INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 8 (2008), #A10

HUMAN PROOFS OF IDENTITIES BY OSBURN AND SCHNEIDER

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Received: 10/1/07, Accepted: 2/11/08, Published: 2/18/08

Abstract

Osburn and Schneider recently derived several combinatorial identities involving harmonic numbers using the computer programm Sigma. Here, they are derived by partial fraction decomposition and creative telescoping.

1. Introduction

In [4], sums of the type

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\cdot X$$

where computed, where X was one of the list

$$1,\ H_k,\ kH_k,\ kH_{n+k},\ kH_{n-k},\ H_k^{(2)},\dots,$$

with harmonic numbers of first and second order. The powerful computer software Sigma was used to evaluate these sums.

In this note, I want to show a method that humans can use; in this way we obtain "computer-free" proofs (although it is always helpful to use a computer, to ease computations). The method consists of the following steps:

- A rational function, of the form
$$\frac{(z+1)\dots(z+n)}{z(z-1)\dots(z-n)}\cdot Y$$
will undergo *partial fraction decomposition*.
- The resulting equation will be multiplied by z, and the limit $z\to\infty$ will be performed. In the simpler cases, this is already the desired identity.

- If, say, H_k is involved, we write it as

$$H_k = \sum_{j\geq 1} \left[\frac{1}{j} - \frac{1}{j+k}\right] = \sum_{j\geq 1} \frac{k}{j(j+k)}.$$

  We use an individual term $\frac{z}{j(j+z)}$ as part of the rational function. The resulting identities will be summed over all $j \geq 1$.
- To obtain the final result, these infinite series have to be evaluated. One possible route would be to use hypergeometric series and the huge collection of identities and transformations that is known [2]; the desired formula usually comes out as a limiting case.

  However, here, we decided to use *creative telescoping*, as popularized in the book [3].

This method was partially inspired by [1].

I don't want to go through the whole list of identities by Osburn and Schneider, but a fair amount of them of increasing difficulty will be treated, to convince the reader, that there are no principal limitations to the approach.

Factorials are, whenever necessary, defined by the Gamma function. In particular, $1/(-n)!$, for $n \in \mathbb{N}_0$, must be interpreted, as usual, by 0.

2. Identities

2.1.

Consider

$$\frac{(z+1)\dots(z+n)}{z(z-1)\dots(z-n)} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{1}{z-k}.$$

Multiplying this by z, and letting $z \to \infty$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k} = 1.$$

2.2.

Consider

$$\frac{(z+1)\dots(z+n-1)}{z(z-1)\dots(z-n)}\frac{1}{z+n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{1}{(n+k)^2}\frac{1}{z-k} + \frac{(n-1)!^2}{(2n)!}\frac{1}{z+n}.$$

The limit form is

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{1}{(n+k)^2} = -\frac{(n-1)!^2}{(2n)!}.$$

2.3.

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)}\frac{1}{j(j+z)} = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k}{j(j+k)}\frac{1}{z-k} + \frac{(j-1)!^2}{(j-n-1)!(n+j)!}\frac{1}{j+z}.$$

The limit form is

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k}{j+k} = -\frac{(j-1)!^2}{(j-n-1)!(n+j)!} + \frac{1}{j}.$$

Summing on $j \geq 1$ (and shifting the index), we get

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_k = \sum_{j\geq 0}\left[\frac{1}{j+1} - \frac{j!^2}{(j-n)!(n+1+j)!}\right].$$

Denote the summand of the last expression by $F(n,j)$. Then we have

$$F(n+1,j) - F(n,j) = G(n,j+1) - G(n,j),$$

with

$$G(n,j) = \frac{2j!^2}{(j-n-1)!(n+1+j)!(n+1)}.$$

Now set $S(n) := \sum_{j\geq 0} F(n,j)$, then

$$S(n+1) - S(n) = \lim_{J\to\infty}[G(n,J) - G(n,0)] = \frac{2}{n+1}.$$

Together with $S(0) = 0$, this leads to $S(n) = 2H_n$, and the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_k = 2H_n.$$

2.4.

Consider

$$A := \frac{(z+1)\dots(z+n)}{z\dots(z-n)}\frac{n+z}{j(j+n+z)}.$$

Then

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{n+k}{j(j+n+k)}\frac{1}{z-k} = A - \frac{(n+j-1)!^2}{(j-1)!(2n+j)!}\frac{1}{j+n+z}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{n+k}{j(j+n+k)} = \frac{1}{j} - \frac{(n+j-1)!^2}{(j-1)!(2n+j)!}.$$

Summing,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_{n+k} = \sum_{j\geq 0}\left[\frac{1}{j+1} - \frac{(n+j)!^2}{j!(2n+1+j)!}\right].$$

Denoting the term in the sum again by $F(n,j)$, we find

$$F(n+1,j) - F(n,j) = G(n,j+1) - G(n,j),$$

with

$$G(n,j) = \frac{(n+j)!^2(3n+3+2j)}{(j-1)!(2n+2+j)!(n+1)}.$$

For the summatory functions, we thus get

$$S(n+1) - S(n) = \frac{2}{n+1},$$

with the same result as before, and henceforth the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_{n+k} = 2H_n.$$

2.5.

Consider

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)}\frac{z}{j(j+z)}$$
$$= \sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k^2}{j(j+k)}\frac{1}{z-k} - \frac{(j-1)!j!}{(j-n-1)!(n+j)!}\frac{1}{j+z} + \frac{1}{j}$$

and the limiting form

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k^2}{j(j+k)} = \frac{n(n+1)}{j} - 1 + \frac{(j-1)!j!}{(j-n-1)!(n+j)!}.$$

Summing on j:

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_k=\sum_{j\geq 0}\left[\frac{n(n+1)}{j+1}-1+\frac{j!(j+1)!}{(j-n)!(n+1+j)!}\right].$$

Hence

$$(n+2)F(n,j)-nF(n+1,j)=G(n,j+1)-G(n,j)-2,$$

with

$$G(n,j)=\frac{2j!(j+1)!}{(j-n-1)!(n+1+j)!}.$$

Thus

$$(n+2)S(n)-nS(n+1)=\lim_{J\to\infty}[G(n,J)-2J-G(n,0)]=-2n(n+2),$$

and furthermore

$$\frac{S(n+1)}{(n+1)(n+2)}-\frac{S(n)}{n(n+1)}=\frac{2}{n+1}.$$

Unwinding this, we get

$$\frac{S(n)}{n(n+1)}=\frac{S(1)}{2}+\sum_{k=1}^{n-1}\frac{2}{k+1}=1+2H_n-2.$$

So we found $S(n)=n(n+1)(2H_n-1)$, and the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_k=n(n+1)(2H_n-1).$$

2.6.

Consider

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)}\frac{n+z}{j(j+n+z)}$$
$$=\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(n+k)}{j(j+n+k)}\frac{1}{z-k}-\frac{(n+j-1)!(n+j)!}{(j-1)!(2n+j)!}\frac{1}{j+n+z}+\frac{1}{j}$$

and the limiting form

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(n+k)}{j(j+n+k)}=\frac{n(n+1)}{j}-1+\frac{(n+j-1)!(n+j)!}{(j-1)!(2n+j)!}$$

which we sum:

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_{n+k}=\sum_{j\geq 0}\left[\frac{n(n+1)}{j+1}-1+\frac{(n+j)!(n+1+j)!}{j!(2n+1+j)!}\right].$$

Hence

$$(n+2)F(n,j)-nF(n+1,j)=G(n,j+1)-G(n,j)-2,$$

with

$$G(n,j)=\frac{(n+j)!(n+1+j)!(3n+4+2j)}{(j-1)!(2n+2+j)!}.$$

Consequently

$$(n+2)S(n)-nS(n+1)=\lim_{J\to\infty}[G(n,J)-2J-G(n,0)]=-n(2n+3).$$

The solution of this is $S(n)=2n(n+1)H_n-n^2$, and thus we have the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_{n+k}=2n(n+1)H_n-n^2.$$

2.7.

Consider

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)}\frac{n-z}{j(j+n-z)}$$
$$=\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(n-k)}{j(j+n-k)}\frac{1}{z-k}+\frac{(2n+j)!(j-1)!}{(n+j-1)!(n+j)!}\frac{1}{j+n-z}+\frac{1}{j}$$

and the limiting form

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(n-k)}{j(j+n-k)}=\frac{n(n+1)}{j}+1-\frac{(2n+j)!(j-1)!}{(n+j-1)!(n+j)!},$$

which we sum:

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_{n-k}=\sum_{j\geq 0}\left[\frac{n(n+1)}{j+1}+1-\frac{(2n+j+1)!j!}{(n+j)!(n+j+1)!}\right].$$

We have

$$-nF(n+1,j)+(n+2)F(n,j)=G(n,j+1)-G(n,j)+2,$$

with

$$G(n,j)=-\frac{(2n+j+1)!j!(3n+2+2j)}{(n+j)!(n+j+1)!}.$$

Hence

$$-nS(n+1)+(n+2)S(n)=\lim_{J\to\infty}[G(n,J)+2J-G(n,0)]$$
$$=-(n+2)(2n+1)+\frac{(2n+1)!}{n!(n+1)!}(3n+2),$$

or

$$-\frac{S(n+1)}{(n+1)(n+2)}+\frac{S(n)}{n(n+1)}=-\frac{2n+1}{n(n+1)}+\frac{(2n+1)!}{(n+1)!(n+2)!n}(3n+2),$$

and

$$-\frac{S(n)}{n(n+1)}+\frac{S(1)}{2}=-\sum_{k=1}^{n-1}\frac{2k+1}{k(k+1)}+\sum_{k=1}^{n-1}\frac{(2k+1)!}{(k+1)!(k+2)!k}(3k+2).$$

Thus

$$S(n) = 2n(n+1)H_n - (n+1)^2 - n(n+1)\sum_{k=1}^{n-1}\frac{(2k+1)!}{(k+1)!(k+2)!k}(3k+2).$$

So we derived the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}kH_{n-k}$$

$$= 2n(n+1)H_n - (n+1)^2 - n(n+1)\sum_{k=1}^{n-1}\frac{(2k+1)!}{(k+1)!(k+2)!k}(3k+2).$$

The paper [4] gives the different evaluation

$$2n(n+1)H_n - (n+1)^2 + (2n+1)\binom{2n}{n} - \frac{3}{2}n(n+1)\sum_{k=1}^{n}\frac{1}{k}\binom{2k}{k},$$

but it is a routine check to prove that they are the same.

2.8.

Consider

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)}\frac{2j+z}{j^2(j+z)^2} = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(k+2j)}{j^2(j+k)^2}\frac{1}{z-k}$$
$$+\frac{(j-1)!^2}{(n+j)!(j-n-1)!}\frac{1}{(j+z)^2}$$
$$+\frac{(j-1)!^2}{(n+j)!(j-n-1)!}(H_{j+n}+H_{j-n-1}-2H_{j-1})\frac{1}{j+z}$$

and the limiting form

$$\frac{1}{j^2} = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\frac{k(k+2j)}{j^2(j+k)^2} + \frac{(j-1)!^2}{(n+j)!(j-n-1)!}(H_{j+n}+H_{j-n-1}-2H_{j-1}).$$

Summing,

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_k^{(2)} = \sum_{j\geq 0}\left[\frac{1}{(j+1)^2} - \frac{j!^2}{(n+1+j)!(j-n)!}(H_{j+1+n}+H_{j-n}-2H_j)\right]$$
$$= \sum_{j\geq 0}\left[\frac{1}{(j+1)^2} + \frac{d}{dx}\frac{(j+x)!^2}{(n+1+j+x)!(j+x-n)!}\bigg|_{x=0}\right].$$

Now set

$$F(n,j) := \frac{x}{(j+1)^2} + \frac{(j+x)!^2}{(n+1+j+x)!(j+x-n)!},$$

and

$$G(n,j) = \frac{2(j+x)!^2}{(n+1+j+x)!(j+x-n-1)!},$$

then

$$(n+1)F(n,j)-(n+1)F(n+1,j)=G(n,j+1)-G(n,j).$$

Summing,

$$(n+1)S(n)-(n+1)S(n+1)=\lim_{J\to\infty}[G(n,J)-G(n,0)]=2-\frac{2x!^2}{(n+1+x)!(x-n-1)!}.$$

Further,

$$S(n)-S(n-1)=-\frac{2}{n}+\frac{2x!^2}{(n+x)!(x-n)!n}.$$

Now we differentiate this relation w.r.t. x and set $x=0$. With an obvious notation, this yields

$$T(n)-T(n-1)=\frac{2(-1)^{n-1}}{n^2},$$

and

$$T(n)=2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}.$$

Hence we have proved the identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_k^{(2)}=2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}.$$

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