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ANALOGUES OF TWO CLASSICAL THEOREMS ON THE REPRESENTATIONS OF A NUMBER

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Abstract

Jacobi's classical four-square theorem gives the number of representations of a positive integer as the sum of four squares. A theorem of Legendre gives the number of representations of a positive integer as the sum of four triangular numbers. In this paper we give analogous results that involve triangular numbers, squares, pentagonal numbers, and octagonal numbers.

1. Introduction

Let $\sigma(n)$ denote the sum of the divisors of n. Following convention, $\sigma(n) = 0$ if n is not a positive integer. Then two classical theorems of Jacobi and Legendre, respectively, can be stated thus:

Theorem 1 (Jacobi [7]). *The number of representations of $n \geq 1$ as the sum of four squares is*

$$r\left\{\square+\square+\square+\square\right\}(n) = 8\sum_{d|n,\,4\nmid d} d = 8\left(\sigma(n) - 4\sigma(n/4)\right).$$

Theorem 2 (Legendre [9]). *The number of representations of $n \geq 1$ as the sum of four triangular numbers is*

$$r\left\{\triangle+\triangle+\triangle+\triangle\right\}(n) = \sigma(2n+1).$$

To state another old result, which we require in the sequel, we first define

$$\chi(n) = \begin{cases} +1, & n \equiv \pm 1 \pmod 6, \\ -1, & n \equiv \pm 2 \pmod 6, \\ 0, & n \equiv 0 \pmod 3. \end{cases}$$

Then

Theorem 3. *The number of representations of $n \geq 1$ as the sum of a square plus a square plus three times a square plus three times a square is*

$$r\left\{\square + \square + 3\square + 3\square\right\}(n) = (-1)^{n-1} 4 \sum_{d|n} d\chi(d). \tag{1}$$

This result has a long history. It was stated without proof by Liouville, and proved by Pepin, Bachmann, and Kloosterman. The references to the relevant works of these authors are given in [1], where the result is listed as Theorem 1.9. A proof can also be found in [4, p. 76]. Ramanujan also listed it as Entry 3 (iv) in Chapter 19 of his second notebook. See [2, p. 223].

With the same notation, Hirschhorn [5] recently proved the following:

$$r\left\{\triangle + \triangle + \square + \square\right\}(n) = \sigma(4n+1),\ n \geq 1, \tag{2}$$

$$r\left\{\triangle + \triangle + 2\triangle + 2\triangle\right\}(n) = \sigma(4n+3)/4,\ n \geq 1. \tag{3}$$

While Hirschhorn did not state (3) explicitly, it follows immediately from his equation (3.20). In further work, presently unpublished, Hirschhorn discovered two further results that involve pentagonal and octagonal numbers. Both are valid for $n \geq 1$, and can be stated thus:

$$r\left\{3\triangle + 3\triangle + \mathrm{D} + \mathrm{D}\right\}(n) = \sigma(6n+5)/6, \tag{4}$$

$$r\left\{3\square + 3\square + 8 + 8\right\}(n) = \left(\sigma(3n+2) - 4\sigma((3n+2)/4)\right)/3. \tag{5}$$

In this paper we prove the following results, each of which being valid for $n \geq 1$:

$$r\left\{\triangle + \triangle + \triangle + 4\triangle\right\}(n) = \sigma(8n+7)/8, \tag{6}$$

$$r\left\{\triangle + \triangle + \triangle + 2\square\right\}(n) = \sigma(8n+3)/4, \tag{7}$$

$$r\left\{\square+\square+3\square+8\right\}(n)=(-1)^n\sum_{d|3n+1}d\chi(d), \tag{8}$$

$$r\left\{\square+\square+8+8\right\}(n)=(-1)^{n+1}\sum_{d|3n+2}d\chi(d), \tag{9}$$

$$r\left\{\triangle+\triangle+3\triangle+3\triangle\right\}(n)=-\sum_{d|8n+8,\,d\nmid 2n+2}d\chi(d)/12, \tag{10}$$

$$r\left\{\triangle+\triangle+3\triangle+\mathrm{D}\right\}(n)=-\sum_{d|24n+16,\,d\nmid 6n+4}d\chi(d)/24, \tag{11}$$

$$r\left\{\triangle+\triangle+\mathrm{D}+\mathrm{D}\right\}(n)=-\sum_{d|24n+8,\,d\nmid 6n+2}d\chi(d)/12, \tag{12}$$

$$r\left\{3\square+\mathrm{D}+\mathrm{D}+8\right\}(n)=\sigma(12n+5)/6, \tag{13}$$

$$r\left\{6\triangle+\mathrm{D}+\mathrm{D}+2\mathrm{D}\right\}(n)=\sigma(12n+11)/12. \tag{14}$$

A theta function identity that implies (7) was given by H. Y. Lam in [8, Eq. 9]. Lam has informed the present author that, with the aforementioned identity, he was able to prove (7), and that it appears in his thesis (cited in [8]) as identity (6.4.37).

A theta function identity that implies (10) was given by Ramanujan, who listed it as Entry 3 (iii) in Chapter 19 of his second notebook. See [2, p. 223].

2. Preliminary Results

Define

$$\psi(q)=\sum_{n\geq 0}q^{(n^2+n)/2},\quad \phi(q)=\sum_{-\infty}^{\infty}q^{n^2},$$

which are generating functions of the triangular numbers and the squares, respectively. In addition let

$$P(q)=\sum_{-\infty}^{\infty}q^{(3n^2+n)/2},\quad X(q)=\sum_{-\infty}^{\infty}q^{3n^2+2n},$$

generating functions of the pentagonal and octagonal numbers, respectively.

The following three relationships, which are easy to prove, are given in [3], [5], and [6]:

$$\begin{aligned}\phi(q) &= \phi\left(q^4\right) + 2q\psi\left(q^8\right), \\ \phi(q) &= \phi\left(q^9\right) + 2qX\left(q^3\right), \\ \psi(q) &= P\left(q^3\right) + q\psi\left(q^9\right).\end{aligned}$$

We use the next identity, established by Hirschhorn [5], in the proof of (10).

$$\psi(q)\,\psi\left(q^3\right) = \phi\left(q^6\right)\psi\left(q^4\right) + q\phi\left(q^2\right)\psi\left(q^{12}\right). \tag{15}$$

We also require the following two results.

$$P\left(q^2\right)X(q) = P(q)\,\psi\left(q^3\right), \tag{16}$$

$$P(q)^2 = P\left(q^2\right)\phi\left(q^3\right) + 2qX(q)\,\psi\left(q^6\right). \tag{17}$$

We have not cited (16) in the literature, while (17) appears in [3, p. 19].

Let a and q be complex numbers with $|q| < 1$, and set

$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1 - aq^j\right).$$

With the use Jacobi's triple-product identity, together with simple properties of the infinite product above, it can be shown that

$$\psi(q) = \frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty}, \quad \phi(q) = \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2\left(q^4;q^4\right)_\infty^2},$$

and

$$P(q) = \frac{\left(q^2;q^2\right)_\infty\left(q^3;q^3\right)_\infty^2}{(q;q)_\infty\left(q^6;q^6\right)_\infty}, \quad X(q) = \frac{\left(q^2;q^2\right)_\infty^2\left(q^3;q^3\right)_\infty\left(q^{12};q^{12}\right)_\infty}{(q;q)_\infty\left(q^4;q^4\right)_\infty\left(q^6;q^6\right)_\infty}.$$

Identity (16) follows upon substitution of the relevant expressions.

3. Proofs

Jacobi's result is equivalent to

$$\phi(q)^4 = \left(\phi\left(q^{16}\right) + 2q\psi\left(q^8\right) + 2q^4\psi\left(q^{32}\right)\right)^4 = 1 + 8\sum_{n\geq 1}\left(\sum_{d|n,\, 4\nmid d} d\right)q^n. \tag{18}$$

In the expansion of $\left(\phi\left(q^{16}\right) + 2q\psi\left(q^8\right) + 2q^4\psi\left(q^{32}\right)\right)^4$ only the product $64q^7\psi\left(q^8\right)^3\psi\left(q^{32}\right)$ contains powers of q that are congruent to 7 (mod 8). Therefore from the right side of (18) we extract those terms in which the power of q is 7 (mod 8), divide by q^7, and replace q^8 by q to obtain

$$64\psi(q)^3\psi\left(q^4\right) = 8\sum_{n\geq 0}\left(\sum_{d|8n+7} d\right)q^n = 8\sum_{n\geq 0}\sigma(8n+7)q^n,$$

which is (6).

Here we have used dissection of series to prove (6). Indeed, we use this technique in all the proofs that follow (in Section 4 we give a short commentary on other types of identities that can be proved with this technique). The following table will assist the reader to keep track of this process. The first column gives the identity being proved. The second column gives the identity (or its equivalent) that is being dissected, while the third column gives the powers of q that are (in the first instance) being extracted.

Identity	Dissected Series	Terms Extracted
6	Theorem 1	q^{8n+7}
7	Theorem 1	q^{8n+3}
8	Theorem 3	q^{3n+1}
9	Theorem 3	q^{3n+2}
10	Theorem 3	q^{4n}
11	23	q^{3n+1}
12	23	q^{3n}
13	2	q^{3n+1}
14	3	q^{3n+2}

Once again, in the expansion of $\left(\phi\left(q^{16}\right) + 2q\psi\left(q^8\right) + 2q^4\psi\left(q^{32}\right)\right)^4$ only the product $32q^3\psi\left(q^8\right)^3\phi\left(q^{16}\right)$ contains powers of q that are congruent to 3 (mod 8). From the right side of (18) we extract those terms in which the power of q is 3 (mod 8), divide by q^3, and replace q^8 by q to obtain

$$32\psi(q)^3\phi\left(q^2\right) = 8\sum_{n\geq 0}\left(\sum_{d|8n+3} d\right)q^n = 8\sum_{n\geq 0}\sigma(8n+3)q^n,$$

which is (7).

With the same expansion we also obtain (3), discovered by Hirschhorn. To accomplish this we consider the powers of q congruent to 6 (mod 8), and make use of the result $\phi(q)\psi\left(q^2\right) = \psi(q)^2$, which was recorded by Ramanujan as Entry 25 (iv) in Chapter 16 of his second notebook. See [2, p. 40].

To prove (8) and (9) we consider (1), which is equivalent to

$$\begin{aligned}\phi(q)^2\phi\left(q^3\right)^2 &= \left(\phi\left(q^9\right) + 2qX\left(q^3\right)\right)^2\phi\left(q^3\right)^2 \\ &= 1 + \sum_{n\geq 1}\left((-1)^{n-1}4\sum_{d|n} d\chi(d)\right)q^n. \qquad (19)\end{aligned}$$

In the expansion of $\left(\phi\left(q^9\right) + 2qX\left(q^3\right)\right)^2\phi\left(q^3\right)^2$ only the product $4q\phi\left(q^3\right)^2\phi\left(q^9\right)X\left(q^3\right)$ contains powers of q that are congruent to 1 (mod 3). Accordingly, from the right side of (19) we extract those terms in which the power of q is 1 (mod 3), divide by $4q$, and replace q^3 by q to obtain

$$\phi(q)^2\phi\left(q^3\right)X(q) = \sum_{n\geq 0}\left((-1)^n\sum_{d|3n+1} d\chi(d)\right)q^n,$$

which is equivalent to (8).

In the same manner, those terms in which the power of q is 2 (mod 3) yield

$$\phi(q)^2X(q)^2 = \sum_{n\geq 0}\left((-1)^{n+1}\sum_{d|3n+2} d\chi(d)\right)q^n,$$

which is equivalent to (9).

Once again we turn to (1), which we write in an alternative manner as

$$\begin{aligned}\phi(q)^2\phi\left(q^3\right)^2 &= \left(\phi\left(q^4\right)+2q\psi\left(q^8\right)\right)^2\left(\phi\left(q^{12}\right)+2q^3\psi\left(q^{24}\right)\right)^2 \\ &= 1+\sum_{n\geq 1}\left((-1)^{n-1}4\sum_{d|n}d\chi(d)\right)q^n. \end{aligned} \tag{20}$$

Of the nine products obtained by expanding, three consist of powers of q that are congruent to 0 (mod 4). Extracting these powers of q from both sides we have

$$\begin{aligned}&\phi\left(q^4\right)^2\phi\left(q^{12}\right)^2+16q^4\psi\left(q^4\right)^2\psi\left(q^{12}\right)^2+16q^8\psi\left(q^8\right)^2\psi\left(q^{24}\right)^2 \\ &\qquad = 1+\sum_{n\geq 1}\left(-4\sum_{d|4n}d\chi(d)\right)q^{4n}. \end{aligned} \tag{21}$$

Next, we use (1) to substitute for $\phi\left(q^4\right)^2\phi\left(q^{12}\right)^2$, replace q^4 by q, and divide by $16q$ to obtain

$$\psi\left(q\right)^2\psi\left(q^3\right)^2+q\psi\left(q^2\right)^2\psi\left(q^6\right)^2=\sum_{n\geq 1}\left(\left(-\sum_{d|4n}d\chi(d)+(-1)^n\sum_{d|n}d\chi(d)\right)/4\right)q^{n-1}. \tag{22}$$

Now, identity (15) allows us to substitute for $\psi\left(q\right)^2\psi\left(q^3\right)^2$, so the left side of (22) can be replaced by $3q\psi\left(q^2\right)^2\psi\left(q^6\right)^2+\phi\left(q^6\right)^2\psi\left(q^4\right)^2+q^2\phi\left(q^2\right)^2\psi\left(q^{12}\right)^2$. Then, from both sides of the identity that results, we extract the powers of q that are 1 (mod 2), divide by $3q$, and replace q^2 by q to obtain

$$\begin{aligned}\psi\left(q\right)^2\psi\left(q^3\right)^2 &= \sum_{n\geq 1}\left(\left(-\sum_{d|8n}d\chi(d)+\sum_{d|2n}d\chi(d)\right)/12\right)q^{n-1} \\ &= \sum_{n\geq 1}\left(-\sum_{d|8n,\,d\nmid 2n}d\chi(d)/12\right)q^{n-1}, \end{aligned} \tag{23}$$

which is equivalent to (10).

The left side of (23) is $\psi\left(q^3\right)^2\left(q\psi\left(q^9\right)+P\left(q^3\right)\right)^2$, which expands to

$$\psi\left(q^3\right)^2P\left(q^3\right)^2+2q\psi\left(q^3\right)^2\psi\left(q^9\right)P\left(q^3\right)+q^2\psi\left(q^3\right)^2\psi\left(q^9\right)^2. \tag{24}$$

Therefore, if from the right side of (23) we extract those terms in which the power of q is 1 (mod 3), divide by $2q$, and replace q^3 by q, we obtain

$$\psi(q)^2\,\psi\left(q^3\right)P(q)=\sum_{n\geq 0}\left(-\sum_{d\mid 24n+16,\,d\nmid 6n+4}d\chi(d)/24\right)q^n,$$

which is equivalent to (11).

In view of (24), from the right side of (23) we extract those terms in which the power of q is 0 (mod 3), and replace q^3 by q to obtain

$$\psi(q)^2\,P(q)^2=\sum_{n\geq 0}\left(-\sum_{d\mid 24n+8,\,d\nmid 6n+2}d\chi(d)/12\right)q^n,$$

which is equivalent to (12).

To prove (13) we begin with (2), writing it as

$$\begin{aligned}\psi(q)^2\phi(q)^2 &= \left(q\psi\left(q^9\right)+P\left(q^3\right)\right)^2\left(\phi\left(q^9\right)+2qX\left(q^3\right)\right)^2\\ &= \sum_{n\geq 0}\sigma(4n+1)q^n. \qquad (25)\end{aligned}$$

Three products in the expansion of $\left(q\psi\left(q^9\right)+P\left(q^3\right)\right)^2\left(\phi\left(q^9\right)+2qX\left(q^3\right)\right)^2$ consist of powers of q that are 1 (mod 3), their sum being

$$4q\phi\left(q^9\right)P\left(q^3\right)^2X\left(q^3\right)+2q\psi\left(q^9\right)\phi\left(q^9\right)^2P\left(q^3\right)+4q^4\psi\left(q^9\right)^2X\left(q^3\right)^2. \qquad (26)$$

Now, observing that

$$\phi\left(q^3\right)P(q)^2X(q)=\psi\left(q^3\right)\phi\left(q^3\right)^2P(q)+2q\psi\left(q^3\right)^2X(q)^2,$$

which follows immediately from (16) and (17), we see that the sum in (26) is $6q\phi\left(q^9\right)P\left(q^3\right)^2X\left(q^3\right)$. We therefore extract from the right side of (25) those terms in which the power of q is 1 (mod 3) to obtain

$$6q\phi\left(q^9\right)P\left(q^3\right)^2X\left(q^3\right)=\sum_{n\geq 0}\sigma(12n+5)q^{3n+1},$$

from which (13) follows.

To prove (14) we begin with (3), writing it as

$$\begin{aligned}\psi(q)^2\psi(q^2)^2 &= \left(q\psi\left(q^9\right)+P\left(q^3\right)\right)^2\left(q^2\psi\left(q^{18}\right)+P\left(q^6\right)\right)^2\\ &= \sum_{n\geq 0}\left(\sigma(4n+3)/4\right)q^n. \end{aligned} \tag{27}$$

In the expansion of $\left(q\psi\left(q^9\right)+P\left(q^3\right)\right)^2\left(q^2\psi\left(q^{18}\right)+P\left(q^6\right)\right)^2$, three terms consist of powers of q that are 2 (mod 3), their sum being

$$2q^2\psi\left(q^{18}\right)P\left(q^3\right)^2P\left(q^6\right)+q^2\psi\left(q^9\right)^2P\left(q^6\right)^2+2q^5\psi\left(q^9\right)\psi\left(q^{18}\right)^2P\left(q^3\right). \tag{28}$$

Observing that

$$\psi\left(q^6\right)P\left(q\right)^2P\left(q^2\right)=\psi\left(q^3\right)^2P\left(q^2\right)^2+2q\psi\left(q^3\right)\psi\left(q^6\right)^2P\left(q\right),$$

which follows from (16) and (17), we see that (28) is $3q^2\psi\left(q^{18}\right)P\left(q^3\right)^2P\left(q^6\right)$. Accordingly, from the right side of (27) we extract those terms in which the power of q is 2 (mod 3) to obtain

$$3q^2\psi\left(q^{18}\right)P\left(q^3\right)^2P\left(q^6\right)=\sum_{n\geq 0}\left(\sigma(12n+11)/4\right)q^{3n+2},$$

from which (14) follows.

4. Final Comments

The method of proof used in this paper is dissection of series. To obtain our results we considered products of the form

$$\psi\left(q^a\right)^r\phi\left(q^b\right)^sP\left(q^c\right)^tX\left(q^d\right)^u, \tag{29}$$

in which $r+s+t+u=4$. This led to analogues of the classical results of Jacobi and Legendre.

In [5] Hirschhorn considered certain products (29) in which $t=u=0$. In so doing he reproduced some classical results, and discovered new ones. In [6] he introduced the generating

functions P and X and exploited them similarly to obtain representation results involving triangular, square, pentagonal, and octagonal numbers. In both these works Hirschhorn used dissection of series. Indeed, [5] and [6] were the starting points of our present paper.

Let $r_3(n)$ denote the number of representations of the positive integer n as a sum of three squares. In [3] the authors proved a large number of results in which the generating function of $r_3(an+b)$ is a simple infinite product. They proved eleven results of Hurwitz, and another twelve results of the same type. They also proved eighty infinite families of similar results, and stated as conjectures a further eighty-two infinite families of results. Here the starting point was $r_3(n) = \phi(q)^3$, and the method employed was dissection of series.

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