INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 8 (2008), #A38

THE OVERPARTITION FUNCTION MODULO 128

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Received: 10/25/07, Revised: 7/15/08, Accepted: 8/7/08, Published: 9/4/08

Abstract

Let $\overline{p}(n)$ denote the overpartition function. In a recent paper, K. Mahlburg showed that $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1. In this paper, we will prove that $\overline{p}(n) \equiv 0 \pmod{128}$ for almost all integers n.

1. Introduction

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ be the number of overpartitions of an integer n. For convenience, define $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ because there are 8 overpartitions of 3: $3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1$.

We observe that the overlined parts form a partition into distinct parts and that the un-overlined parts form an ordinary partition. Thus, the generating function for overpartitions is

$$\overline{P}(q) = \sum_{n\geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Here we use the following standard q-series notation:

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1,\end{aligned}$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n,\ |q|<1.$$

The overpartition function can be used to interpret identities arising from basic hypergeometric series. For more information and references, see the works of S. Corteel, J. Lovejoy and A.J. Yee [4], [5], [17].

S. Treneer [16] showed that the coefficients of a wide class of weakly holomorphic modular forms have infinitely many congruence relations for powers of every prime p except 2 and 3. For example, Treneer showed that $\overline{p}(5l^3n) \equiv 0 \pmod 5$ for all n which are coprime to l, where l is a prime such that $l \equiv -1 \pmod 5$. However, much less is known modulo 2 and 3. For powers of 2, two different approaches have been used. One method is to find the generating function for an arithmetic progression by using q-series identities. For example, the identity $\sum_{n\geq 0} \overline{p}(8n+7)q^n = 64\frac{(q^2,q^2)_\infty^{22}}{(q,q)_\infty^{23}}$ implies that $\overline{p}(8n+7) \equiv 0 \pmod{64}$. For more information, see J.-F. Fortin, P. Jacob and P. Mathieu [6] and M.D. Hirschhorn and J. Sellers [10]. Another way is to use relations between $\overline{p}(n)$ and the number of representations of n as a sum of squares. In this direction, K. Mahlburg [14] showed that $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1. This approach uses the fact that the generating function for the overpartition function can be represented by one of Ramanujan's classical theta functions. Here we will follow the method of Mahlburg [14] in order to prove the following.

Theorem 1. $\overline{p}(n) \equiv 0 \pmod{128}$ for a set of integers of arithmetic density 1.

Mahlburg conjectured that for all positive integers k, $\overline{p}(n) \equiv 0 \pmod{2^k}$ for almost all integers n. The method here, like Mahlburg's method, relies on an ad-hoc argument, and therefore seems unlikely to generalize to arbitrary powers of 2. In general, 2-adic properties of coefficients of modular form of half-integral weight are somewhat mysterious. For example, it has long been conjectured that

$$\frac{|\{n \leq x : p(n) \equiv 0 \pmod 2\}|}{x} \sim \frac{1}{2},$$

where $p(n)$ is the number of ordinary partition of n. (See [1], for example, for references regarding this problem). This stands in contrast to the behavior exhibited by $\overline{p}(n)$.

We will conclude by proving the explicit example: $\overline{p}(10672200n+624855) \equiv 0 \pmod{128}$.

2. Proof of Theorem

Let $\theta(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$, $\psi(q) = \sum_{n=0}^{\infty} q^{(n^2+n)/2}$, and $\varphi(q) = \sum_{n=1}^{\infty} q^{n^2}$.

The coefficients $r_k(n)$ of $\theta(q)^k = \sum_{n\geq 0} r_k(n)q^n$ are the number of representations of n as the sum of k squares, where different orders and signs are counted as different. Similarly, the coefficients of $\varphi(q)^k = \sum_{n\geq 0} c_k(n)q^n$ are the number of representations of $n = n_1^2 + \cdots + n_k^2$ where each n_i is a positive integer.

From Mahlburg's paper [14], we have

$$\overline{P}(q) = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} (-1)^{n+k} c_k(n) q^n. \tag{1}$$

Reducing this expression modulo 128 we obtain

$$\begin{aligned}\overline{P}(q) \equiv & 1 + 2\sum_{n=1}^{\infty}(-1)^{n+1}c_1(n)q^n + 4\sum_{n=1}^{\infty}(-1)^n c_2(n)q^n + 8\sum_{n=1}^{\infty}(-1)^{n+1}c_3(n)q^n \\ & + 16\sum_{n=1}^{\infty}(-1)^n c_4(n)q^n + 32\sum_{n=1}^{\infty}(-1)^{n+1}c_5(n)q^n + 64\sum_{n=1}^{\infty}(-1)^n c_6(n)q^n \pmod{128}.\end{aligned}$$

We will show that in each of the six sums, the coefficient of q^n is zero modulo 128 for a set of arithmetic density 1.

The following lemma and its proof summarize results of Mahlburg [14].

Lemma 2. For almost all integers n, $c_1(n)$, $c_2(n)$, $r_1(n)$ and $r_2(n)$ are zero. If k is a fixed positive integer, then $c_3(n)$, $c_4(n)$, $r_3(n)$ and $r_4(n)$ are almost always divisible by 2^k.

Sketch of proof. First, note that by a simple combinatorial argument, we have

$$r_k(n) = 2^k c_k(n) + \sum_{i=1}^{k}\binom{k}{i}(-1)^{i-1}r_{k-i}(n). \tag{2}$$

In [14], Mahlburg showed that $c_1(n)$, $c_2(n)$, $r_1(n)$ and $r_2(n)$ are almost always zero.

For $c_3(n)$ and $r_3(n)$, note that $c_3(n) = r_3(n)/8$ for almost all integers n by (2). By the famous result of Gauss [7], we have

$$r_3(n) = \begin{cases} 12H(-4n), & \text{if } n \equiv 1,\, 2,\, 5,\, 6 \pmod 8, \\ 24H(-n), & \text{if } n \equiv 3 \pmod 8, \\ r_3(n/4), & \text{if } n \equiv 0,\, 4 \pmod 8. \end{cases}$$

Here $H(-n)$ is the Hurwitz class number of positive definite binary quadratic forms. If $2^m \| H(-n)$, then m is at least or equal to the number of distinct odd primes dividing the squarefree part of n. Thus if n has at least l distinct odd primes in its squarefree part, then $r_3(n)$ is divisible by 2^l. By (8) of [14], if $\sigma_l(x)$ is the number of integers $n \le x$ having at most l distinct odd prime factors, then asymptotically

$$\sigma_l(x) \sim \frac{x(\log\log x)^{(l-1)}}{(l-1)!\log x}. \tag{3}$$

Since $\sigma_l(x)/x$ tends to 0 as x tends to infinity, for a fixed positive integer k, $r_3(n)$ is divisible by 2^k for almost all n and so the same is true for $c_3(n)$, since $c_3(n) = r_3(n)/8$ for almost all integers n.

For $c_4(n)$ and $r_4(n)$, define $\sigma'(n) = \sum_{d|n,4\nmid d} d$. Then, by [3], $r_4(n) = 8\sigma'(n)$. Since $r_3(n)$ is almost always divisible by 2^k, by (2), $c_4(n) \equiv \frac{1}{2}\sigma'(n) \pmod{2^k}$ for almost all integers n. By (11) of [14], we have

$$\sigma'(n) = C \cdot \sum_{i=0}^{a_1} p_1^i \cdots \sum_{i=0}^{a_m} p_m^i, \tag{4}$$

where $n = 2^{a_0}p_1^{a_1}\cdots p_m^{a_m}$ and $C = 1$ or 3 according to $a_0 = 0$ or not. Let $w(n) =$ the number of distinct prime factors of n with odd exponents. Then, by (4), we have $2^{w(n)}|\sigma'(n)$. Thus if there are at least l distinct odd primes with odd exponent in the factorization of n, then $2^l|\sigma'(n)$. Since the complement of this set is the set of integers whose squarefree parts have at most l distinct odd prime factors, by (3), for a fixed positive integer k, $2^k|\sigma'(n)$ for almost all integers n. This completes the proof. □

By Lemma 2, it remains to show that $c_5(n) \equiv 0 \pmod 4$ and $c_6(n) \equiv 0 \pmod 2$ for almost all integers n. Let us show that $c_5(n) \equiv 0 \pmod 4$ for almost all integers n.

First, note that

$$\varphi^5(q) \equiv \varphi^2(q^2)\varphi(q) \pmod 4. \tag{5}$$

Let $\varphi^2(q^2)\varphi(q) = \sum_{n=1}^{\infty} R(n)q^n$. Then $R(n)$ is the number of representations of n of the form $n = x^2 + 2y^2 + 2z^2$ where x, y and z are positive integers and different orders are counted as different. It suffices to show that $R(n)$ is divisible by 4 for almost all integers n.

Before going further, we need the following lemma.

Lemma 3. Let $r_{1,2}(n)$ be the number of representations of $n = x^2 + 2y^2$, where x and y are integers. Then, we have $r_{1,2}(n) = 0$ for almost all integers n.

Sketch of proof. By [3, Theorem 3.7.3], we have $r_{1,2}(n) = 2(d_{1,8}(n)+d_{3,8}(n)-d_{5,8}(n)-d_{7,8}(n))$, where $d_{j,8}(n)$, $j = 1, 3, 5, 7$, is the number of positive divisors d of n such that $d \equiv j \pmod 8$. Thus $r_{1,2}(n) > 0$ if and only if $n = 2^a\mu\nu^2$, where μ is a product of primes congruent to 1 or 3 (mod 8) and ν is a product of primes congruent to 5 or 7 (mod 8). We denote primes $8n+1$, $8n+3$, $8n+5$ and $8n+7$ by q, r, u, and v, respectively.

Since the remainder of the proof is very similar to a proof by E. Landau [12] that $r_2(n)$ is almost always 0, we will follow the idea of this proof as given in G.H. Hardy's book *Ramanujan* [8, Sect. 4.5 and Sect. 4.6] and give only a very brief sketch here.[1] Define $b(n)$ as 1 when $r_{1,2}(n) > 0$ and 0 otherwise. Then consider the functions:

$$f(s) = \sum \frac{b(n)}{n^s} = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1-u^{-2s}} \prod_v \frac{1}{1-v^{-2s}},$$

$$\zeta(s) = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1-u^{-s}} \prod_v \frac{1}{1-v^{-s}},$$

$$L(s,\chi) = \sum \frac{\chi(n)}{n^s} = \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1+u^{-s}} \prod_v \frac{1}{1+v^{-s}},$$

where $\chi(n)$ is a Dirichlet character of conductor 8. Thus we have

$$f(s)^2 = \xi(s)\zeta(s)L(s,\chi),$$

where $\xi(s) = (1-2^{-s})^{-1}\prod_u(1-u^{-2s})^{-1}\prod_v(1-v^{-2s})^{-1}$.

[1] As G. Hardy indicated, the idea of the proof is very similar to the proof of the prime number theorem.

It is well known that neither $\zeta(s)$ nor $L(s,\chi)$ vanishes in a region D, stretching to the left of $\sigma = 1$, of type

$$\sigma > 1 - \frac{A}{\{\log(|t|+2)\}^A},$$

where, as usual, $s = \sigma + it$ [14]. Finally, note that $\zeta(s)$ and $L(s,\chi)$ are $O((\log|t|)^A)$, as $|t|$ tends to infinity in D and that $\xi(s)$ has no zeros for $\sigma > \frac{1}{2}$. It follows that $f(s) = (s-1)^{1/2}g(s)$, where $g(s)$ is analytic in D and $g(1) = (L(1,\chi)\xi(1))^{1/2}$. If $B(x) = \sum_{n\le x} b(n)$, then $B(x)$ is the number of representable numbers up to x, and we can approximate $B(x)$ by examining the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}ds$$

for $c > 1$. We can transform the path of integration into a path, stretching to the left of $\sigma = 1$ in the zero-free region D. By approximating this integral, we can conclude that

$$B(x) = \sum_{n\le x} b(n) \ll \frac{x}{\sqrt{\log x}}.$$

Therefore, $b(n) = 0$ for almost all n. □

Let $R(n,Q)$ be the number of essentially distinct representations of n by the quadratic form $Q = ax^2 + by^2 + cz^2$ where a, b and c are positive integers and x, y and z are integers. Let $r(n,Q)$ be the number of essentially distinct primitive representations of n by Q. Recall that essentially distinct representation means that different orders and signs are counted as the same and primitive representation means that the greatest common divisor of x, y and z is 1.

Then, by [15], we have

$$R(n,Q) = \sum_{d^2|n} r(\frac{n}{d^2}, Q). \tag{6}$$

Note that in $R(n, x^2+2y^2+2z^2)$, x, y or z could be 0 and a change of order between y and z is considered as the same. On the other hand, $R(n)$ is the number of representations of n of the form $n = x^2+2y^2+2z^2$, where x, y and z are positive integers and different orders are counted as different. By Lemma 2 and Lemma 3, we know that $R(n, 2x^2+2y^2)$, $R(n,x^2)$, $R(n,x^2+2y^2)$ and $R(n,2x^2)$ are almost always 0. Therefore, we can conclude that $R(n) = 2R(n,x^2+2y^2+2z^2) - R(n,x^2+4y^2)$ for almost all n. By Lemma 3 again, we have

$$R(n) = 2R(n,x^2+2y^2+2z^2) \text{ for almost all integers } n, \tag{7}$$

because $x^2+4y^2 = x^2+(2y)^2$.

For future use, we recall Theorem 86 of W. Jones [11].

Lemma 4. Let Q be a ternary form in a genus consisting of a single class. Let d be the determinant of Q and Ω be the greatest common divisor of the two-rowed minor determinants

of Q. Then, for all $n \neq \pm 1$ which are coprime to $2d$, we have

$$r(n, Q) = h(-4nd/\Omega^2)2^{-t(d/\Omega^2)}\rho \text{ or } 0,$$

where $h(D)$ is the discriminant D class number, $t(\omega)$ is the number of odd prime factors of ω and $\rho = 1/8$, $1/6$, $1/4$, $1/3$, $1/2$, 1 or 2.

Let Q be the ternary quadratic form $x^2 + 2y^2 + 2z^2$. Then Q is a ternary form in a genus consisting of a single class. Thus, we can use Lemma 4 whenever n is an odd integer.

Therefore, for odd integer n, we have

$$r(n, Q) = h(-4n)\rho \text{ or } 0.$$

Let $t(n)$ be the number of odd primes dividing the squarefree part of n. Then, by simple genus theory, recall that the exponent of 2 in $h(-n)$ is greater than or equal to $t(n) - 1$. Thus if n has at least 5 distinct odd primes in its squarefree part, then $r(n, Q)$ is divisible by 2. Thus, by (6) and (7), we see that $R(n)$ is divisible by 4 for such n. By (3), $R(n) \equiv 0 \pmod 4$ for almost all n. Thus for odd integer n, we are done.

Let us consider the case $n \equiv 2 \pmod 4$. Since $n = x^2 + 2y^2 + 2z^2$ and n is divisible by 2, x must be an even number. Thus, we can write $n' = 2x'^2 + y^2 + z^2$, where $n = 2n'$ and $x = 2x'$. Set $Q = x^2 + y^2 + 2z^2$. Then Q is a ternary form in a genus consisting of a single class. Since n' is odd, the result follows again from Lemma 4.

For the case $n \equiv 0 \pmod 4$, we need the following identities.

$$\theta(q) = \theta(q^4) + 2q\psi(q^8), \tag{8}$$
$$\theta^2(q) = \theta^2(q^2) + 4q\psi^2(q^4). \tag{9}$$

Define U by

$$U\sum_{n\geq 0} a(n)q^n = \sum_{n\geq 0} a(4n)q^n.$$

Note that $\varphi(q) = \frac{1}{2}(\theta(q) - 1)$. Thus we have

$$\begin{aligned}\varphi^2(q^2)\varphi(q) &= (\frac{1}{2}(\theta(q^2) - 1))^2\frac{1}{2}(\theta(q) - 1)\\ &= \frac{1}{8}(\theta^2(q^2) - 2\theta(q^2) + 1))(\theta(q) - 1)\\ &= \frac{1}{8}\{\theta^2(q^2)\theta(q) - 2\theta(q^2)\theta(q) + \theta(q) - \theta^2(q^2) + 2\theta(q^2) - 1\}.\end{aligned}$$

Therefore, the coefficient of q^n in $\varphi^2(q^2)\varphi(q)$ is almost always the same as the coefficient of q^n in $\frac{1}{8}\theta^2(q^2)\theta(q)$ because the coefficients of the other terms are almost always 0 by Lemma 2 and Lemma 3.

Let $\theta^2(q^2)\theta(q) = \sum_{n\geq 0} R'(n)q^n$. Then, by (5), it suffices to show that $R'(n) \equiv 0 \pmod{32}$ for almost all integers n. By (8) and (9), we have

$$\begin{aligned} U\theta^2(q^2)\theta(q) &= U((\theta^2(q^4) + 4q^2\psi^2(q^8))(\theta(q^4) + 2q\psi(q^8)) \\ &= U(\theta^3(q^4) + 2q\theta^2(q^4)\psi(q^8) + 4q^2\theta(q^4)\psi^2(q^8) + 8q^3\psi^3(q^8)) \\ &= \theta^3(q), \\ U\theta^3(q) &= \theta^3(q). \end{aligned}$$

From this, we have $R'(4^\lambda n) = r_3(4^{\lambda-1}n) = \cdots = r_3(n)$, where n is not divisible by 4. Thus by Lemma 2, $c_5(n) \equiv 0 \pmod 4$ for almost all integers $n \equiv 0 \pmod 4$. From the last three cases, we conclude that $c_5(n) \equiv 0 \pmod 4$ for almost all integers n.

Finally, we need to show that $c_6(n)$ is almost always even. To show this, note that

$$\varphi^6(q) \equiv \varphi^3(q^2) \pmod 2.$$

Then,

$$c_6(n) \equiv \begin{cases} 0 & \pmod 2, \quad \text{if } n \text{ is odd}, \\ c_3(n/2) & \pmod 2, \quad \text{if } n \text{ is even}. \end{cases} \tag{10}$$

Therefore, by Lemma 2 again, we conclude that $c_6(n)$ is almost always even. This completes the proof of Theorem 1. □

We conclude with an example. When $n \equiv 7 \pmod 8$, $c_1(n), c_2(n), c_3(n)$ and $c_5(n)$ are zero. Moreover, $c_6(n)$ is always even by (10). Thus, if $c_4(n)$ is divisible by 8, then $\overline{p}(n) \equiv 0 \pmod{128}$ for such n. Recall the proof of Lemma 2. We have $c_4(n) = \frac{1}{2}\sigma'(n)$, where $\sigma'(n) = \sum_{d|n, 4\nmid d} d$. By (4), to guarantee that $\frac{1}{2}\sigma'(n)$ is divisible by 16, n must have at least four distinct odd prime factors with odd exponent.

We solve the following system of congruences:

$$\begin{aligned} n &\equiv 7 \pmod 8, \\ n &\equiv 3 \pmod{3^2}, \\ n &\equiv 5 \pmod{5^2}, \\ n &\equiv 7 \pmod{7^2}, \\ n &\equiv 11 \pmod{11^2}. \end{aligned}$$

By a simple calculation, we find that $n \equiv 624855 \pmod{10672200}$. Since $3\|n$, $5\|n$, $7\|n$, $11\|n$ and $n \equiv 7 \pmod 8$, the proof of Theorem 1 implies that $\overline{p}(10672200n + 624855) \equiv 0 \pmod{128}$.

Acknowledgments

The author would like to thank Scott Ahlgren for his encouragement and help. The author is also grateful to Bruce Berndt and the referee for their helpful comments.

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