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GENERALIZED CONVOLUTION IDENTITIES FOR STIRLING NUMBERS OF THE SECOND KIND

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Abstract

We prove an identity for sums of products of an arbitrary fixed number of Stirling numbers of the second kind; this can be seen as a generalized convolution identity. As a consequence we obtain two polynomial identities that also involve Stirling numbers of the second kind.

1. Introduction

The *Stirling numbers of the second kind*, $S(n,k)$, belong to the most basic and important objects in combinatorics. They can be defined as the number of ways to partition a set of n elements into k nonempty subsets. One of their important uses is as coefficients connecting the two most fundamental bases of the vector space of single-variable polynomials, namely

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1). \tag{1}$$

We note in passing that the inverse transformation between these two bases is given by the Stirling numbers *of the first kind*; we will not be dealing with these numbers here. The numbers $S(n,k)$ have the explicit expansion

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n, \tag{2}$$

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or equivalently,

$$S(n,k) = \frac{(-1)^k}{(k-1)!}\sum_{j=1}^{k}(-1)^j\binom{k-1}{j-1}j^{n-1}. \tag{3}$$

Two of the most basic properties are the recurrence relation

$$S(n,k) = \sum_{j=k-1}^{n-1}\binom{n-1}{j}S(j,k-1), \qquad (n\geq 2, k\geq 1),$$

and the "triangular" or "Pascal-type" recurrence relation

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k). \tag{4}$$

These and numerous other properties can be found, e.g., in the books [1, Ch. 24], [5], [6], or in the on-line resources [10] or [9, A008277]. Although there are some advantages to the bracket notation used in [6] (see also [7]) we use here the main competing notation $S(n,k)$.

Among the numerous known identities satisfied by Stirling numbers there are classes of convolution formulas. Several of these can be found, e.g., in [6] or [1, p. 825]. A particularly simple such identity is

$$\binom{m}{r}S(n,m) = \sum_{k=m-r}^{n-r}\binom{n}{k}S(n-k,r)S(k,m-r), \tag{5}$$

which is meaningful for $r\leq m\leq n$; see [1, p. 825], or [6, p. 272] for generalizations.

An apparently new convolution identity recently arose in connection with the authors' study of sums of products of Bernoulli numbers [2]:

$$\begin{aligned}\frac{(k-1)!(m-1)!}{(k+m-1)!}S(k+m,d+1) &= \sum_{i=0}^{d-1}\frac{S(k,i+1)S(m,d-i)}{d\binom{d-1}{i}} \\ &+ \sum_{j=d+1}^{k+m-1}\left((-1)^m\binom{k-1}{j-1} + (-1)^k\binom{m-1}{j-1}\right)\frac{B_{k+m-j}}{k+m-j}S(j,d+1),\end{aligned} \tag{6}$$

valid for all integers $k,m\geq 1$ and $d\geq 0$ (but note that the identity becomes trivial for $d\geq k+m$). The right-most summation in (6) can be considered a "correction term" which disappears when $d\geq k$ and $d\geq m$. In order to set the stage for our main result, we set $k_1 := k-1$, $k_2 := m-1$, and $r := d+1$. Then with a small shift in the order of summation we get from (6),

$$\frac{k_1!k_2!(r-1)!}{(k_1+k_2+1)!}S(k_1+k_2+2,r) = \sum_{i=1}^{r-1}(i-1)!(r-i-1)!S(k_1+1,i)S(k_2+1,r-i), \tag{7}$$

valid for $r\geq \max\{k_1,k_2\}+2$ since in this case the last sum on the right-hand side of (6) vanishes. Note that in (7) the convolution is with respect to the second parameters of the Stirling numbers, while in (5) it is with respect to the first parameters.

It is the purpose of this paper to generalize the identity (7) to sums of products of an arbitrary fixed number m of Stirling numbers of the second kind, summed over all compositions of r with exactly m positive parts. This will be done in Section 2. In Section 3 we apply this to obtain two polynomial identities involving Stirling numbers of the second kind.

The last result plays an essential role in a forthcoming paper on sums of products of Bernoulli numbers [3]. However, we believe that the results in this paper are of independent interest.

2. The Main Result

The main result of this section is a direct generalization of the convolution identity (7).

Theorem 1. *Let $m \geq 2$ and $k_1, \ldots, k_m \geq 0$ be integers, and set $s_m := k_1 + \cdots + k_m$. Then for all integers $r \geq \max_{1\leq j\leq m}\{s_m - k_j\} + m$ we have*

$$\frac{k_1! \cdots k_m!(r-1)!}{(s_m+m-1)!} S(s_m+m, r) = \sum_{\substack{i_1+\ldots+i_m=r \\ i_1,\ldots,i_m\geq 1}} \left(\prod_{j=1}^{m} \big((i_j-1)! S(k_j+1, i_j) \big) \right). \tag{8}$$

The proof of this result will be done by induction on m, where $m = 2$, and thus (7), is the base case. Therefore we begin by briefly sketching the proof of (6) for completeness; details can be found in [2].

Proof of (6) *(sketch).* A known formula for "convolved power sums" (see [8] or [4]) can be rewritten as

$$\sum_{j=1}^{k+m+1} \left[(-1)^m \binom{k-1}{j-1} + (-1)^k \binom{m-1}{j-1} \right] \frac{B_{k+m-j}}{k+m-j} a^{j-1} \tag{9}$$
$$= \frac{(k-1)!(m-1)!}{(k+m-1)!} a^{k+m-1} - \sum_{r=1}^{a-1} r^{k-1}(a-r)^{m-1}.$$

For each $a = 1, 2, \ldots, d+1$ we multiply both sides of (9) by $(-1)^{d+1+a}\binom{d}{a-1}\frac{1}{d!}$ and sum over all these a. Then by (3) we get $S(j, d+1)$ in place of a^{j-1} in (9), and $S(k+m, d+1)$ in place of a^{k+m-1}; note that the first sum now starts only at $j = d+1$ since $S(j, d+1) = 0$ for $j \leq d$. The sum of the last term in (9) is determined by the identity

$$\sum_{i=0}^{d-1} \frac{S(k, i+1)S(m, d-i)}{d\binom{d-1}{i}} = \sum_{a=2}^{d+1} \frac{(-1)^{d+1-a}}{(a-1)!(d+1-a)!} \sum_{r=1}^{a-1} r^{k-1}(a-r)^{m-1}, \tag{10}$$

valid for $k, m \geq 1$ and $d \geq 0$. This formula follows from substituting the expression (3) into the left-hand side of (10) and using some combinatorial identities. □

Proof of Theorem 1. First note that for $r > s_m + m$ the identity (8) is trivially correct. Indeed, since $S(n,k) = 0$ whenever $k > n$, the left-hand side clearly vanishes. Also, for each composition $r = i_1 + \cdots + i_m$ there will be at least one index j with $i_j > k_j + 1$. This means that all the products on the right-hand side also vanish.

We now prove the result by induction on m. For $m = 2$ this is just (7). Suppose now that (8) holds for a certain $m \geq 2$ and for all $k_1, \ldots, k_m$ and r as in the theorem. Our aim is to show that (8) then also holds for $m+1$ parameters $k_1, \ldots, k_m, k_{m+1}$. Let therefore $k_1, \ldots, k_{m+1}$ be given, and let r be such that

$$r \geq \max\{s_{m+1} - k_1, \ldots, s_{m+1} - k_{m+1}\} + (m+1). \tag{11}$$

To simplify notation we set $i := i_{m+1}$ and $k := k_{m+1}$. Then

$$\begin{aligned} S_{m+1} &:= \sum_{\substack{i_1+\cdots+i_{m+1}=r \\ i_1,\ldots,i_{m+1}\geq 1}} \prod_{j=1}^{m+1} \big((i_j-1)!S(k_j+1,i_j)\big) \\ &= \sum_{i=1}^{r-1} (i-1)!S(k+1,i) \sum_{\substack{i_1+\cdots+i_m=r-i \\ i_1,\ldots,i_m\geq 1}} \prod_{j=1}^{m} \big((i_j-1)!S(k_j+1,i_j)\big). \end{aligned}$$

Since $i \leq k+1$ (otherwise the right-hand side vanishes), we have by (11),

$$\begin{aligned} r - i \geq r - k - 1 &\geq \max\{s_{m+1} - k_1, \ldots, s_{m+1} - k_m\} - k_{m+1} + m \\ &= \max\{s_m - k_1, \ldots, s_m - k_m\} + m. \end{aligned}$$

Therefore the induction hypothesis applies, and we get

$$\begin{aligned} S_{m+1} &= \sum_{i=1}^{r-1} (i-1)!S(k+1,i) \frac{k_1! \cdots k_m!(r-i-1)!}{(s_m+m-1)!} S(s_m+m, r-i) \\ &= \frac{k_1! \cdots k_m!}{(s_m+m-1)!} \sum_{i=1}^{r-1} (i-1)!S(k+1,i)(r-i-1)!S(s_m+m,r-i). \end{aligned}$$

Since by the assumption (11) we have

$$\begin{aligned} r &\geq \max\{s_{m+1} - k_1, \ldots, s_{m+1} - k_{m+1}\} + (m+1) \\ &\geq \max\{k, s_m + m - 1\} + 2, \end{aligned}$$

the identity (7) applies and we finally get

$$\begin{aligned} S_{m+1} &= \frac{k_1! \cdots k_m!}{(s_m+m-1)!} \frac{k!(s_m+m-1)!}{(k+(s_m+m-1)+1)!} S(k+(s_m+m-1)+2, r) \\ &= \frac{k_1! \cdots k_m! k_{m+1}!}{(s_{m+1}+(m+1)-1)!} S(s_{m+1}+m+1, r), \end{aligned}$$

and this concludes the proof by induction. □

3. Polynomial Identities

As indicated in the introduction, Stirling numbers (in particular those of the second kind) occur in connection with the Bernoulli numbers B_n which can be defined by way of the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!},\quad |x|<2\pi. \tag{12}$$

It is occasionally useful or necessary to consider the mth derivative of the generating function (12); this can be expressed by

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=(-1)^m\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j)x-mS(m,j)}{(e^x-1)^j}; \tag{13}$$

see [2]. The right-hand side of (13) now gives the motivation for the following result.

Theorem 2. *Let $m\geq 2$ and $k_1,\ldots,k_m\geq 1$ be integers, and set $s_m:=k_1+\cdots+k_m$. Then for all integers $\max_{1\leq j\leq m}\{s_m-k_j\}+m\leq r\leq s_m+m-1$ we have*

$$\sum_{\substack{i_1+\cdots+i_m=r\\ i_1,\ldots,i_m\geq 1}}\left(\prod_{j=1}^{m}(i_j-1)!\left[S(k_j+1,i_j)x-k_jS(k_j,i_j)\right]\right) \tag{14}$$

$$=k_1!\cdots k_m!(r-1)!\sum_{\nu=0}^{s_m+m-r}(-1)^{\nu}\binom{m}{\nu}\frac{S(s_m+m-\nu,r)}{(s_m+m-1-\nu)!}x^{m-\nu}.$$

Proof. We consider the coefficients of each power of x separately, with particular attention to the product

$$\prod_{j=1}^{m}\left[S(k_j+1,i_j)x-k_jS(k_j,i_j)\right]. \tag{15}$$

The coefficient of x^m in (15) is the product $S(k_1+1,i_1)\cdots S(k_m+1,i_m)$. Then (8) clearly gives us the coefficient of x^m on the right-hand side of (14).

Next, the coefficient of x^{m-1} in (15) is the sum of m products of the type $S(k_1+1,i_1)\cdots S(k_m+1,i_m)$, with $S(k_j+1,i_j)$ replaced by the corresponding $-k_jS(k_j,i_j)$ for each $j=1,\ldots,m$. Then (8) gives

$$-m\frac{k_1!\cdots k_m!(r-1)!}{(s_m+m-2)!}S(s_m+m-1,r),$$

which is the coefficient of x^{m-1} on the right-hand side of (14).

In general, the coefficient of $x^{m-\nu}$ in (15) is the sum of $\binom{m}{\nu}$ products of the type $S(k_1+1,i_1)\cdots S(k_m+1,i_m)$, with ν different factors $S(k_j+1,i_j)$ replaced by the corresponding

$-k_jS(k_j,i_j)$. Then (8) gives again

$$(-1)^\nu\binom{m}{\nu}\frac{k_1!\cdots k_m!(r-1)!}{(s_m+m-1-\nu)!}S(s_m+m-\nu,r).$$

This is clearly the coefficient of $x^{m-\nu}$ on the right-hand side of (14). The proof is now complete. □

In view of the identity (13) it may seem more natural to deal with the linear polynomial

$$T(n,j):=(j-1)!\big[S(n+1,j)x-nS(n,j)\big], \tag{16}$$

where for simlicity of notation we suppress the variable x in $T(n,j)$. Then (13) becomes simply

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=(-1)^m\sum_{j=1}^{m+1}\frac{T(m,j)}{(e^x-1)^j}. \tag{17}$$

The following result leads to applications via (17).

Corollary 1. *Let $m\ge 2$ and $k_1,\dots,k_m\ge m-1$ be integers, and set $s_m:=k_1+\cdots+k_m$. Then for all integers $r\ge s_m+1$ we have*

$$\sum_{\substack{i_1+\cdots+i_m=r\\ i_1,\dots,i_m\ge 1}}\prod_{j=1}^{m}\frac{T(k_j,i_j)}{k_j!}=\sum_{\nu=1}^{s_m+m+1-r}(-1)^{\nu-1}\binom{m-1}{\nu-1}\frac{T(s_m+m-\nu,r)}{(s_m+m-\nu)!}x^{m-\nu}. \tag{18}$$

Proof. As was the case with Theorem 1, the identity (18) is trivially correct when $r>s_m+m$; in that case the sum on the right is empty, and each summand on the left vanishes. Hence we restrict our attention to $s_m+1\le r\le s_m+m$.

We divide both sides of (14) by $k_1!\cdots k_m!$; then with (16) the left-hand side of (14) becomes the left-hand side of (18). Next we use the binomial identity $\binom{m}{\nu}=\binom{m-1}{\nu}+\binom{m-1}{\nu-1}$ to rewrite

$$\begin{aligned}(-1)^\nu\binom{m}{\nu}&\frac{S(s_m+m-\nu,r)}{(s_m+m-1-\nu)!}x^{m-\nu}\\ &=(-1)^\nu\binom{m-1}{\nu}\frac{x^{m-1-\nu}}{(s_m+m-1-\nu)!}S(s_m+m-\nu,r)x\\ &\quad-(-1)^{\nu-1}\binom{m-1}{\nu-1}\frac{x^{m-\nu}}{(s_m+m-\nu)!}(s_m+m-\nu)S(s_m+m-\nu,r).\end{aligned}$$

Summing this for $\nu=0,1,\dots,s_m+m-r$ and shifting the summation of the first term on the right, we see that the sum on the right of (14) becomes

$$\begin{aligned}\sum_{\nu=1}^{s_m+m+1-r}&(-1)^{\nu-1}\binom{m-1}{\nu-1}\frac{x^{m-\nu}}{(s_m+m-\nu)!}\\ &\times\left[S(s_m+m-\nu,r)x-(s_m+m-\nu)S(s_m+m-\nu,r)\right].\end{aligned}$$

Finally we multiply this by $(r-1)!$ and use (16). Then we immediately get the right-hand side of (18) from the right-hand side of (14). This proves (18), which holds for $r \geq s_m + 1$ since $k_j \geq m-1$ for all j. □

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