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MAXIMUM STIRLING NUMBERS OF THE SECOND KIND

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Abstract

Say an integer n is *exceptional* if the maximum Stirling number of the second kind $S(n,k)$ occurs for two (of necessity consecutive) values of k. We prove that the number of exceptional integers less than or equal to x is $O(x^{1/2+\epsilon})$, for any $\epsilon > 0$. We derive a similar result for partitions of n into exactly k integers.

1. Introduction

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty pairwise disjoint subsets of $[n]$, called *blocks*, whose union equals $[n]$. The Stirling number of the second kind, $S(n,k)$, is the number of partitions of n into k blocks. It is well-known that the ratio $S(n,k+1)/S(n,k)$ is strictly decreasing. (The function $S(n,k)$ is log-concave, in other words.) So either there is a unique maximum Stirling number

$$S(n,K_n) > S(n,k), \text{ for all } k \neq K_n$$

or else there are two consecutive peaks

$$S(n,K_n) = S(n,K_n+1) > S(n,k), \text{ for all } k \notin \{K_n, K_n+1\}.$$

Define the exceptional set E to be those n for which the second alternative holds. Canfield and Pomerance [2] use this terminology. Based on computation through $n = 10^6$ reported in [2] it is possible that $E = \{2\}$.

Quite a lot is known about K_n. Canfield and Pomerance [2] give the most extensive list of references known to us. For example they show that for n sufficiently large and r defined by $re^r = n$ the relation

$$K_n \in \{\lfloor e^r - 1 \rfloor, \lceil e^r - 1 \rceil\}$$

holds for n sufficiently large and also for $1 \leq n \leq 1200$. It is well-known and easy to check from the previous information that $K_n \sim n/\ln n$.

Let $E(x)$ be defined by

$$E(x) = \#\{n : n \leq x \text{ and } n \in E\}.$$

The purpose of this paper is to prove

Theorem 1. *For any $\epsilon > 0$,*

$$E(x) = O(x^{1/2+\epsilon}).$$

Our proof of this theorem uses different methods than the proof of Canfield and Pomerance [2] that $E(x) = O(x^{3/5+\epsilon})$.

The first difference is that Canfield and Pomerance use a result of Huxley [9] on counting integer points near curves. We use a result of Bombieri and Pila [1] on counting integer points on the graph of a many-times differentiable strictly convex function. We now state the Bombieri-Pila result. The numbers d and D below are related by $D = \frac{1}{2}(d+1)(d+2)$. Bombieri and Pila let I denote a closed subinterval of $[0, N]$. Our theorem 2 is theorem 8 of [1]. Note that $c(d)$ below is independent of f. This seems relevant for similar problems. A very attractive feature of the Bombieri-Pila theorem for us is while deep diophantine analysis is required for its proof the theorem itself has an easy to understand statement.

Theorem 2 (Bombieri and Pila). *Suppose $d \geq 4$, $N \geq 1$, and let $f(x) \in C(I)$ be a strictly convex function with $|f'| \leq 1$. Suppose $f^{(D)}(I)$ has at most M zeros. Let Γ be the graph of f. Then*

$$\left|\Gamma \cap \mathbb{Z}^2\right| \leq (M+1)c(d)N^{1/2+3/(d+3)}.$$

We shall apply this theorem to intervals of the form $I = I_{C,F} = [C, F]$ and a particular function $f(x)$ which we specify later. We will also need to derive the asymptotic behaviour of the ith derivative of $f(x)$ as $x \to \infty$. This will show that $f^{(i)}(x)$ is non-zero on $I_{C,F}$ where C, F depend upon i. Thus, in Theorem 2, M will be zero if $x \in I_{C,F}$. Since d goes to infinity with D the bound in Theorem 2 is $O(N^{1/2+\epsilon})$ where ϵ can be made arbitrarily small by choosing D large enough.

Another difference our paper has with the Canfield-Pomerance paper is that we define the Stirling number $S(n, k)$ for complex values of n and k in a way analogous to the way Euler defined $n!$ for real values of n: we represent $S(n, k)$ as a contour integral which is defined for complex values of n and k. D. Knuth et al [12] posed defining $S(n, k)$ for complex values. Flajolet and Prodinger [7] gave the first good solution. Merlini and Richmond [15]

also gave a solution which if interpreted properly is equivalent to that in [7] and which we shall use here. (Merlini and Richmond [15] did not interpret their definition to discover the Flajolet-Prodinger definition.) We then define a curve in the real euclidean plane implicitly by setting

$$\frac{S(n,k)}{S(n,k-1)} = 1.$$

We show that $n = n(k)$ and $k = k(n)$ are analytic functions, at least for $k \sim n/\ln n$. The lattice points of this curve with n and k positive are the elements of E as defined above so we may apply the Bombieri-Pila result. We shall depend heavily on the asymptotic behaviour of the numbers $S(n,k)$ as determined by Moser and Wyman [16]. This paper is a fine example of the saddle-point method. The survey article by Odlyzko [17] gives an elegant introduction to the saddle-point method.

Since we need information about the D-th derivative of f it will be convenient for us that an asymptotic formula for an analytic function in the form of a sum of analytic functions followed by an analytic error term can be differentiated term by term. We include a proof of this fact even though it is a conventional proof because books on asymptotic analysis often gloss over this; only giving an example to show that one may not be justified in differentiating term by term.

Perhaps the example of binomial coefficients is interesting here. Let $\binom{n}{i}$ be defined by

$$\binom{n}{i} = \frac{\Gamma(n+1)}{\Gamma(i+1)\Gamma(n-i+1)}.$$

Then we could define a curve in the Euclidean plane by

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k} = 1.$$

We can define exceptional points as before and they are $k = (n+1)/2$ if n is odd and if n is even there are no exceptional points. In the previous context there are $N/2$ exceptional points if N is even and $(N-1)/2$ if N is odd.

If in the above discussion we instead let $S(n,k)$ denote the Stirling number of the first kind, the number of permutations of n objects with k cycles, and define the exceptional set E in the same way, then it is known that $E = \{2\}$. This was proved by Erdős [6].

2. The Asymptotics of the Stirling Numbers

We use the asymptotic behaviour of $S(n,k)$ (also denoted $S(n,m)$) in the form given by Moser and Wyman [16]. Our paper has been written assuming that the reader has a copy of this paper available. (We expect a copy of Gardy's paper [8] will also be very helpful). We provide a fairly detailed sketch of the results in the Moser-Wyman paper since we need

to refine their analysis slightly. We need to estimate certain derivatives of the terms in the Moser-Wyman asymptotic expansion. Moser-Wyman show that from the well-known generating function for the Stirling numbers

$$\sum_{n=0}^{\infty} \frac{S(n,m)}{n!} x^n = \frac{(e^x-1)^m}{m!}$$

and Cauchy's integral formula for the coefficients of an analytic function it follows that

$$S(n,m) = \frac{n!(e^R-1)^m}{2\pi R^n m!} \int_{-\pi}^{\pi} \exp(mg(\theta,R))d\theta \tag{1}$$

where

$$\begin{aligned} g(\theta,R) &= \ln(\exp(Re^{i\theta})-1) - \ln(e^R-1) - i(n/m)\theta \\ &= \sum_{k=1}^{\infty} C_k(R)(i\theta)^k, \\ &= iB\theta - RH\theta^2 + \sum_{k=3}^{\infty} C_k(R)(i\theta)^k, \\ B &= R(1-e^{-R})^{-1} - n/m, \\ H &= e^R(e^R-1-R)/2(e^R-1)^2 \end{aligned}$$

and R is the unique solution of

$$B = R(1-e^{-R})^{-1} - (n/m) = 0.$$

They point out that $C_k(R)$ can be given explicitly in terms of the operator $\Psi = R(\frac{d}{dR})$ by

$$C_k(R) = \Psi^k \ln(e^R-1)/k!.$$

Thus

$$\begin{aligned} C_1(R) &= \frac{Re^R}{e^R-1} = \frac{R}{1-e^{-R}}, \\ 2C_2(R) &= \frac{R}{1-e^{-R}} - \frac{R^2e^{-R}}{(1-e^{-R})^2}, \\ 6C_3(R) &= \frac{R}{1-e^{-R}} - 3\frac{R^2e^{-R}}{(1-e^{-R})^2} + \frac{R^3e^{-R}}{(1-e^{-R})^2} + \frac{2R^3e^{-2R}}{(1-e^{-R})^3} \end{aligned}$$

and so on. Thus, each $C_k(R)$ can be written as

$$\sum_{j=0}^{\infty} \frac{A_{k,j}(R)}{e^{jR}},$$

where each $A_{k,j}(R)$ is a polynomial function of R with rational coefficients. Moser and Wyman also prove in their Lemma 3.2 that there is a constant M independent of k and R such that

$$|C_k(R)| \leq MR \text{ and } |C_k'(R)| \leq M.$$

Their argument can be extended as follows to show that for higher derivatives, $\left|C_k^{(l)}(R)\right| \leq M$ $(= M_l)$. They show that $\frac{d}{dR}g(\theta, R)$ is an analytic function of θ for the domain $|\theta| \leq 1$, $R \geq 0$. For R near 0 this function can be written as $(e^{i\theta} - 1)/2$ plus a power series in R with coefficients being polynomials in $e^{i\theta}$. For $R \to \infty$ this function can be written as $e^{i\theta} - 1$ plus a power series in $\exp(-Re^{i\theta})$. One can then deduce that all derivatives $\frac{d^l}{dR^l}g(\theta, R)$ are bounded and so the coefficients $C_k^{(l)}(R)$ are also bounded.

Furthermore Moser and Wyman write for $\epsilon = (mR)^{-3/8}$

$$J = \int_\epsilon^\pi \exp(mg(\theta, R))d\theta$$

and their Lemma 4.1 shows that for some constant $K > 0$

$$|\exp(mg(\theta, R))| \leq \exp(-K(mR)^{1/4}) \text{ for } \theta \in [\epsilon, \pi].$$

Now for small R (the only case requiring thought since if R is not small $dg/dR = O(1)$), using $\sinh x = (e^x - e^{-x})/2$

$$\begin{aligned} \left|\frac{dg}{dR}\right| &= \left|\frac{\exp(i\theta + \frac{R}{2}e^{i\theta})\sinh(\frac{R}{2}) - \exp(\frac{R}{2})\sinh(\frac{R}{2}e^{i\theta})}{2\sinh(\frac{R}{2}e^{i\theta})\sinh(\frac{R}{2})}\right| \\ &= O\left(\frac{1}{R}\right) = O\left(\frac{m}{mR}\right) = o(m). \end{aligned}$$

The last equality follows from the fact that $mR \to \infty$, which is proved by Moser and Wyman (Lemma 3.1). (In fact we shall see that mR is of order n in our application.) Combining the above estimates we find

$$\begin{aligned} \left|\frac{dJ}{dR}\right| &\leq m\int_\epsilon^\pi |\exp(mg(\theta, R))| \left|\frac{dg}{dR}\right| d\theta \\ &= m\pi \exp(-K(mR)^{1/4})o(m) \\ &= O(n^2)\exp(-K(mR)^{1/4}). \end{aligned}$$

In general, for $l \geq 0$ one may show by this argument

$$\left|\frac{d^l}{dR^l}J\right| = O\left(n^{2l}\exp(-K(mR)^{1/4})\right). \tag{2}$$

Moser and Wyman go on to let $z = (mR)^{-1/2}$, $\phi = \theta(mRH)^{1/2}$, $h = \epsilon(mRH)^{1/2} = (mR)^{1/8}H^{1/2}$, $a_k = C_{k+2}(R)(i\phi)^{k+2}/R(H)^{(k+2)/2}$, $f = f(z, R, \phi) = \sum_{k=1}^\infty a_k z^k$. Then $mg(\theta, R) =$

$-\phi^2 + f$ and so (1) can be rewritten

$$S(n,m) = \frac{n!(e^R-1)^m}{2\pi R^n m!(mRH)^{1/2}} \int_{-h}^{h} e^{-\phi^2+f} d\phi + \frac{n!(e^R-1)^m}{2\pi R^n m!} \left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) e^{mg(\theta,R)} d\theta.$$

We shall estimate these last two integrals by (2) with $l = 0$ shortly. (They are negligible.) For the integral from $-h$ to h we write it as Moser and Wyman do,

$$\sum_{k=0}^{s-1} \int_{-h}^{h} \left(e^{-\phi^2} b_k d\phi \right) z^k + R_s \tag{3}$$

where

$$\sum_{k=0}^{\infty} b_k z^k = \exp(f) = \exp\left(\sum_{k=0}^{\infty} a_k z^k \right),$$

$$R_s = \int_{-h}^{h} e^{-\phi^2} \sum_{k=s}^{\infty} b_k z^k d\phi$$

and

$$|R_s| \le z^s \int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|) d\phi$$

and where $P_s(|\phi|)$ is a polynomial in $|\phi|$. Indeed Moser and Wyman let $h \to \infty$ and write

$$S(n,m) = \frac{n!\left(e^R-1\right)^m}{2\pi R^n m!(mRH)^{1/2}} \sum_{k=0}^{s-1} \left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k \right) z^k + R_s$$

where $R_s = O\left(z^s\right)$. This corresponds to a Poincaré asymptotic expansion of $f(x)$ of the form

$$f(x) = \sum_i a_i x^i$$

with $x = (mR)^{-1/2}$. Two such asymptotic expansions may be divided as the Poincaré series are divided.

We note now that the Moser-Wyman equation (1) allows us to define $S(n,m)$ for n and m arbitrary positive real numbers using a standard result on functions defined by contour integrals. This was first done by Flajolet and Prodinger [7] we believe. On page 195 of Dieudonné's book [4] we find the following theorem.

Theorem 3 (Dieudonné). *Let $\gamma : I = [a,b] \to C$ be a path in C. Let D be an open set in C and suppose we are given a function $(z,u) \to h(z,u)$ in $D \times \gamma(I)$ having the following properties:*

1. For every $u \in \gamma(I)$, the function $z \to h(z,u)$ is analytic in D.

2. The functions $(z,u) \to h(z,u)$ and $(z,u) \to \frac{\partial}{\partial z}h(z,u)$ are continuous in $D \times \gamma(I)$. Under these conditions the function

$$f(z) = \int_{\gamma} h(z,u)\, du$$

is analytic in D.

We can let $h(z,u) = \exp(ng(\theta,R)), u = \theta$ and $[a,b] = [-\pi,\pi]$. Note that setting the Moser-Wyman B equal to 0 defines R as an analytic function of n/m for $n/m \in (0,\infty)$ by the inverse function theorem since

$$\frac{d}{dR}\left(\frac{R}{1-e^{-R}}\right) > 0.$$

It follows routinely from this theorem that the integral in the formula for $S(n,m)$ is an analytic function of n/m, hence this integral is an analytic function of the complex variables n and m. It now follows that $S(n,m)$ is an analytic function of n and m. (We have defined $S(n,m)$ in an open set containing the ordered pairs of positive reals.) The singularities of $S(n,m)$, as Flajolet and Prodinger point out, are at the zeros of $e^R - 1$ if m is not an integer since $1/\Gamma(n+1)$ is entire. Thus $S(n,m)$ has no singularities for positive n and m. Perhaps now is a convenient time to point out that eqs (1) and (3) express $S(n,m)$ as sum of s analytic terms plus the analytic R_s(the error term).

Note the Moser-Wyman path of integration in eq (1) is a straight line with real part equal to 0 and imaginary part part going from $-\pi$ to π. This is a circle in the $u = e^{i\theta}$ plane so it can be deformed into a Hankel contour circling the origin and going to $-\infty$ above and below the negative real axis in such a way that the distance from the negative reals is less than 2π, this is the Flajolet-Prodinger contour. The Flajolet-Prodinger contour is very convenient for proving identities since it does not depend upon n and m as Flajolet and Prodinger point out. See the Kemkes, Lee, Merlini and Richmond paper [10] for examples in addition to the Flajolet-Prodinger examples. To get asymptotic information about $S(n,m)$ it is convenient to use the Moser-Wyman contour as it goes through the saddle point of the integrand. It's fortunate that the definitions agree because some identities in [10] have infinite sums and asymptotic estimates are required to prove convergence. It's also very convenient for us that the asymptotic formulas of Moser and Wyman for positive integral n and k hold for positive real n and k, it's only necessary to note that the Moser-Wyman proofs hold for these values. Professor Donald Knuth pointed out [13] that there is an error in the paper [3] concerning some bounds. This paper proves a uniform asymptotic expansion for $S(n,k)$ due to Temme. The proof shows that Temme's estimate agrees with Moser and Wyman's estimate for all positive ranges of n and k. The proof is a calculation expressing the Moser-Wyman results in Temme's form, the error comes from forgetting the factor $n!/m!$ in the definition of $S(n,m)$. Since the Moser-Wyman results are correct and the calculations expressing Temme's formula in terms of these results are correct the results in Chelluri et al [22] are correct. This explains the excellent agreement of the Stirling numbers with Temme's formula noted in Temme [22].

3. The Exceptional Points

We now consider in detail the implicitly defined curve

$$F(n,m) = \frac{S(n,m)}{S(n,m-1)} = 1.$$

We will show that $n \sim m \ln m$ on this curve and that we can find the asymptotic behaviour of the derivatives of n with respect to m by differentiating $m \ln m$. In fact, in order to satisfy the assumptions of the Bombieri-Pila theorem we shall consider the reflection of this curve about the line $n = m$, i.e., the graph of the inverse of $n = n(m)$, since we will see that $\frac{dn}{dm} \sim \ln m$ hence $n'(m) > 1$. Of course there is a 1-1 correspondence between lattice points of $n = n(m)$ and $m = m(n)$. We define the curve for now as above, however.

The lattice points on this curve are the exceptional points as defined by Canfield and Pomerance. We have seen that Moser and Wyman have given the complete asymptotic expansion of $S(n,m)$. Since asymptotic expansions may quite generally be divided we will be able to work out the asymptotic behaviour of this curve. (This is usually only proved in texts for asymptotic expansions in terms of power series; however, the Moser-Wyman asymptotics are in powers of mR so the standard proofs justify this statement.) We begin by showing that n is defined implicitly as an analytic function of m. This will follow from the implicit function theorem provided that on the curve

$$\frac{\partial}{\partial m} F(n,m) \neq 0.$$

Now

$$\begin{aligned}
\frac{\partial}{\partial m} F &= \frac{\frac{\partial}{\partial m} S(n,m)}{S(n,m-1)} - \frac{S(n,m)\frac{\partial}{\partial m} S(n,m-1)}{S^2(n,m-1)} \\
&= \frac{S(n,m)}{S(n,m-1)} \left[\frac{\partial}{\partial m} \ln S(n,m) - \frac{\partial}{\partial m} \ln S(n,m-1) \right] \\
&= \frac{\partial}{\partial m} \ln S(n,m) - \frac{\partial}{\partial m} \ln S(n,m-1).
\end{aligned}$$

It has been proved by D. Merlini and B. Richmond [15] that $\ln S(n,m)$ is a concave function (see Example 4.1) for $n^\epsilon \leq m \leq n$. Thus the partial is nonzero for these m and by the implicit function theorem we have that n is an analytic function of m for this range of m. We now find the asymptotic behaviour of n in terms of m on this curve.

We begin by finding the behaviour of the R in the Moser-Wyman formula in terms of n and m.

Lemma 1. *The solution for R to*

$$z = \frac{R}{1 - e^{-R}}$$

is an analytic function of z, with domain including the positive reals, with the asymptotic behaviour

$$R = z - ze^{-z} - \left(z^2 - z\right) e^{-2z} + O\left(z^3 e^{-3z}\right).$$

Proof. Since

$$\frac{R}{1 - e^{-R}}$$

is a monotonic function for real positive R the statement that R is an analytic function of z follows immediately from the implicit function theorem. The asymptotic behaviour of R follows by a standard application of the boot-strapping method; since $z = R + Re^{-R} + Re^{-2R} + \cdots$ we see immediately that $R = z + O\left(ze^{-z}\right)$. Then we find that if $R = z - ze^{-z} + g(z)$ then $g(z) = \left(z - z^2\right) e^{-2z} + O\left(z^3 e^{-3z}\right)$ and so on. □

Indeed it's not difficult to see that

$$R = z - ze^{-z} + \sum_{i=2}^{I} P_i(z) e^{-iz} + O\left(z^{I+1} e^{-(I+1)x}\right)$$

where $P_i(z)$ is a polynomial of degree i.

We now consider the behaviour of $S(n,m)/S(n,m-1)$. We let

$$\frac{n}{m} = \frac{R}{1 - e^{-R}}, \quad \frac{n}{m-1} = \frac{S}{1 - e^{-S}}.$$

We also let $R + \Delta R = S$.

From the Moser-Wyman results we have (the first term approximation),

$$\frac{S(n,m-1)}{S(n,m)} = \frac{\left(\frac{n!\left(e^S - 1\right)^{m-1}}{2\pi S^n (m-1)!}\right)}{\left(\frac{n!(e^R - 1)^m}{2\pi R^n m!}\right)} \frac{\left(\frac{1}{(m-1)SH_0}\right)^{1/2}}{\left(\frac{1}{mRH_1}\right)^{1/2}} \left(1 + O\left(\frac{1}{mR}\right)^{1/2}\right) \tag{4}$$

where

$$H_0 = \frac{e^S \left(e^S - 1 - S\right)}{2\left(e^S - 1\right)^2},$$
$$H_1 = \frac{e^R \left(e^R - 1 - R\right)}{2\left(e^R - 1\right)^2},$$

and from Lemma 1, $mR \sim mz \sim n$. Since R and S tend to ∞ if $m = o(n)$ we see that $H_0, H_1 \sim 1/2$ if $m = o(n)$.

We now use the asymptotic formula in eq (4) to find the asymptotic behaviour of n in terms of m on the curve we have defined implicitly. From our Lemma 1 we find that

$$R = \frac{n}{m}\left[1 - e^{-n/m} + O\left(\frac{n}{m} e^{-2n/m}\right)\right].$$

Furthermore since

$$\frac{n}{m-1} = \frac{n}{m} + \frac{n}{m^2} + O\left(\frac{n}{m^3}\right)$$

and

$$\begin{aligned} e^{-\frac{n}{m-1}} &= e^{-n/m-n/m^2+O(n/m^3)} \\ &= e^{-n/m}\left(1 - n/m^2 + n^2/m^4 + O\left(n/m^3 + n^3/m^6\right)\right) \end{aligned}$$

we find that

$$\begin{aligned} \frac{n}{m-1}e^{-\frac{n}{m-1}} &= \left(n/m + n/m^2\right)e^{-n/m} - \frac{n}{m^2}\left(n/m + n/m^2\right)e^{-n/m} \\ &\quad + O(n^2/m^5)e^{-n/m} + O\left(n/m^3\right)e^{-n/m} \\ &\quad + O\left(n^3/m^5\right)e^{-n/m} + O\left(n^3/m^6\right)e^{-n/m} + O\left(n^2/m^4\right)e^{-n/m}. \end{aligned}$$

We now suppose $(1-\epsilon)n/\ln n \le m \le (1+\epsilon)n/\ln n$ (recall that $m \sim n/\ln n$ for all exceptional points). We shall find the following idea of Canfield and Pomerance [2] very convenient. Let us say

$$f(x) = O_*(g(x))$$

if for a sufficiently large constant C

$$|f(x)| \le C(\ln x)^C g(x) \text{ for } x \ge C.$$

Thus since $\Delta R = S - R$

$$\Delta R = \frac{n}{m^2}\left(1 - e^{-n/m}\right) + O_*\left(\frac{1}{n^2}\right) + O_*\left(\frac{n^2}{m^3}e^{-n/m}\right) + O_*\left(e^{-2n/m}\right)$$

and if $\epsilon \le 1/4$

$$\frac{\Delta R}{R} = \frac{1}{m}\left[1 + O_*\left(\frac{1}{n^{3/5}}\right)\right].$$

Thus in eq (4) we may use

$$\begin{aligned} \left(\frac{R}{S}\right)^n &= \left(\frac{1}{1+\frac{\Delta R}{R}}\right)^n \\ &= e^{-n\frac{\Delta R}{R}}\left(1 + O_*\left(n\left(\frac{\Delta R}{R}\right)^2\right)\right) \\ &= e^{-n/m}\left(1 + O_*\left(n^{-1}\right)\right). \end{aligned}$$

We therefore use in (4) for $3n/(4\ln n) \le m \le 5n/(4\ln 5\ln n)$

$$\left(\frac{R}{S}\right)^n = e^{-n/m}\left(1 + O_*\left(n^{-1}\right)\right). \tag{5}$$

Furthermore we will use in (4)

$$\frac{e^S - 1}{e^R - 1} = 1 + \frac{e^{\Delta R} - 1}{1 - e^{-R}} = 1 + O_*\left(n^{-1}\right). \tag{6}$$

Using these equations in (4) set equal to 1 gives

$$\frac{S(n, m-1)}{S(n, m)} = me^{-n/m}\left(1 + O_*(n^{-1})\right).$$

We may rewrite our curve as

$$me^{-n/m} = 1 + O_*\left(n^{-1}\right)$$

which we may rewrite as

$$n = m \ln m + O_*(1) \tag{7}$$

or, by bootstrapping,

$$m = \frac{n}{\ln n}\left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right).$$

Let us first recall that (7) is just the first term in a complete asymptotic expansion of $S(n, m-1)/S(n, m)$. The Moser-Wyman formula for the asymptotic behaviour $S(n, m)$ is a sum of as many terms as we want and each term is an analytic function of n and m plus an analytic error term. Moser and Wyman show that each function term is smaller than the previous term by the fraction $(mR)^{-1/2}$. Erdélyi's book [5] shows in Section 2.4 that asymptotic expansions may be divided so $S(n, m-1)/S(n, m)$ has a complete asymptotic expansion and each term in the expansion is smaller than the previous term by the just-mentioned ratio. More than this, however, our previous arguments show that the terms in this expansion are analytic functions. We now give a slightly modified proof of Erdélyi showing that we can get the asymptotic behaviour of the derivatives of n with respect to m by differentiating the asymptotic expansion of n with respect to m.

Lemma 2. *Suppose we have that (for example in the Moser-Wyman expansion for $S(n, m)$)*

$$f(x) = f_1(x) + f_2(x) + \cdots + f_s(x) + E(x)$$

where the f_i and $E(x)$ are analytic functions of x, and x is in the interior of the domain of f. In our case x can be a positive real. Then

$$f'(x) = f_1'(x) + f_2'(x) + \cdots + f_s'(x) + E'(x).$$

Furthermore this gives a complete asymptotic expansion for $S'(n, m)$.

Proof. From Cauchy's integral formula for the derivative

$$f'(x) = \frac{1}{2\pi i}\int_C \frac{f(x)}{(x-z)^2} dz$$

where C is in the interior of the domain of $f(x)$. We may take, as Erdélyi does, C to be a circle, $z = x + \epsilon x e^{it}$ and $0 \le t \le 2\pi$ so that the circle is in the interior of the region where f is analytic provided ϵ is small enough. If it is, we immediately have

$$f'(x) = f_1'(x) + \cdots + f_s'(x) + E'(x).$$

□

From the definition of the derivative we have

$$f(x+h) - f(x) = hf'(x) + o(h)f'(x). \tag{8}$$

In our present case we let $f(x) = \frac{\partial S(n,m)}{\partial m}$. We wish to get a lower bound on h if $f(x+h) = 0$. We find the following theorem of Gardy [8] convenient:

Theorem 4 (Gardy). *Let $g(z)$ have positive coefficients with constant term and the coefficient of $z \neq 0$. Assume that $\Delta g(z) = z\frac{g'(z)}{g(z)} = n/m$ has a real positive solution ρ smaller than the radius of convergence of g and Ψ. Suppose these radii are positive, suppose $\Psi(0) = 0$ and that Ψ has positive coefficients. Then for $n, d \to \infty$ and $\eta \le n/m \le M$, where η and M are positive constants,*

$$[z^n]\{g^d(z)\Psi(z)\} = \frac{g^d(\rho)\Psi(\rho)}{\rho^{n+1}\sqrt{2\pi d\delta g(\rho)}}(1 + o(1)).$$

The condition that Ψ has positive coefficients can be replaced in our present application. Here we shall have

$$\Psi(z) = \ln\left(\frac{e^z - 1}{z}\right)$$

which is analytic in $|z| \le 2\pi$. Also the significant range of integration in the proof of Gardy's theorem(is determined by $g(z)$) and is $O_*(n^{-1/2})$ so $\Psi(z) \sim \Psi(\rho)$ over this range of integration, indeed one can find a complete asymptotic expansion for the coefficient in Gardy's theorem. Merlini and Richmond [15] show that the conditions of Gardy's theorem hold for $g(z) = (e^z - 1)/z$. Moreover, they apply this theorem to $S(n,m)$; they denote $S(n,m)$ by $a_{y,x}$, and begin with

$$S(n,m) = \frac{n!}{m!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-n(\alpha+i\theta)+m\ln g(e^{\alpha+i\theta})}d\theta$$

where $\Delta g(e^\alpha) = n/m$. They considered $\frac{\partial}{\partial m}$; however, in the present case their method begins by writing

$$\frac{\partial}{\partial n}S(n,m) = \frac{\partial \ln n!}{\partial n}S(n,m)$$
$$+\frac{n!}{m!}\int_{-\pi}^{\pi} e^{-n(\alpha+i\theta)+m\ln g\left(e^{\alpha+i\theta}\right)}\left((-\alpha+i\theta) - n\frac{\partial\alpha}{\partial n} + m\frac{g'\left(e^{\alpha+i\theta}\right)e^{\alpha+i\theta}}{g\left(e^{\alpha+i\theta}\right)}\frac{\partial\alpha}{\partial n}\right)d\theta.$$

Since $\Delta g\left(e^{\alpha}\right) = n/m$ identically we see that the coefficient of $\frac{\partial \alpha}{\partial n}$ is zero and we may apply Gardy's theorem (the minor modifications described in [15] must be made again) and find that

$$\frac{\partial \ln S(n,m)}{\partial n} = \left(\left(\frac{\partial n!}{\partial n}\right) - \alpha\right)\left(1 + O\left(\frac{(\ln n)^2}{n}\right)\right);$$

and since

$$\frac{\partial^2 \ln S(n,m)}{\partial^2 n} = \frac{(S(n,m))''}{S(n,m)} - \left(\frac{(S(n,m)'}{S(n,m)}\right)^2$$

we find that

$$\begin{aligned}\frac{\partial^2 S(n,m)}{\partial^2 n} =& \frac{n!}{m!}\frac{\partial^2(\ln n!)}{\partial^2 n} S(n,m) \\ &+ 2\frac{n!}{m!}\frac{\partial \ln n!}{\partial n}\frac{\partial S(n,m)}{\partial n} \\ &+ \frac{n!}{m!}\frac{\partial^2 S(n,m)}{\partial^2 n}.\end{aligned}$$

Now we find that

$$\frac{\partial^2 S(n,m)}{\partial^2 n} = S(n,m)\left(\frac{1}{n} + 2\ln n\left(\ln n - \alpha\right) + \alpha^2 + \frac{\partial \alpha(n)}{\partial n}\right)\left(1 + O\left(\frac{(\ln n)^2}{n}\right)\right).$$

Since $e^{\alpha}h'(e^{\alpha})/h(e^{\alpha}) = n/m$ we have $e^{\alpha} = (n/m)\left(1 + O\left(e^{-\alpha}\right)\right)$ and $\alpha \sim \ln(n/m) \sim \ln\ln n$. We find upon differentiating with respect to n that $\frac{\partial \alpha}{\partial n} = \frac{1}{m}\left(1 + O\left(e^{-n/m}/m\right)\right)$. Thus

$$\frac{\partial^2 \ln S(n,m)}{\partial^2 n} \sim \ln^2 n.$$

Thus if $3n/(4\ln n) \le m \le 5n/(4\ln n)$ then

$$B\frac{1}{\ln n} \le \frac{\left(\frac{\ln S(n,m)}{\partial n}\right)}{\left(\frac{\partial^2 \ln S(n,m)}{\partial^2 n}\right)} \le C\frac{1}{\ln n}.$$

We now get a lower bound on the h in eq (8). If $f(x+h) = 0$ with $f(x) = \frac{\partial S(n,m)}{\partial n}$ then it follows from the immediately preceding bounds that h can be bounded away from zero by an absolute constant times $(\ln n)^{-1}$ (at least for m in the given range). Thus in Cauchy's integral formula for the derivative we may take C to a circle of radius $C/\ln n$ and so all of the derivatives in Lemma 2 are $O\left(\ln n\right)^2$ of the function being differentiated. Thus Lemma 2 implies that we have a complete asymptotic expansion for $n'(m)$ and all of the higher derivatives if we wish. Since we may simply differentiate the leading term in the asymptotic expansion of $n(m)$ we find easily that

$$n'(m) \sim \ln m, \quad \text{and for } s \ge 2 \;\; n^s(m) \sim (-1)^s \frac{(s-2)!}{m^{s-1}}.$$

To find the asymptotic behaviour of $m^s(n)$, the formula in terms of Bell polynomials [19] for the derivative of an inverse function works. If g_i denotes $\frac{g^i(n)}{d^i(n)}$ and $G(n)$ denotes the inverse of $n(m)$ then, for example,

$$G_4 = -g_4 g_1^{-5} + 10 g_3 g_2 g_1^{-6} - 15 g_2^3 g_1^{-7}.$$

From the estimates above for the derivatives of g we see that the first term is $O\left(m^{-3}\left(\ln m\right)^{-5}\right)$ and the others are $O\left(m^{-3}\left(\ln m\right)^{-6}\right)$ and $O\left(m^{-3}\left(\ln m\right)^{-7}\right)$. Since $m \sim n/\ln n$, we find that $G_4 \sim -2m^{-3}(\ln m)^{-5}$. Hence, $m^4(n) \sim -2n^{-3}(\ln n)^{-2}$. More generally, all of the terms in G_s have a factor m^{-s+1} and the largest term is the first one, because of the powers of g_1. It's not difficult to see that

$$m'(n) \sim \frac{1}{\ln n}, \quad \text{and for } s \geq 2, \;\; m^s(n) \sim (-1)^{s-1}\frac{(s-2)!}{n^{s-1}\ln^2 n}.$$

This agrees with the results we get by differentiating $m = n/\ln n$, of course. The terms in the asymptotic expansion, including the error term, are all analytic functions, so we may apply Lemma 2. The radius of the circle in this lemma is at least $1/\ln n$ so the error term when differentiated is still negligible being $O(n/(\ln n)^N)$ where N is as large as we want. The conditions of the Bombieri-Pila theorem are therefore satisfied. Note also for n sufficiently large the M in the Bombieri-Pila theorem is zero. Since we can consider arbitrarily large derivatives our Theorem 1 is proved for $\frac{3n}{4\ln n} \leq m \leq \frac{5n}{4\ln n}$. As we mentioned before there are no exceptional points outside this range ($K_n \sim n/\ln n$) so Theorem 1 is proved.

3.1. Integer Partitions

The proofs we have given may apply fairly generally, in our opinion, because the conditions in the Bombieri-Pila theorem are easy to understand and verify. The following example illustrates this. Consider the number, $p(m,n)$, of partitions of n into exactly m parts. Szekeres [21] showed that $p(m,n)$ is unimodal so that we may say that n is exceptional if $p(m,n) = p(m-1,n)$ for some m. We will derive a $O(x^{1/2+\epsilon})$ result for the exceptional n associated with $p(m,n)$. We only sketch the proof because it is so similar to our treatment of $S(n,m)$ and because our results are likely very far from best possible.

Let $P(n,k)$ denote the number of partitions of n into at most k integer parts. In the line just before his eq (14), Szekeres represents $P(n,k)$ as a contour integral which is analogous to that of Merlini and Richmond [15] for $S(n,m)$. (Szekeres's ρ is e^{α} in [15].) Using $p(m,n) = P(n-m,n)$ we have a representation of $p(m,n)$ as a contour integral hence as an analytic function of the two complex variables m and n. This is completely analogous to our representation of $S(n,m)$ as a function of the complex variables n and m. Using his complete asymptotic expansion for $P(n,k)$ (see his eq (24)), Szekeres shows how to derive a complete asymptotic expansion for $p(m,n)$. See his eq (32) for the first few terms. This is analogous to the complete asymptotic expansion of $S(n,m)$ by Moser and Wyman.

Let $\bar{m}$ denote the m which maximizes $p(n,m)$. The results of Szekeres allow one to prove that $p(m,n)$ is log-concave for $m \leq \bar{m}$ and log-convex for $m \geq \bar{m}$. See [11] for details. Hence

as before the curve $p(m,n)/p(m-1,n)=1$ defines $m(n)$ and $n(m)$ as analytic functions. Szekeres shows that as a function of n the maximum of $p(m,n)$ occurs for

$$\bar{m} = cn^{1/2}L + c^2\left(3/2 + 3L/2 - L^2/4\right) - 1/2 + O\left(n^{-1/2}\ln^4 n\right)$$

where $L = \ln\left(cn^{1/2}\right)$ and $c = 6^{1/2}/\pi$. Szekeres also shows that this $\bar{m}$ has an asymptotic expansion as a sum of analytic functions of n, each one $O_*\left(n^{-1/2}\right)$ of the previous one. He does not explicitly state this; however, his method proves this. (One can see this by bootstrapping to solve $p(m,n) = p(m-1,n)$). Finally the results mentioned above show that the radius of the circle in Lemma 2 can be up to $n^{-1/2-\epsilon}$. This means that we differentiate the asymptotic expansion for m given above as many times as we want and still have an error term of $O_*\left(n^{-N}\right)$ for any positive N provided we keep enough terms in the asymptotic expansion. We find that the derivatives of $m(n)$ behave rather like the derivatives of the $m(n)$ we had with the Stirling numbers,

$$m^k(n) \sim (-1)^{k+1}\frac{3\times 5\times 7\times\cdots\times(2k-3)}{2^k}n^{-(2k-1)/2}L.$$

We thus deduce as before that at most $O\left(N^{1/2+\epsilon}\right)$ of the integers up to N are exceptional.

4. Future Research

We think that if the combinatorial numbers $a(m,n)$ are log-concave and the exceptional integers are defined by $a(m-1,n) = a(m,n)$ then there is a 'reasonable' chance of proving by the methods we used here that the bound in this paper for the exceptional numbers holds. Of course one must have an explicit generating function for the $a(m,n)$ and one must be able to determine the asymptotic behavior of the relevant derivatives and have an estimate for the m that maximizes the $a(m,n)$ similar to that of Szekeres. The results in [18] include log-concavity results for $p_A(m,n)$ defined by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_A(m,n)u^m t^n = \prod_{r=1}^{\infty}\frac{1}{1-ut^{a_r}}$$

where $A = \{a_1, a_2, \cdots\}$ is a sequence of increasing integers satisfying certain mild restrictions. So, one can define a curve by $p_A(m,n)/p_A(m-1,n) = 1$. The reason that we cannot obtain a $O(x^{1/2+\epsilon})$ result is that the complete asymptotic expansion of $p_A(m,n)$ is still not available; only the first term is known.

As for $S(n,m)$ and $p(m,n)$, perhaps if one could prove the uniform estimates that Bombieri and Pila refer to in their paper one could get an estimate of $O(x^{\epsilon})$. Unfortunately to do this requires a much deeper understanding of diophantine approximation than we have it seems.

Acknowledgments

This paper is a completely revised version of an earlier paper. We thought, in error, that Canfield and Pomerance estimated the number of lattice points near the curve we define here. They did not, so we cannot use their results.

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References

[1] E. Bombieri and J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J., 59 (1989), pp. 337–357.

[2] E.R. Canfield and C. Pomerance, *On the problem of uniqueness for the maximal Stirling number(s) of the second kind*, Integers, 2 (2002), paper A1.

[3] R. Chelluri, L. B. Richmond and Temme, *Asymptotic Estimates for Generalized Stirling Numbers*, Analysis, 20(2000), pp. 1–13.

[4] J. Dieudonné, *Infinitesimal Calculus*, Hermann, Paris, 1971.

[5] A. Erdélyi, *Asymptotic Expansions*, Dover Publications, New York, 1956.

[6] P. Erdős, *On a conjecture of Hammersley*, J. London Math. Soc., 28 (1953), pp. 232–236.

[7] P. Flajolet and H. Prodinger, *On Stirling Numbers for Complex Arguments and Hankel Contours*, SIAM J. of Discrete Math., 12(1999), 155–159.

[8] D. Gardy, *Some results on the asymptotic behaviour of coefficients of large powers of functions*, Discrete Math., 139 (1995), pp. 189–217.

[9] M. N. Huxley, *The integer points close to a curve III*, Number Theory in Progress, Walter de Gruyter, Berlin (1999) pp. 911–940.

[10] G. Kemkes, C. Lee, D. Merlini and B. Richmond, *Stirling Numbers for Complex Arguments: Asymptotics and Identities*, SIAM J. Discrete Math., 16 (2003), pp. 179–191.

[11] A. Knopfmacher and B. Richmond, *Compositions with distinct parts*, Aequationes Math., 49 (1995), pp. 86–97.

[12] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison Wesley, Reading, MA, 1989.

[13] D. Knuth, *Private Communication.*

[14] V. V. Menon, *On the maximum of Stirling numbers of the second kind*, J. Comb. Theory (A), 15 (1973), pp. 11–24.

[15] D. Merlini and B. Richmond, *Stirling Numbers for complex arguments,* SIAM J. Discrete Math., 10(1) 1997, pp 73–82.

[16] L. Moser and M. Wyman, *Stirling numbers of the second kind*, Duke Mathematical Journal, 25 (1958), pp. 29–43.

[17] A. Odlyzko, *Asymptotic enumeration methods*, in Handbook of Combinatorics, vol. 2 (R. L. Graham, M. Groetschel, and L. Lovasz, eds.), North-Holland, 1995, pp. 1063–1229.

[18] B. Richmond , *Some general problems on the number of parts in partitions*, Acta Arithmetica, LXVI.4 (1994), pp. 297–313.

[19] J. Riordan, *Combinatorial Identities*, J. Wiley and Sons, 1968.

[20] W.M. Schmidt, *Integer points on curves and surfaces*, Monatshefte für Math., 99 (1985), pp. 45–72.

[21] G. Szekeres, *An asymptotic formula in the theory of partitions*, Quarterly Journal of Mathematics, Oxford series (2), 4 (1953), pp. 96–111.

[22] N. M. Temme, *Asymptotic estimates of Stirling numbers* , Studies in Applied Math. 89 (1993), pp. 233–243.