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ON THE FROBENIUS PROBLEM FOR GEOMETRIC SEQUENCES

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Abstract

Let a, b, k be positive integers, with $\gcd(a, b) = 1$, and let $\mathcal{A}$ denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$. Let $\Gamma(\mathcal{A})$ denote the set of integers that are expressible as a linear combination of elements of $\mathcal{A}$ with non-negative integer coefficients. We determine $g(\mathcal{A})$ and $n(\mathcal{A})$ which denote the *largest* (respectively, the *number* of) positive integer(s) not in $\Gamma(\mathcal{A})$. We also determine the set $\mathcal{S}^\star(\mathcal{A})$ of positive integers not in $\Gamma(\mathcal{A})$ which satisfy $n + \Gamma^\star(\mathcal{A}) \subset \Gamma^\star(\mathcal{A})$, where $\Gamma^\star(\mathcal{A}) = \Gamma(\mathcal{A}) \setminus \{0\}$.

1. Introduction

For a sequence of relatively prime positive integers $A = a_1, a_2, \ldots, a_k$, let $\Gamma(A)$ denote the set of all integers of the form $\sum_{i=1}^{k} a_i x_i$ where each $x_i \geq 0$. It is well known and not difficult to show that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is a *finite* set. The *Coin Exchange Problem* of Frobenius is to determine the *largest* integer in $\Gamma^c(A)$. This is denoted by $g(A)$, and called the Frobenius number of A. The Frobenius number is known in the case $k = 2$ to be $g(a_1, a_2) = a_1a_2 - a_1 - a_2$, but is generally otherwise unsolved except in some special cases. A related problem is the determination of the number of integers in $\Gamma^c(A)$, which is denoted by $n(A)$ and known in the case $k = 2$ to be given by $n(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. More complete information on the Frobenius problem may be found in [3].

Ong and Ponomarenko recently determined the Frobenius number for geometric sequences in [2]. If we denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$ by $\mathcal{A}_k(a, b)$, and the corresponding Frobenius number by $\mathrm{G}_k = g\big(\mathcal{A}_k(a, b)\big)$, Ong & Ponomarenko proved their claim by showing that the sequence $\{\mathrm{G}_k\}_{k\geq 1}$ satisfies a certain first order recurrence, and then using induction. The main purpose of this note is to show that both the Frobenius number $g(A)$ and $n(A)$ follow in the case of geometric sequences from an old reduction formula due to Johnson [1] and Rödseth [4]. We further determine the set $\mathcal{S}^\star$, introduced in [5], in the case of geometric sequences. This gives another proof of the result for the Frobenius number since $g(A)$ is the largest integer in $\mathcal{S}^\star\big(A\big)$.

2. Main Results

Throughout this section, for positive integers a, b, k with $\gcd(a,b) = 1$, we denote by $\mathcal{A}_k(a,b)$ the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$. We derive the values of both $g\big(\mathcal{A}_k(a,b)\big) := \mathrm{G}_k$ and $n\big(\mathcal{A}_k(a,b)\big) := \mathrm{N}_k$ by two methods. We first use a well-known reduction formula to derive recurrence relations for the two sequences $\{\mathrm{G}_k\}_{k\geq 1}$ and $\{\mathrm{N}_k\}_{k\geq 1}$, and then use telescoping sums to solve each recurrence. The second method to derive $g\big(\mathcal{A}_k(a,b)\big)$ consists in showing that $\mathcal{S}^\star\big(\mathcal{A}_k(a,b)\big)$ has exactly one element, which must then be $g\big(\mathcal{A}_k(a,b)\big)$. Our second proof of the result for $n\big(\mathcal{A}_k(a,b)\big)$ is indirect; we show that $2\,n\big(\mathcal{A}_k(a,b)\big) - 1 = g\big(\mathcal{A}_k(a,b)\big)$. We first recall the reduction formula that is central to our first derivation.

Lemma 1. ([1, 4]) *Let $a_1, a_2, \ldots, a_k$ be positive integers. If $\gcd(a_2, \ldots, a_k) = d$ and $a_j = da_j'$ for each $j > 1$, then*

(a) $g(a_1, a_2, \ldots, a_k) = d\, g(a_1, a_2', \ldots, a_k') + a_1(d-1)$;

(b) $n(a_1, a_2, \ldots, a_k) = d\, n(a_1, a_2', \ldots, a_k') + \frac{1}{2}(a_1 - 1)(d-1)$.

Theorem 1. *Let a, b, k be positive integers, with $\gcd(a,b) = 1$. Let $\mathcal{A}_k(a,b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a,b)$ denote the sum of the integers in $\mathcal{A}_k(a,b)$. Then*

(a) $g(\mathcal{A}_k(a,b)) = \sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1})$;

(b) $n(\mathcal{A}_k(a,b)) = \frac{1}{2}\big\{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}) + 1\big\}$.

Proof.

(a) For $k \geq 1$, by Lemma 1, with $a_1 = a^k$ and $d = b$, we have

$$\begin{aligned} g\big(\mathcal{A}_k(a,b)\big) &= b\, g(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b-1) \\ &= b\, g(a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b-1) \\ &= b\, g\big(\mathcal{A}_{k-1}(a,b)\big) + a^k(b-1). \end{aligned}$$

If we write $g\big(\mathcal{A}_k(a,b)\big) := \mathrm{G}_k$, then the sequence $\{\mathrm{G}_n\}_{n\geq 1}$ satisfies the first order recurrence

$$\mathrm{G}_n = b\,\mathrm{G}_{n-1} + a^n(b-1), \quad \mathrm{G}_1 = g(a,b) = ab - a - b.$$

Dividing both sides of the recurrence by b^n, summing from $n = 2$ to $n = k$ and simplifying, we get

$$\frac{\mathrm{G}_k}{b^k} = \frac{\mathrm{G}_1}{b} + a^2(b-1)\frac{b^{k-1} - a^{k-1}}{b^k(b-a)},$$

so that

$$\begin{aligned} g\big(\mathcal{A}_k(a,b)\big) &= \mathrm{G}_k = a^2(b-1)\frac{b^{k-1} - a^{k-1}}{b-a} + b^{k-1}(ab - a - b) \\ &= \sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}). \end{aligned}$$

(b) This is similar to part (a). For $k \geq 1$, by Lemma 1, with $a_1 = a^k$ and $d = b$, we have

$$\begin{aligned} n\big(\mathcal{A}_k(a,b)\big) &= b\, n(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k-1)(b-1) \\ &= b\, n(a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k-1)(b-1) \\ &= b\, n\big(\mathcal{A}_{k-1}(a,b)\big) + \frac{1}{2}(a^k-1)(b-1). \end{aligned}$$

If we write $n\big(\mathcal{A}_k(a,b)\big) := \mathrm{N}_k$, then the sequence $\{\mathrm{N}_n\}_{n\geq 1}$ satisfies the first order recurrence

$$\mathrm{N}_n = b\,\mathrm{N}_{n-1} + \frac{1}{2}(a^n-1)(b-1), \quad \mathrm{N}_1 = n(a,b) = \frac{1}{2}(a-1)(b-1).$$

Dividing both sides of the recurrence by b^n, summing from $n = 2$ to $n = k$ and simplifying, we get

$$\frac{\mathrm{N}_k}{b^k} = \frac{\mathrm{N}_1}{b} + \frac{1}{2}a^2(b-1)\frac{b^{k-1}-a^{k-1}}{b^k(b-a)} - \frac{1}{2}\frac{b^{k-1}-1}{b^k},$$

so that

$$\begin{aligned} n\big(\mathcal{A}_k(a,b)\big) &= \mathrm{N}_k = \frac{1}{2}a^2(b-1)\frac{b^{k-1}-a^{k-1}}{b-a} - \frac{1}{2}(b^{k-1}-1) + \frac{1}{2}b^{k-1}(a-1)(b-1) \\ &= \frac{1}{2}\big\{1 + g(\mathcal{A}_k(a,b))\big\} \\ &= \frac{1}{2}\big\{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1}+b^{k+1}) + 1\big\}. \end{aligned}$$

□

The formulae for both $g\big(\mathcal{A}_k(a,b)\big)$ and $n\big(\mathcal{A}_k(a,b)\big)$ in Theorem 1 display a nice symmetry in the variables a, b. From Theorem 1 we have $n\big(\mathcal{A}_k(a,b)\big) = \frac{1}{2}\big\{1 + g(\mathcal{A}_k(a,b))\big\}$. If m, n are integers with sum $g(\mathcal{A}_k(a,b))$, then it is easy to see that *at most* one of m, n can belong to $\Gamma(\mathcal{A}_k(a,b))$. On the other hand, if for some such pair m, n, neither belongs to $\Gamma(\mathcal{A}_k(a,b))$, there would be less than $\frac{1}{2}\big\{1 + g(\mathcal{A}_k(a,b))\big\}$ integers in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. Thus, for every pair of non-negative integers m, n with sum $g(\mathcal{A}_k(a,b))$, *exactly* one of m, n belong to $\Gamma^c(\mathcal{A}_k(a,b))$. We use this to derive $n(\mathcal{A}_k(a,b))$, giving a second proof of the assertion in the second part of Theorem 1.

Theorem 2. *Let a, b, k be positive integers, with $\gcd(a,b) = 1$. Let $\mathcal{A}_k(a,b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a,b)$ denote the sum of the integers in $\mathcal{A}_k(a,b)$. If $m + n = g(\mathcal{A}_k(a,b))$, then $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$ if and only if $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$.*

Proof. Let $m + n = g(\mathcal{A}_k(a,b))$. If $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$, then $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$, for otherwise $m + n = g(\mathcal{A}_k(a,b)) \in \Gamma\big(\mathcal{A}_k(a,b)\big)$, which is impossible.

Conversely, suppose $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$. If $n < 0$, then $m > g(\mathcal{A}_k(a,b))$ and so $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$. We may therefore assume that $1 \leq n \leq g(\mathcal{A}_k(a,b))$ since both 0 and any integer greater than

$g(\mathcal{A}_k(a,b))$ belong to $\Gamma\big(\mathcal{A}_k(a,b)\big)$. Since $n+\lambda b^k \in \Gamma\big(a^k, a^{k-1}b, \ldots, ab^{k-1}\big)$ for all sufficiently large integer λ and $n \notin \Gamma\big(a^k, a^{k-1}b, \ldots, ab^{k-1}\big)$, we may write $n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k$, where $x_i \geq 0$ for $0 \leq i \leq k-1$ and $x_k \geq 1$. If $x_0 > b$ in this representation, by repeatedly using the identity $a^k(x_0-b)+a^{k-1}b(x_1+a) = a^k x_0 + a^{k-1}bx_1$ we may assume that $0 \leq x_0 < b$ while maintaining $x_1 \geq 0$. Assuming that $x_0, x_1, \ldots, x_{j-1}$ are all non-negative integers less than b for some $j < k$, by repeatedly using the identity $a^{k-j}b^j(x_j-b)+a^{k-j-1}b^{j+1}(x_{j+1}+a) = a^{k-j}b^j x_j + a^{k-j-1}b^{j+1}x_{j+1}$, we may assume that $0 \leq x_j < b$ and still have $x_{j+1} \geq 0$. Thus we may write

$$n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k,$$

with $0 \leq x_i \leq b-1$ for $0 \leq i \leq k-1$, and since $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$, also $x_k \geq 1$. Writing $g(\mathcal{A}_k(a,b)) = (b-1)\sum_{i=0}^{k-1} a^{k-i}b^i - b^k$, we have

$$m = g(\mathcal{A}_k(a,b)) - n = \sum_{i=0}^{k-1} (b-1-x_i)a^{k-i}b^i + (x_k-1)b^k \in \Gamma\big(\mathcal{A}_k(a,b)\big).$$

This completes the proof. □

Corollary 1. *Let a, b, k be positive integers, with $\gcd(a,b) = 1$. Then*

$$n\big(\mathcal{A}_k(a,b)\big) = \frac{1}{2}\left\{1 + g\big(\mathcal{A}_k(a,b)\big)\right\}.$$

Proof. Consider pairs $\{m,n\}$ of integers in the interval $[0, g\big(\mathcal{A}_k(a,b)\big)]$ with $m+n = g\big(\mathcal{A}_k(a,b)\big)$. By Theorem 2, *exactly* one integer from each such pair is in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. This completes the proof since no integer greater than $g\big(\mathcal{A}_k(a,b)\big)$ is in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. □

Remark 1. *Let a, b, k be positive integers, with $\gcd(a,b) = 1$. Then $g\big(\mathcal{A}_k(a,b)\big)$ is an odd integer.*

The evaluation of g given in Theorem 1 can also be derived by explicitly determining the set $\mathcal{S}^\star$, introduced in [5], since $g(a_1, a_2, \ldots, a_k)$ is the largest element in $\mathcal{S}^\star(a_1, a_2, \ldots, a_k)$. For positive and coprime integers $a_1, a_2, \ldots, a_k$, let $\Gamma(a_1, a_2, \ldots, a_k)$ denote the non-negative integers in the set $\{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0\}$, let m_j denote the least positive integer in $\Gamma(a_1, a_2, \ldots, a_k)$ that is congruent to j mod a_1 for $1 \leq j \leq a_1 - 1$, and let $\Gamma^\star(a_1, a_2, \ldots, a_k) = \Gamma(a_1, a_2, \ldots, a_k) \setminus \{0\}$. Then

$$\begin{aligned} \mathcal{S}^\star(a_1, a_2, \ldots, a_k) &:= \{n \notin \Gamma(a_1, \ldots, a_k) : n + \Gamma^\star(a_1, \ldots, a_k) \subset \Gamma^\star(a_1, \ldots, a_k)\} \\ &\subseteq \{m_j - a_1 : 1 \leq j \leq a_1 - 1\}. \end{aligned}$$

Moreover,

$$m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k) \Longleftrightarrow m_j + m_i > m_{j+i} \text{ for } 1 \leq i \leq a_1 - 1. \tag{1}$$

We refer to [5] for more notations and results. With the notations above, we show that $\mathcal{S}^\star\big(\mathcal{A}_k(a,b)\big) = \{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1}+b^{k+1})\}$. Since $g(a_1, a_2, \ldots, a_k) \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$, this further verifies the first result of Theorem 1.

Lemma 2. *Let $a_1, a_2, \ldots, a_k$ be positive integers with $\gcd(a_2, \ldots, a_k) = d$. Define, $a'_j = a_j/d$ for $2 \le j \le k$. Let m_j (respectively, m'_j) denote the least positive integer in $\Gamma(a_1, a_2, \ldots, a_k)$ (resp., in $\Gamma(a_1, a'_2, \ldots, a'_k)$) that is congruent to j mod a_1. Then $m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ if and only if $m'_j - a_1 \in \mathcal{S}^\star(a_1, a'_2, \ldots, a'_k)$ for $1 \le j \le a_1 - 1$.*

Proof. Let A denote the sequence $a_1, a_2, \ldots, a_k$ and A' the sequence $a_1, a'_2, \ldots, a'_k$. Since each m_j and m'_j must also be representable as a non-negative linear combination of $a_2, \ldots, a_k$ and $a'_2, \ldots, a'_k$ respectively, it follows that $\{m_j : 1 \le j \le a_1 - 1\} = \{dm'_j : 1 \le j \le a_1 - 1\}$. Therefore, by (1), $m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ if and only if $m_j + m_i > m_{j+i}$ for $1 \le i \le a_1 - 1$ if and only if $m'_j + m'_i > m'_{j+i}$ for $1 \le i \le a_1 - 1$ if and only if $m'_j - a_1 \in \mathcal{S}^\star(a_1, a'_2, \ldots, a'_k)$. This completes the proof. □

Theorem 3. *Let a, b, k be positive integers, with $\gcd(a, b) = 1$. Let $\mathcal{A}_k(a, b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a, b)$ denote the sum of the integers in $\mathcal{A}_k(a, b)$. Then $\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big) = \big\{\sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1})\big\}$ for $k \ge 1$.*

Proof. We apply Lemma 2 with $A = \mathcal{A}_k(a, b)$ and $a_1 = a^k$. Then $d = b$ and $m_j - a^k \in \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$ if and only if $\frac{1}{b}m_j - a^k \in \mathcal{S}^\star\big(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}\big) = \mathcal{S}^\star\big(\mathcal{A}_{k-1}(a, b)\big)$. Therefore, by Theorem 1 in [5], $\big|\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)\big| = \big|\mathcal{S}^\star\big(\mathcal{A}_1(a, b)\big)\big| = 1$ for each $k > 1$. Since we have $g(\mathcal{A}_k(a, b)) \in \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$, there can be no other integer in this set. □

Corollary 2. *Let a, b, k be positive integers, with $\gcd(a, b) = 1$. Then*

$$g\big(\mathcal{A}_k(a, b)\big) = \max \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}).$$

Remark 2. *The proof of Theorem 3 shows that the sequence of Frobenius numbers $\{g(\mathcal{A}_k(a, b))\}_{k\ge1}$ satisfies the recurrence $\mathrm{G}_k = b\,\mathrm{G}_{k-1} + a^k(b-1)$ since $g(\mathcal{A}_k(a, b)) = m_j - a^k$ is the only element in $\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$. This result coincides with the result in the first part of Theorem 1.*

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