpha^{1/3}\\ &= \left((\omega-1)\;\omega\;\left(\left(\frac{-\omega}{2}\right)^n - \left(\frac{-\omega^2}{2}\right)^n\right) + \frac{n\delta_3}{2^n\alpha^{1/3}} + \frac{\delta_4}{2^{2n}}\right)\alpha^{1/3},\end{aligned}$$

where $|\delta_3| \le 8$ and $|\delta_4| \le 48$. Note that we have used (11) to replace the ratios β_2/β_1 and β_3/β_1 in the final expression. □

Remark: Note that for $n \ge 3$ and $\alpha > 2^{4n}$, we have the following:

$$(13)\qquad 2^n\left(\frac{A_{n,2,1}}{A_{n,3,1}\,\alpha^{1/3}} - 1\right) = \begin{cases} -3 + \frac{K_n}{2^n}, & n \equiv 1,\,2\,(\mod 6),\\ 0 + \frac{K_n}{2^n}, & n \equiv 3,\,6\,(\mod 6),\\ 3 + \frac{K_n}{2^n}, & n \equiv 4,\,5\,(\mod 6),\end{cases}$$

where $|K_n| < 61$.

3. Approximating the Real Root of an Arbitrary Cubic

If the zeros of $a\,x^3 + bx^2 + cx + d$ are β_1, β_2 and β_3, then the zeros of $x^3 + (9ac - 3b^2)x + 2b^3 - 9abc + 27a^2d$ are $3a\beta_1 + b$, $3a\beta_2 + b$ and $3a\beta_3 + b$. Thus, in finding the roots of a general cubic equation, it is sufficient to study cubics of the form $f(x) = x^3 - px - q$. For simplicity, here we restrict to the case $p > 0$, $q > 0$ and $27q^2 - 4p^3 > 0$, so that $f(x)$ has exactly one real root, which is largest in absolute value. We have the following theorem.

Theorem 8. *Let $p > 0$, $q > 0$ be integers such that $27q^2 - 4p^3 > 0$. Define*

$$a_n = \sum_{2i+3j\le n} \binom{i+j}{j}\binom{n-i-2j}{i+j} 3^{n-2i-3j}(3-p)^i(q-p+1)^j.$$

Then

$$(14)\qquad -1 + \lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \frac{(2/3)^{1/3}\,p}{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}} + \frac{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}}{2^{1/3}\,3^{2/3}},$$

is the real root of $f(x) = x^3 - px - q$.

Proof. As before, let $\omega = \exp(2\pi\imath/3)$ and set

$$A = \begin{pmatrix} 1 & p & q\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}.$$

Define

$$\alpha = \frac{(2/3)^{1/3}\, p}{\left(9\,q - \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}}, \qquad \beta = \frac{\left(9\,q - \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}}{2^{1/3}\,3^{2/3}}.$$

The eigenvalues of A are

$$\begin{aligned} \gamma_1 &= 1 + \alpha + \beta, \\ \gamma_2 &= 1 + \alpha\,\omega^2 + \beta\,\omega, \\ \gamma_3 &= 1 + \alpha\,\omega + \beta\,\omega^2. \end{aligned} \tag{15}$$

Set

$$M = \begin{pmatrix} (\alpha+\beta)^2 & (\beta\,\omega + \alpha\,\omega^2)^2 & (\alpha\,\omega + \beta\,\omega^2)^2 \\ \alpha+\beta & \beta\,\omega + \alpha\,\omega^2 & \alpha\,\omega + \beta\,\omega^2 \\ 1 & 1 & 1 \end{pmatrix}$$

and then

$$M^{-1} A\, M = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}.$$

Here we use the facts that $q = \alpha^3 + \beta^3$ and $p = 3\alpha\beta$. Clearly

$$A^n = M \begin{pmatrix} \gamma_1^{\,n} & 0 & 0 \\ 0 & \gamma_2^{\,n} & 0 \\ 0 & 0 & \gamma_3^{\,n} \end{pmatrix} M^{-1}.$$

As before, let $A_{n,\,i\,j}$ denote the $(i,\,j)$ entry of A^n. It is straightforward to show (preferably after using a computer algebra system like *Mathematica* to perform the matrix multiplications) that

$$\begin{aligned} A_{n,2,1} &= \frac{(-1+\gamma_1)\ {\gamma_1}^n}{(\gamma_1-\gamma_2)\ (\gamma_1-\gamma_3)} + \frac{(-1+\gamma_3)\ {\gamma_3}^n}{(\gamma_1-\gamma_3)\ (\gamma_2-\gamma_3)} + \frac{(-1+\gamma_2)\ {\gamma_2}^n}{(\gamma_1-\gamma_2)\ (-\gamma_2+\gamma_3)}, \\ A_{n,3,1} &= \frac{-\left({\gamma_2}^n\,\gamma_3\right) + \gamma_2\,{\gamma_3}^n + {\gamma_1}^n\ (-\gamma_2+\gamma_3) + \gamma_1\ ({\gamma_2}^n - {\gamma_3}^n)}{(\gamma_1-\gamma_2)\ (\gamma_1-\gamma_3)\ (-\gamma_2+\gamma_3)}. \end{aligned}$$

Since $\gamma_1 > |\gamma_2|,\ |\gamma_3|$, it follows that

$$\lim_{n\to\infty} \frac{A_{n,\,2,\,1}}{A_{n,\,3,\,1}} = \gamma_1 - 1 = \alpha + \beta. \tag{16}$$

Next, the real zero of $x^3 - p\,x - q = 0$ is

$$\frac{(2/3)^{1/3}\, p}{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}} + \frac{\left(9\,q + \sqrt{81\,q^2 - 12\,p^3}\right)^{1/3}}{2^{1/3}\,3^{2/3}},$$

and some simple algebraic manipulation shows that this is equal to $\alpha + \beta$, so that the limit at (16) is indeed equal to this real zero.

Finally, the characteristic polynomial of A is $X^3 = 3\,X^2 - (3-P)X + q + 1 - p$, so that Corollary 1 gives, after setting $t = 3$, $d = q + 1 - p$ and $s = 3 - p$,

$$a_n = \sum_{2i+3j \le n} \binom{i+j}{j} \binom{n-i-2j}{i+j} 3^{n-2i-3j} (3-p)^i (q-p+1)^j$$

and $\epsilon_n = (1 - p + q)\ a_{n-3} + (-2 + p)\ a_{n-2} + a_{n-1}$, that

$$A^n = \begin{pmatrix} \epsilon_n & (q-p)\ a_{n-2} + p\, a_{n-1} & q\ (a_{n-1} - a_{n-2}) \\ a_{n-1} - a_{n-2} & \epsilon_n & q\, a_{n-2} \\ a_{n-2} & a_{n-1} - a_{n-2} & \epsilon_n - p\, a_{n-2} \end{pmatrix}.$$

The result now follows, after comparing $\lim_{\to\infty} A_{n,2,1}/A_{n,3,1}$ in the matrix above with the limit found at (16). □

4. Approximating roots of Arbitrary order of a positive integer

Khovanskii shows that the method of section 2 extends to roots of arbitrary order m, by considering the $m \times m$ matrix

$$A = \begin{pmatrix} a & \alpha & \alpha & \alpha & \dots & \alpha \\ 1 & a & \alpha & \alpha & \dots & \alpha \\ 1 & 1 & a & \alpha & \dots & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & \alpha \\ 1 & 1 & 1 & \dots & 1 & a \end{pmatrix}. \tag{17}$$

Again his result is dependent on the existence of $\lim_{n\to\infty} A_{n,i,j}/A_{n,u,v}$, for various pairs (i,j) and (u,v), but he does not suggest any criteria which guarantee these limits exist. We make his statement more precise in the following theorem.

Theorem 9. *Let A be the matrix defined above at* (17)*. Let $A_{n,i,j}$ denote the $(i,\ j)$ entry of A^n and suppose $a > 0$. Then*

$$\lim_{n\to\infty} \frac{A_{n,i,j}}{A_{n,u,v}} = \alpha^{(j+u-i-v)/m}. \tag{18}$$

Proof. Let ω_m be a primitive m-th root of unity. Define the matrix M by $(M)_{i,j} = \alpha^{(m-i)/m} \omega_m^{(m-j+1)i}$. Then $(M^{-1})_{i,j} = \frac{1}{m\,(M)_{j,i}}$. (We omit the proof of this statement. It can easily be checked by showing that multiplying M and the claimed inverse together gives the $m \times m$ identity matrix.)

It is now not difficult to show that $M^{-1} A\, M = \text{ diag } (\beta_1, \beta_2, \dots \beta_m)$, where $\text{ diag } (\beta_1, \beta_2, \dots \beta_m)$ is the matrix with $\beta_1, \beta_2, \dots \beta_m$ along the main diagonal and zeroes elsewhere. Here

$$\beta_i = a + \sum_{j=1}^{m-1} (\omega_m^{i-1} \alpha^{1/m})^j, \qquad i = 1, 2, \dots m,$$

are the eigenvalues of A. For $a > 0$, there is clearly a dominant eigenvalue, namely β_1.

(This condition could be relaxed to allow a to take some negative values, but the precise lower bound which makes $\beta_1 > |\beta_j|$, $j \neq 1$, is not so easy to determine in the case of arbitrary m.)

Next, it is clear that $A^n = M\text{diag}\,(\beta_1^n, \beta_2^n, \ldots \beta_m^n)M^{-1}$, and it is simple algebra to show that

$$A_{n,\,i\,j} = \frac{\alpha^{(j-i)/m}}{m} \sum_{k=1}^{m} \omega_m^{(m-k+1)(i-j)} \beta_k^n.$$

The result now follows, upon using the fact that β_1 is the dominant eigenvalue. □

Note, as in Theorem 6, that the limit is independent of the choice of a.

Theorem 5 could be used to produce results similar to those in Theorem 6 and its various corollaries, but the statements of these results become much more complicated with increasing m.

Also, we have not been able to determine the optimum choice of a that gives the most rapid convergence in (18). One difference between the $m = 3$ case and the general case is that the sub-dominant eigenvalues in the general case need not necessarily all have the same absolute value.

5. Concluding Remarks

For completeness we include the following neat construction by Khovanskii, one that enables good approximations to a root of an arbitrary polynomial to be found in many cases. Let

$$A = \begin{pmatrix} k & l\,a_m & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & k & l\,a_m & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & l\,a_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & k & l\,a_m & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & k & 0 & l\,a_m \\ -l\,a_0 & -l\,a_1 & -l\,a_2 & \ldots & -l\,a_{m-4} & -l\,a_{m-3} & k - l\,a_{m-1} & -l\,a_{m-2} \\ 0 & 0 & 0 & \ldots & 0 & 0 & l\,a_m & k \end{pmatrix}.$$

Here k and l are non-zero. If $\lim_{n\to\infty} A_{n,\,i,\,1}/A_{n,\,m,\,1}$ exists and equals, say, β_i, for $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots a_1 x + a_0$.

This can be seen as follows. Since the limits exist and $\beta_m = 1$, we get the system of equations

$$\begin{aligned}
\beta_i &= \frac{k\,\beta_i + l\,a_m\beta_{i+1}}{l\,a_m\beta_{m-1} + k}, \qquad 1 \le i \le m-3,\\
\beta_{m-2} &= \frac{k\,\beta_{m-2} + l\,a_m}{l\,a_m\beta_{m-1} + k}\\
\beta_{m-1} &= \frac{-l\,a_0\beta_1 - l\,a_1\beta_2 - \cdots - l\,a_{m-3}\beta_{m-2} + (k - l\,a_{m-1})\beta_{m-1} - l\,a_{m-2}}{l\,a_m\beta_{m-1} + k}.
\end{aligned}$$

This system of equations leads to the system $\beta_{m-1}\beta_{m-2} = 1$, $\beta_{i+1} = \beta_{m-1}\beta_i$, $1 \le i \le m-3$, and $a_m\beta_{m-1}^2 + a_{m-1}\beta_{m-1} + a_{m-2} + a_{m-3}\beta_{m-2} + \cdots + a_1\beta_2 + a_0\beta_1 = 0$. The result now follows, after multiplying the last equation by β_{m-1}^{m-2} and using the equations preceding it to eliminate β_i, $i \ne m-1$.

This situation is of course even more difficult to analyze: $f(x)$ may not even have real zeroes, or it may have multiple real zeroes, or even if it has a single real zero, this may not be enough to guarantee that the limits $\lim_{n\to\infty} A_{n,i,1}/A_{n,m,1}$, $1 \le i \le m$, exist,.

It would be interesting to find and prove general criteria, based on the entries of the matrix A, which guarantee that this method of Khovanskii's does lead to convergence to one of the roots.

References

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