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COUNTING FUNCTIONS AND FINITE DIFFERENCES

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Abstract

Any increasing function $p(d)$ on the natural numbers has an associated counting function $\pi(n)$ that yields the number of inputs d for which $p(d) \leq n$. In this article we derive three formulas that relate a sequence to its finite difference sequence by way of counting functions and the technique of summation by parts. We demonstrate our formulas by using them to produce several identities for Fibonacci numbers and binomial coefficients.

1. Introduction

In this article we use counting functions and summation by parts to establish three formulas that relate a sequence to its finite difference sequence. The first formula shows how to convert the sum of a sequence $\{y_k\}_{k=0}^{\infty}$ of real numbers indexed by another increasing sequence into a weighted sum of the finite difference sequence $\{\Delta y_k\}$, where $\Delta y_k = y_{k+1} - y_k$. The third shows how to do the same conversion for the sum of y_k's when they are indexed by a decreasing sequence. The second is a generalization of the summation by parts formula that allows for a certain kind of function composition. We illustrate our formulas by using them to derive identities for Fibonacci numbers and binomial coefficients. (We do not claim that these identities are new; the purpose of deriving them is to show the use of our results.) We also address a question of Koshy concerning weighted Fibonacci sums.

Section 2 contains our results on counting functions. These consist of Theorems 2.3 and 2.16; the special cases of these results for certain types of two-term recurrences, the Fibonacci numbers, and columns of binomial coefficients; and several identities involving Fibonacci numbers and binomial coefficients derived from these corollaries. Section 3 contains the

corresponding results on inverted counting functions, including Theorem 3.2, its corollaries, and associated identities. Section 4 consists of the proofs of our main results: Theorems 2.3, 2.16, and 3.2.

2. Counting Functions

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $p : \mathbb{N} \mapsto \mathbb{N}_0$ be an increasing function. Then let $\pi : \mathbb{N}_0 \mapsto \mathbb{N}_0 \cup \{\infty\}$ be defined by $\pi(i) = \#\{d \,|\, p(d) \leq i\}$, which we call a *counting function* for p.

We make extensive use of summation by parts, sometimes known as Abel's Lemma ([5, pp. 197–198]):

Lemma 2.1. $\sum_{i=1}^{n} f(i)\Delta g(i) = f(n+1)g(n+1) - f(1)g(1) - \sum_{i=1}^{n} g(i+1)\Delta f(i)$.

It is clear from the definition of p and its associated counting function π that the following is true:

Lemma 2.2. *For $i \geq 1$, if $\pi(i) < \infty$, then $p(d) = i \iff \pi(i-1) < d \leq \pi(i)$.*

Given p and its associated counting function π, then, we have the following.

Theorem 2.3. *If $p : \mathbb{N} \mapsto \mathbb{N}_0$ is an increasing function and $\pi(k) < \infty$, then*

$$\sum_{d=1}^{\pi(k)} y_{p(d)} = \pi(k)y_{k+1} - \sum_{d=0}^{k} \pi(d)\Delta y_d.$$

Proof. See Section 4. □

If the form of Δy_k is known then Theorem 2.3 can be used to generate an infinite number of identities involving the sequence $\{y_k\}$: Choose an increasing function p, determine π, and then apply the theorem. We illustrate the use of Theorem 2.3 by considering some special cases. For instance, if y_k satisfies a recurrence relation of the form $\Delta y_k = y_{k+1} - y_k = f(k-1)y_{k-1} + h(k-1)$ for some functions f and h and $k \geq 1$, then Theorem 2.3 becomes the following:

Corollary 2.4. *If $\pi(k) < \infty$, then*

$$\sum_{d=1}^{\pi(k)} y_{p(d)} = \pi(k)y_{k+1} - \pi(0)(y_1 - y_0) - \sum_{d=0}^{k-1} \pi(d+1)(f(d)y_d + h(d)).$$

Without knowing more about the form of f and h it is difficult to simplify this expression any further. However, an important special case of Corollary 2.4 has $f(k) = 1$, $h(k) = 0$, $y_0 = 0$, and $y_1 = 1$. Then $y_k = F_k$, the kth Fibonacci number. In this situation Corollary 2.4 becomes

Corollary 2.5. *If $\pi(k) < \infty$, then*

$$\sum_{d=1}^{\pi(k)} F_{p(d)} = \pi(k)F_{k+1} - \pi(0) - \sum_{d=1}^{k-1} \pi(d+1)F_d.$$

Another important special case of Theorem 2.3 has y_k equal to the binomial coefficient $\binom{k}{j}$, for fixed $j \geq 0$. Here $\Delta y_k = \binom{k+1}{j} - \binom{k}{j} = \binom{k}{j-1}$ (where $\binom{k}{-1}$ is taken to be 0 for $k \geq 0$). Then Theorem 2.3 becomes

Corollary 2.6. *If $\pi(k) < \infty$, then*

$$\sum_{d=1}^{\pi(k)} \binom{p(d)}{j} = \pi(k)\binom{k+1}{j} - \sum_{d=0}^{k} \pi(d)\binom{d}{j-1}.$$

Corollaries 2.5 and 2.6 can be used to generate an infinite number of identities involving Fibonacci numbers and binomial coefficients. Here are some examples for which $p(d)$ and $\pi(i)$ are simple enough to obtain fairly clean identities.

- If $p(d) = d$, then $\pi(i) = i$. With $k = n$, we have

  $$F_1 + F_2 + F_3 + \cdots + F_n = nF_{n+1} - (2F_1 + 3F_2 + \cdots + nF_{n-1}).$$

  Rearranging this equation yields

  Identity 2.7.

  $$3F_1 + 4F_2 + 5F_3 + \cdots + (n+1)F_{n-1} = nF_{n+1} - F_n.$$

  We also have

  $$\binom{1}{j} + \binom{2}{j} + \cdots + \binom{n}{j} = n\binom{n+1}{j} - \left(\binom{1}{j-1} + 2\binom{2}{j-1} + \cdots + n\binom{n}{j-1}\right).$$

  Reindexing and using the known result that $\sum_{d=0}^{n} \binom{d}{j} = \binom{n+1}{j+1}$ ([3, p. 160]) we can express this as

  Identity 2.8.

  $$\sum_{d=0}^{n} d\binom{d}{j} = n\binom{n+1}{j+1} - \binom{n+1}{j+2}.$$

- If $p(d) = 2d$, we have $\pi(i) = \lfloor i/2 \rfloor$. If $k = 2n$, then

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = nF_{2n+1} - (F_1 + F_2 + 2F_3 + 2F_4 + \cdots + nF_{2n-1}),$$

or

Identity 2.9.

$$F_1 + 2F_2 + 2F_3 + 3F_4 + 3F_5 + \cdots + nF_{2n-1} = nF_{2n+1} - F_{2n}.$$

For the binomial coefficients we have

Identity 2.10.

$$\sum_{d=1}^{n} \binom{2d}{j} = n\binom{2n+1}{j} - \sum_{d=0}^{2n} \left\lfloor \frac{d}{2} \right\rfloor \binom{d}{j-1}.$$

- In general, for $p(d) = ad$, where a is a positive integer, $\pi(i) = \lfloor i/a \rfloor$. For $k = an$, we have

Identity 2.11.

$$\sum_{d=1}^{an-1} \left(\left\lfloor \frac{d+1}{a} \right\rfloor + [a|d] \right) F_d = nF_{an+1} - F_{an},$$

where $[a|d]$ is 1 if a divides d and 0 otherwise, and

Identity 2.12.

$$\sum_{d=1}^{n} \binom{ad}{j} = n\binom{an+1}{j} - \sum_{d=0}^{an} \left\lfloor \frac{d}{a} \right\rfloor \binom{d}{j-1}.$$

- If $p(d) = \lfloor d/a \rfloor$, $\pi(i) = ai + a - 1$. For $k = n$, we have

$$a\,(F_1 + F_2 + \cdots + F_n) = -\left[(3a-1)F_1 + (4a-1)F_2 + \cdots ((n+1)a-1)F_{n-1}\right] + (an + a - 1)F_{n+1} - (a-1),$$

or

Identity 2.13.

$$\sum_{d=1}^{n-1} ((d+3)a - 1)\, F_d = (an + a - 1)F_{n+1} - aF_n - a + 1.$$

(The binomial coefficient identity for this p and π reduces quickly to the identity in the $p(d) = d$ case.)

Here are two additional examples using well-known increasing functions.

- For $p(d) = a^d$, where a is a positive integer larger than 1, $\pi(i) = \begin{cases} \lfloor \log_a i \rfloor, & i \neq 0; \\ 0, & i = 0. \end{cases}$

  When $k = a^n$, we have

  Identity 2.14.

$$\sum_{d=1}^{n} F_{a^d} = nF_{a^n+1} - \sum_{d=1}^{a^n-1} \lfloor \log_a(d+1) \rfloor F_d,$$

  and

  Identity 2.15.

$$\sum_{d=1}^{n} \binom{a^d}{j} = n\binom{a^n+1}{j} - \sum_{d=1}^{a^n} \lfloor \log_a d \rfloor \binom{d}{j-1}.$$

- We could also take $p(d)$ to be the dth prime and $\pi(i)$ to be the prime-counting function.

If we require p to be strictly increasing then we can use counting functions to generalize the summation by parts formula (Lemma 2.1, Abel's Lemma):

Theorem 2.16. *If $p : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function and $\pi(k) < \infty$, then, for any function g,*

$$\sum_{d=1}^{\pi(k)} g(p(d))\Delta y_d = g(k+1)y_{\pi(k)+1} - g(1)y_1 - \sum_{d=1}^{k} y_{\pi(d)+1}\Delta g(d).$$

Summation by parts is the special case $p(d) = \pi(d) = d$.

Proof. See Section 4. □

If y_k satisfies a two-term recurrence relation of the form $\Delta y_k = y_{k+1} - y_k = f(k-1)y_{k-1} + h(k-1)$ then Theorem 2.16 becomes

Corollary 2.17. *If $p : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function and $\pi(k) < \infty$, then, for any function g,*

$$\sum_{d=1}^{\pi(k)} g(p(d))(f(k-1)y_{k-1} + h(k-1)) = g(k+1)y_{\pi(k)+1} - g(1)y_1 - \sum_{d=1}^{k} y_{\pi(d)+1}\Delta g(d).$$

If $f(k) = 1$, $g(k) = 0$, and $y_0 = y_1 = 1$, then $y_k = F_{k+1}$, the $k+1$ Fibonacci number. Corollary 2.17 produces

Corollary 2.18. *If $p : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function and $\pi(k) < \infty$, then, for any function g,*

$$\sum_{d=1}^{\pi(k)} g(p(d))F_d = g(k+1)F_{\pi(k)+2} - g(1) - \sum_{d=1}^{k} F_{\pi(d)+2}\Delta g(d).$$

For the case $y_k = \binom{k}{j}$, $j \geq 0$, Theorem 2.16 yields

Corollary 2.19. *If $p : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function and $\pi(k) < \infty$, then, for any function g,*

$$\sum_{d=1}^{\pi(k)} g(p(d))\binom{d}{j-1} = g(k+1)\binom{\pi(k)+1}{j} - g(1)\binom{1}{j} - \sum_{d=1}^{k}\binom{\pi(d)+1}{j}\Delta g(d).$$

Here are some examples illustrating Corollaries 2.18 and 2.19.

- Let $p(d)$ denote the dth prime number, and let $g(d) = d$. Then, for $k = n$, we have the following identity relating the sum of the products of primes and Fibonacci numbers with the sum of Fibonacci numbers indexed by the prime-counting function:

  Identity 2.20.

  $$\sum_{d=1}^{\pi(n)} p(d)F_d = (n+1)F_{\pi(n)+2} - 1 - \sum_{d=1}^{n} F_{\pi(d)+2}.$$

  We have the same type of result with binomial coefficients, for $j \geq 0$:

  Identity 2.21.

  $$\sum_{d=1}^{\pi(n)} p(d)\binom{d}{j} = (n+1)\binom{\pi(n)+1}{j+1} - \binom{1}{j+1} - \sum_{d=1}^{n}\binom{\pi(d)+1}{j+1}.$$

  The case $j = 0$ of Identity 2.21 produces a particularly clean result:

  Identity 2.22.

  $$\sum_{d=1}^{\pi(n)} p(d) + \sum_{d=1}^{n-1}\pi(d) = n\pi(n).$$

  A variation on Identity 2.22 is perhaps of interest as well. Replacing n by $p(n)$ in Identity 2.22 we obtain

  Identity 2.23.

  $$\sum_{d=1}^{n} p(d) + \sum_{d=1}^{p(n)-1}\pi(d) = np(n).$$

Identities 2.20 – 2.23, of course, hold for any strictly increasing function p and its corresponding π, not just the sequence of primes and the prime-counting function.

- If $g(d) = 1/d$, then for $k = n$, we have, for any p,

  Identity 2.24.
  $$\sum_{d=1}^{\pi(n)} \frac{F_d}{p(d)} = \frac{F_{\pi(n)+2}}{n+1} - 1 + \sum_{d=1}^{n} \frac{F_{\pi(d)+2}}{d(d+1)},$$

  and, for $j \geq 0$,

  Identity 2.25.
  $$\sum_{d=1}^{\pi(n)} \frac{\binom{d}{j-1}}{p(d)} = \frac{\binom{\pi(n)+1}{j}}{n+1} - \binom{1}{j} + \sum_{d=1}^{n} \frac{\binom{\pi(d)+1}{j}}{d(d+1)}.$$

- If $g(d) = 1/F_d^2$, $p(d) = d$, and $k = n$, then Corollary 2.18 yields

  Identity 2.26.
  $$\sum_{d=1}^{n} \frac{1}{F_d} = \frac{F_n}{F_{n+1}^2} + \sum_{d=1}^{n} \left(\frac{F_{d+1}^3 - F_d^3}{F_{d+1}^2 F_d^2} \right).$$

  The derivation of Identity 2.26 is as follows:

$$\begin{aligned}
\sum_{d=1}^{n} \frac{1}{F_d} &= \frac{F_{n+2}}{F_{n+1}^2} - 1 - \sum_{d=1}^{n} F_{d+2} \left(\frac{1}{F_{d+1}^2} - \frac{1}{F_d^2} \right) \\
&= \frac{F_{n+1} + F_n}{F_{n+1}^2} - 1 - \sum_{d=1}^{n} \left(\frac{F_{d+1} + F_d}{F_{d+1}^2} - \frac{F_{d+1} + F_d}{F_d^2} \right) \\
&= \frac{1}{F_{n+1}} + \frac{F_n}{F_{n+1}^2} - 1 - \sum_{d=1}^{n} \left(\frac{1}{F_{d+1}} + \frac{F_d}{F_{d+1}^2} - \frac{F_{d+1}}{F_d^2} - \frac{1}{F_d} \right) \\
&= \frac{1}{F_{n+1}} + \frac{F_n}{F_{n+1}^2} - 1 - \frac{1}{F_{n+1}} + 1 - \sum_{d=1}^{n} \left(\frac{F_d}{F_{d+1}^2} - \frac{F_{d+1}}{F_d^2} \right) \\
&= \frac{F_n}{F_{n+1}^2} + \sum_{d=1}^{n} \left(\frac{F_{d+1}}{F_d^2} - \frac{F_d}{F_{d+1}^2} \right) \\
&= \frac{F_n}{F_{n+1}^2} + \sum_{d=1}^{n} \left(\frac{F_{d+1}^3 - F_d^3}{F_{d+1}^2 F_d^2} \right).
\end{aligned}$$

We know that F_d grows as ϕ^d ([4, p. 240]), where $\phi = (1 + \sqrt{5})/2$. Thus as n approaches ∞, the first term on the right-hand side of Identity 2.26 goes to 0, and we have

Identity 2.27.
$$\sum_{d=1}^{\infty} \frac{1}{F_d} = \sum_{d=1}^{\infty} \left(\frac{F_{d+1}^3 - F_d^3}{F_{d+1}^2 F_d^2} \right).$$

(It is known that $\sum_{d=1}^{\infty} 1/F_d$ converges; see, for example, Brousseau [2].)

We can also use Corollary 2.18 to give a new derivation of the following known result, valid for $a > \phi$. (See Koshy [4, pp. 424–425].)

Identity 2.28.

$$\sum_{d=1}^{\infty} \frac{F_d}{a^d} = \frac{a}{a^2 - a - 1}$$

In the process, we obtain the following formula for the partial sum of the left-hand side of Identity 2.28:

Identity 2.29.

$$\sum_{d=1}^{n} \frac{F_d}{a^d} = \frac{a^{n+1} - aF_{n+1} - F_n}{a^n(a^2 - a - 1)}.$$

From Corollary 2.18, we have, for $g(d) = 1/a^d$, $p(d) = d$, and $k = n$,

$$\begin{aligned}
\sum_{d=1}^{n} \frac{F_d}{a^d} &= \frac{F_{n+2}}{a^{n+1}} - \frac{1}{a} - \sum_{d=1}^{n} F_{d+2}\left(\frac{1}{a^{d+1}} - \frac{1}{a^d}\right) \\
&= \frac{F_{n+2}}{a^{n+1}} - \frac{1}{a} - \sum_{d=1}^{n} F_{d+2}\left(\frac{1-a}{a^{d+1}}\right) \\
&= \frac{F_{n+2}}{a^{n+1}} - \frac{1}{a} - a(1-a)\sum_{d=3}^{n+2} \frac{F_d}{a^d} \\
&= \frac{F_{n+2}}{a^{n+1}} - \frac{1}{a} - a(1-a)\sum_{d=1}^{n} \frac{F_d}{a^d} + a(1-a)\left(\frac{1}{a} + \frac{1}{a^2} - \frac{F_{n+1}}{a^{n+1}} - \frac{F_{n+2}}{a^{n+2}}\right).
\end{aligned}$$

Moving the summation on the right-hand side to the left and combining terms on the right yields

$$\begin{aligned}
(1 + a - a^2)\sum_{d=1}^{n} \frac{F_d}{a^d} &= \frac{F_{n+2}}{a^n} - a + (a-1)\frac{F_{n+1}}{a^n} \\
\Rightarrow \sum_{d=1}^{n} \frac{F_d}{a^d} &= \left(\frac{F_{n+1}}{a^{n-1}} + \frac{F_n}{a^n} - a\right)\frac{1}{1 + a - a^2} \\
\Rightarrow \sum_{d=1}^{n} \frac{F_d}{a^d} &= \frac{a^{n+1} - aF_{n+1} - F_n}{a^n(a^2 - a - 1)}. \qquad (2.1)
\end{aligned}$$

The expression (2.1) is Identity 2.29, the partial sum of the left-hand side of Identity 2.28. Since $a > \phi$, as $n \to \infty$ we have Identity 2.28; namely,

$$\sum_{d=1}^{\infty} \frac{F_d}{a^d} = \frac{a}{a^2 - a - 1}.$$

Finally, we can use Corollary 2.18 to address a question of Koshy concerning weighted Fibonacci sums. Let $m \in \mathbb{N}_0$, and let $S(m) = \sum_{d=1}^{n} d^m F_d$. Koshy ([4, pp. 349–354]) discusses a method by which one can obtain the formula for $S(m-1)$ in terms of $S(m)$. He then states that a formula for $S(m)$ in terms of $S(m-1)$ would be a "tremendous advantage in the study of weighted Fibonacci and Lucas sums" ([4, p. 354]). While we do not have a formula for $S(m)$ in terms of only $S(m-1)$, we can use Theorem 2.16 to obtain a formula for $S(m)$ in terms of $S(m-1), S(m-2), \ldots, S(0)$. With $p(d) = d$, $g(d) = d^m$, and $k = n$, we have

Identity 2.30.

$$\sum_{d=1}^{n} d^m F_d = (n-1)^m F_{n+1} + n^m F_n + (-1)^{m+1} + \sum_{i=0}^{m} \binom{m}{i} (-1)^i (2^i - 1) S(m-i).$$

The derivation is as follows.

$$\begin{aligned}
\sum_{d=1}^{n} d^m F_d =& (n+1)^m F_{n+2} - 1 - \sum_{d=1}^{n} F_{d+2} \left((d+1)^m - d^m\right) \\
=& (n+1)^m F_{n+2} - 1 - \sum_{d=3}^{n+2} F_d \left((d-1)^m - (d-2)^m\right) \\
=& (n+1)^m F_{n+2} - 1 - ((n+1)^m - n^m) F_{n+2} - (n^m - (n-1)^m) F_{n+1} \\
&+ 1 - (-1)^m - \sum_{d=1}^{n} F_d \left((d-1)^m - (d-2)^m\right) \\
=& (n-1)^m F_{n+1} + n^m F_n + (-1)^{m+1} - \sum_{d=1}^{n} F_d \sum_{i=0}^{m} \binom{m}{i} d^{m-i} ((-1)^i - (-2)^i) \\
=& (n-1)^m F_{n+1} + n^m F_n + (-1)^{m+1} + \sum_{d=1}^{n} F_d \sum_{i=0}^{m} \binom{m}{i} d^{m-i} (-1)^i (2^i - 1) \\
=& (n-1)^m F_{n+1} + n^m F_n + (-1)^{m+1} + \sum_{i=0}^{m} \binom{m}{i} (-1)^i (2^i - 1) \sum_{d=1}^{n} F_d \, d^{m-i} \\
=& (n-1)^m F_{n+1} + n^m F_n + (-1)^{m+1} + \sum_{i=0}^{m} \binom{m}{i} (-1)^i (2^i - 1) S(m-i).
\end{aligned}$$

When $i = 0$ in the summation on the right, we have $2^0 - 1 = 0$. Thus $S(m)$ does not appear in this sum, yielding $S(m)$ in terms of $S(m-1), S(m-2), \ldots, S(0)$.

3. Inverted Counting Functions

We can also defined inverted counting functions that produce a result similar to Theorem 2.3. Let $p_D : \mathbb{N} \mapsto \mathbb{N}_0$ be a decreasing function that eventually reaches 0. Define $\pi_D : \mathbb{N}_0 \mapsto \mathbb{N}_0 \cup \{\infty\}$ by $\pi_D(i) = \#\{d \,|\, p(d) \geq i\}$, which we call an *inverted counting function* for p.

As with Lemma 2.2, the definitions of a decreasing function p_D and its inverted counting function π_D imply the following relationship:

Lemma 3.1. $p_D(d) = i \iff \pi_D(i+1) < d \leq \pi_D(i)$.

Given p_D and its associated inverted counting function π_D, we have the following:

Theorem 3.2. *If $n \geq \pi_D(1)$ then*

$$\sum_{d=1}^{n} y_{p_D(d)} = ny_0 + \sum_{d=0}^{\infty} \pi_D(d+1)\Delta y_d.$$

Proof. See Section 4. □

As with Theorem 2.3, we consider special cases of Theorem 3.2 in which y_k satisfies a recurrence relation of the form $y_{k+1} - y_k = f(k-1)y_{k-1} + g(k-1)$, in which $y_k = F_k$, and in which $y_k = \binom{k}{j}$ for fixed $j \geq 0$.

Corollary 3.3. *If y_k satisfies $y_{k+1} - y_k = f(k-1)y_{k-1} + g(k-1)$ then*

$$\sum_{d=1}^{n} y_{p_D(d)} = ny_0 + \pi_D(1)(y_1 - y_0) + \sum_{d=0}^{\infty} \pi_D(d+2)(f(d)y_d + g(d)).$$

Corollary 3.4.

$$\sum_{d=1}^{\infty} F_{p_D(d)} = \pi_D(1) + \sum_{d=1}^{\infty} \pi_D(d+2)F_d.$$

Let $[j = 0]$ be 1 if $j = 0$ and 0 otherwise. Then we have

Corollary 3.5.

$$\sum_{d=1}^{n} \binom{p_D(d)}{j} = n[j=0] + \sum_{d=0}^{\infty} \pi_D(d+1)\binom{d}{j-1},$$

As with the corollaries to Theorem 2.3, some $p_D(d)$ and $\pi_D(i)$ are simple enough that we can obtain fairly clean identities. For example:

- If $p_D(d) = \lfloor n/d \rfloor$ for some n, then $\pi_D(i) = \lfloor n/i \rfloor$. Corollary 3.4 yields

Identity 3.6.

$$\sum_{d=1}^{n}\left(F_{\lfloor n/d\rfloor}-\left\lfloor\frac{n}{d+2}\right\rfloor F_d\right)=n.$$

Corollary 3.5 produces

Identity 3.7.

$$\sum_{d=1}^{n}\left(\binom{\lfloor n/d\rfloor}{j}-\left\lfloor\frac{n}{d}\right\rfloor\binom{d-1}{j-1}\right)=n[j=0].$$

- If $p_D(d)=\max\{n-d,0\}$ for some n, then $\pi_D(i)=\max\{n-i,0\}$. We have

$$\sum_{d=1}^{n}F_{n-d}=n-1+\sum_{d=1}^{n-2}(n-d-2)F_d.$$

Rearranging this equation and reindexing (after the second line) yields

$$\begin{aligned} & F_{n-1}+F_{n-2}+\cdots+F_1=n-1+((n-3)F_1+(n-4)F_2+\cdots+F_{n-3})\\ \Rightarrow\quad & F_n-n+1=(n-4)F_1+(n-5)F_2+\cdots+F_{n-4}\\ \Rightarrow\quad & nF_1+(n-1)F_2+\cdots+F_n=F_{n+4}-n-3. \end{aligned}$$

Thus we have

Identity 3.8.

$$nF_1+(n-1)F_2+\cdots+F_n=F_{n+4}-n-3,$$

which is given in Koshy [4, p. 342].

We also have, from Corollary 3.5,

Identity 3.9.

$$\sum_{d=1}^{n}\left(\binom{n-d}{j}-(n-d)\binom{d-1}{j-1}\right)=n[j=0].$$

- If $p_D(d)=\lfloor n/a^d\rfloor$ for some positive integers $n\geq 1$ and $a\geq 2$, then

$$\pi_D(i)=\begin{cases}\lfloor\log_a n/i\rfloor, & i\leq n;\\ 0, & i>n.\end{cases}$$

We have

Identity 3.10.

$$\sum_{d=1}^{\lfloor\log_a n\rfloor}F_{\lfloor n/a^d\rfloor}=\lfloor\log_a n\rfloor+\sum_{d=1}^{n-2}\left\lfloor\log_a\left(\frac{n}{d+2}\right)\right\rfloor F_d,$$

and

Identity 3.11.

$$\sum_{d=1}^{n}\binom{\lfloor n/a^d\rfloor}{j} = n[j=0] + \sum_{d=0}^{n-1}\left\lfloor \log_a\left(\frac{n}{d+1}\right)\right\rfloor \binom{d}{j-1}.$$

In closing this section we derive from Identity 3.11 an unusual formula for $\epsilon_p(n!)$, the exponent of a prime p in the factorization of $n!$ in primes.

Identity 3.12. *Let n be a positive integer, and let p be any prime with $p \le n$. Let $d_p(x)$ denote for $d \in \mathbb{N}$ the polynomial obtained after replacing p by x in the base-p representation of d. Then*

$$\epsilon_p(n!) = \sum_{d=1}^{\lfloor n/p\rfloor} \lfloor \log_p(n/d)\rfloor = \deg\left(\prod_{d=1}^{\lfloor n/p\rfloor} \lfloor n/d\rfloor_p(x)\right).$$

Proof. The right equality holds because $\deg(d_p(x)) = \lfloor \log_p d\rfloor$ and $\lfloor \log_p \lfloor n/d\rfloor\rfloor = \lfloor \log_p(n/d)\rfloor$. To prove the left equality we consider the case $a = p$, $j = 1$ of Identity 3.11. Since $0 \le \log_p(n/d) < 1$ if and only if $d > n/p$ we obtain

$$\sum_{d=1}^{\infty}\lfloor n/p_d\rfloor = \sum_{d=1}^{n}\lfloor n/p^d\rfloor = \sum_{d=1}^{n}\lfloor \log_p(n/d)\rfloor = \sum_{d=1}^{\lfloor n/p\rfloor}\lfloor \log_p(n/d)\rfloor.$$

The equality on the left now follows from the formula

$$\epsilon_p(n!) = \sum_{d=1}^{\infty}\lfloor n/p^d\rfloor$$

(see, for example, [1, p. 67]). □

4. Proofs of Main Results

Proof of Theorem 2.3.

For $d \le \pi(0)$, $p(d) = 0$. Thus

$$\sum_{d=1}^{\pi(0)} y_{p(d)} = \pi(0)y_0. \tag{4.1}$$

By Lemma 2.2, we have, for $i \geq 1$ and finite $\pi(i)$,

$$\sum_{d=\pi(i-1)+1}^{\pi(i)} y_{p(d)} = y_i \left(\pi(i) - \pi(i-1)\right). \tag{4.2}$$

Summing both sides of (4.2) as i ranges from 1 to k and including (4.1) yields

$$\sum_{d=1}^{\pi(k)} y_{p(d)} = \sum_{i=1}^{k} y_i \left(\pi(i) - \pi(i-1)\right) + \pi(0)y_0.$$

Applying summation by parts (Lemma 2.1) to the sum on the right side of this equation produces

$$\begin{aligned}\sum_{d=1}^{\pi(k)} y_{p(d)} &= \pi(k)\, y_{k+1} - \pi(0)y_1 - \sum_{i=1}^{k} \pi(i)\Delta y_i + \pi(0)y_0 \\ &= \pi(k)\, y_{k+1} - \sum_{i=0}^{k} \pi(i)\Delta y_i,\end{aligned}$$

completing the proof of the theorem.

Proof of Theorem 2.16.

Since p is a strictly increasing function, we have that, for $i \in \mathbb{N}$, $\pi(i) - \pi(i-1) = 1$ if i is in the range of p and 0 otherwise. Also, $\pi(0) = 0$. Therefore,

$$\begin{aligned}\sum_{d=1}^{\pi(k)} g(p(d))\Delta y_d &= \sum_{d=1}^{\pi(k)} g(p(d))(y_{d+1} - y_d) \\ &= \sum_{i=1}^{k} g(i)(y_{\pi(i)+1} - y_{\pi(i-1)+1}) \\ &= g(k+1)y_{\pi(k)+1} - g(1)y_{\pi(0)+1} - \sum_{i=1}^{k} y_{\pi(i)+1}\left(g(i+1) - g(i)\right) \\ &= g(k+1)y_{\pi(k)+1} - g(1)y_1 - \sum_{i=1}^{k} y_{\pi(i)+1}\Delta g(i).\end{aligned}$$

(The second-to-last step follows via Lemma 2.1.)

Proof of Theorem 3.2.

For $d > \pi_D(1)$, $p_D(d) = 0$. Thus

$$\sum_{d=\pi_D(1)+1}^{n} y_{p_D(d)} = y_0(n - \pi_D(1)). \tag{4.3}$$

By Lemma 3.1, we have, for $i \geq 1$,

$$\sum_{d=\pi_D(i+1)+1}^{\pi_D(i)} y_{p_D(d)} = y_i \left(\pi_D(i) - \pi_D(i+1)\right). \tag{4.4}$$

Summing both sides of (4.4) as i ranges from 1 to $k-1$ and including (4.3) yields

$$\sum_{d=\pi_D(k)+1}^{n} y_{p_D(d)} = \sum_{i=1}^{k-1} y_i \left(\pi_D(i) - \pi_D(i+1)\right) + y_0(n - \pi_D(1)).$$

Applying summation by parts (Lemma 2.1) to the sum on the right side of this equation gives us

$$\begin{aligned}\sum_{d=\pi_D(k)+1}^{n} y_{p_D(d)} &= -\pi_D(k)\, y_k + \pi_D(1)y_1 + \sum_{i=1}^{k-1} \pi_D(i+1)\Delta y_i + y_0(n - \pi_D(1)) \\ &= -\pi_D(k)\, y_k + ny_0 + \sum_{i=0}^{k-1} \pi_D(i+1)\Delta y_i.\end{aligned}$$

Letting k be any number larger than $p_D(1)$ yields the theorem.

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