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ON NEWMAN'S CONJECTURE AND PRIME TREES

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Abstract

In 1980, Carl Pomerance and J. L. Selfridge proved D. J. Newman's coprime mapping conjecture: *If n is a positive integer and I is a set of n consecutive integers, then there is a bijection $f:\{1,2,\ldots,n\}\to I$ such that* $\gcd(i,f(i))=1$ *for* $1\le i\le n$. The function f described in their theorem is called a *coprime mapping*. Around the same time, Roger Entringer conjectured that all trees are *prime*; that is, that if T is a tree with vertex set V, then there is a bijection $L:V\to\{1,2,\ldots,|V|\}$ such that $\gcd(L(x),L(y))=1$ for all adjacent vertices x and y in V. So far, little progress towards a proof of this conjecture has been made. In this paper, we extend Pomerance and Selfridge's theorem by replacing I with a set S of n integers in arithmetic progression and determining when there exist coprime mappings $f:\{1,2,\ldots,n\}\to S$ and $g:\{1,3,\ldots,2n-1\}\to S$. We devote the rest of the paper to using coprime mappings to prove that various families of trees are prime, including palm trees, banana trees, binomial trees, and certain families of spider colonies.

1. Introduction

A bijection $f:A\to B$ on two sets of integers A and B is a *coprime mapping* if $\gcd(a,f(a))=1$ for all $a\in A$. Around 1960, D. J. Newman conjectured that if n is a positive integer and I is a set of n consecutive integers (not necessarily positive), then a coprime mapping $f:\{1,2,\ldots,n\}\to I$ always exists. In 1963, D. E. Daykin and M. J. Baines [3] proved the special case of the conjecture where $I=\{n+1,n+2,\ldots,2n\}$. In 1980, Carl Pomerance and J. L. Selfridge [8] proved the complete Newman conjecture by algorithmically constructing the desired coprime mapping f. Their construction relies on an interesting theorem on the distribution of the values of the Euler phi function, which they devote most of their paper to proving. More recently, a simple proof of Newman's Conjecture was claimed in [10], but the main identity used does not appear to be valid.

In Section2, we extend Pomerance and Selfridge's theorem by considering coprime mappings onto sets of integers in arithmetic progression. Specifically, we prove that if n is a positive integer and $S=\{a+tb\,|\,0\le t\le n-1\}$ is a set of n integers in arithmetic progression with leading term a and common difference b, then a coprime mapping $g:\{1,2,\ldots,n\}\to S$ exists if and only if every common prime divisor of a and b is greater than n. We then replace the set $\{1,2,\ldots,n\}$ with the set of the first n

odd integers, and prove that a coprime mapping $h : \{1, 3, \ldots, 2n-1\} \to S$ exists if and only if every common odd prime divisor of a and b is greater than $2n-1$.

Section 3 is devoted to applying these three coprime mapping theorems once or repeatedly to prove that various families of trees are prime. A *prime labeling* of a tree T with vertex set V is a bijection $L : V \to \{1, 2, \ldots, |V|\}$ that satisfies $\gcd(L(x), L(y)) = 1$ for all adjacent vertices x and y in V. If such a labeling exists, then T is said to be *prime*. Around 1980, Entringer conjectured that all trees are prime, but the conjecture remains open. Among the families of trees known to be prime are paths, stars, caterpillars, perfect binary trees, spiders, and all trees with at most 50 vertices [4, 6, 9]. For an up-to-date summary of known results on prime trees, see the dynamic survey by Gallian [5], which contains a section on prime labelings. We use coprime mappings to construct prime labelings of palm trees, certain spider colonies (which we define), binomial trees, and banana trees.

2. Extending Newman's Conjecture

As mentioned above, the following theorem was conjectured by D. J. Newman and proven by Pomerance and Selfridge [8].

Theorem 1. *If n is a positive integer and I is a set of n consecutive integers, then there is a coprime mapping $f : \{1, 2, \ldots, n\} \to I$.*

In this section, we use Theorem 1 to prove two new coprime mapping theorems that extend Newman's conjecture. Namely, in Theorem 2, we replace the set I of n consecutive integers with a set S of n integers in arithmetic progression and determine when a coprime mapping $g : \{1, 2, \ldots, n\} \to S$ exists. In Theorem 3, we replace the set $\{1, 2, \ldots, n\}$ with the set of the first n positive odd integers and determine when a coprime mapping $h : \{1, 3, \ldots, 2n-1\} \to S$ exists.

Theorem 2. *Let n be a positive integer and $S = \{a + tb \,|\, 0 \le t \le n-1\}$ be a set of n integers in arithmetic progression with leading term a and common difference b. Then there is a coprime mapping $g : \{1, 2, \ldots, n\} \to S$ if and only if every common prime divisor of a and b is greater than n.*

Proof. First suppose there is a prime $p \le n$ that divides both a and b. Then p divides every term of the arithmetic progression in S, and so $\gcd(p, g(p)) \ge p$ for every bijection $g : \{1, 2, \ldots, n\} \to S$. Thus a coprime mapping does not exist.

For the converse, suppose that every common prime divisor of a and b is greater than n. Let q be the product of all the primes less than or equal to n that do not divide b. Then b and q are coprime, so there exist integers x and y such that $bx+qy = a$. Let $I = \{x, x+1, \ldots, x+n-1\}$ be the set of n consecutive integers that begins with x. By Theorem 1, there is a coprime mapping $f : \{1, 2, \ldots, n\} \to I$. To prove

Theorem 2 we will show that the bijection $g : \{1, 2, \ldots, n\} \to S$ defined by

$$g(k) = a + (f(k) - x)\, b$$

is a coprime mapping.

We must show that $\gcd(k, g(k)) = 1$ for all $k \in \{1, 2, \ldots, n\}$. Suppose otherwise. Then there is a prime p and an integer $k \in \{1, 2, \ldots, n\}$, such that p divides both k and $g(k)$. Let $f(k) = x + i$. Then $g(k) = a + ib$. Now $p \leq n$, so by the definition of q, p divides either b or q, but not both. If p divides b, then p divides $a = g(k) - ib$, and so p is a common divisor of a and b, which contradicts our assumption. Thus p divides q and p does not divide b. Observe that

$$g(k) = bx + qy + ib = qy + f(k)b.$$

Since p divides $g(k)$ and q, but does not divide b, it follows that p divides $f(k)$. Thus p is a common divisor of k and $f(k)$, which contradicts the fact that f is a coprime mapping. Thus no such prime p exists, and g is indeed a coprime mapping from $\{1, 2, \ldots, n\}$ onto S. □

Theorem 3. *Let n be a positive integer, $O = \{1 + 2t \,|\, 0 \leq t \leq n - 1\}$ be the set of the first n positive odd integers, and $S = \{a + tb \,|\, 0 \leq t \leq n - 1\}$. Then there is a coprime mapping $h : O \to S$ if and only if every common odd prime divisor of a and b is greater than $2n - 1$.*

Proof. As in the proof of Theorem 2, in order for a coprime mapping from O onto S to exist it is necessary that there are no odd primes less than $2n$ that divide both a and b, since otherwise that prime would be an element of O that divides every element in S. To see that this is sufficient, suppose that every common odd prime divisor of a and b is greater than $2n - 1$. Let q be the product of all the odd primes less than $2n$ that do not divide b. Again, b and q are coprime, so there exist integers x and y such that $bx + qy = a$. Now let $I = \{2x, 2x + 1, \ldots, 2x + 2n - 2\}$ be the set of $2n - 1$ consecutive integers that begins with $2x$. By Theorem 1, there is a coprime mapping $f : \{1, 2, \ldots, 2n - 1\} \to I$. Notice that f must map odd integers to even integers since $\{1, 2, \ldots, 2n - 1\}$ contains exactly n odd integers and $n - 1$ even integers and I contains exactly $n - 1$ odd integers and n even integers. Thus if we restrict f to the odd integers in $\{1, 2, \ldots, 2n - 1\}$ and divide every even integer in I by two, we get a coprime mapping $f^* : O \to \{x, x + 1, \ldots, x + n - 1\}$. Now, as in the proof of Theorem 2, the bijection $g : O \to S$ defined by

$$g(k) = a + (f^*(k) - x)\, b$$

is a coprime mapping. We leave the verification to the reader, since essentially the proof given for Theorem 2 will work, noting that any common prime divisor of k and $g(k)$ must be odd in this case. □

3. Applications to Prime Trees

The rest of this paper is devoted to using the above three coprime mapping theorems to prove that various families of trees are prime. Specifically, we use Theorem 1 to prove that palm trees are prime; Theorem 2 to prove that certain families of spider colonies, including binomial trees, are prime; and Theorem 3 to prove that banana trees are prime.

3.1. Palm Trees

The palm tree $PT_{n,k}$ is the tree obtained from the concatenation of n stars with k vertices each by linking one leaf from each star [4] (see Figure 1). In 1994, Fu and Huang [4] proved that if $n \leq 16$ then the palm tree $PT_{n,k}$ is prime for all k. Their proof uses the fact due to Pillai [7] that every set of $m \leq 16$ consecutive integers contains at least one integer that is coprime to all the others in the set. When $m \geq 17$, however, Brauer [1] proved that there exists a set of m consecutive integers that does not share this property, so Fu and Huang's proof does not generalize to arbitrary palm trees. Instead we use coprime mappings to prove that all palm trees are prime. Note that palm trees are a family of firecrackers [2, 5].

Theorem 4. *Palm trees are prime.*

Proof. Let n and k be positive integers, and S_k be a star with k vertices. Then S_k has a vertex of degree $k-1$, which we take as the root, and all other vertices have degree 1. For each j, $1 \leq j \leq k-1$, it follows from Theorem 1 that there is a coprime mapping

$$f_j : \{1, 2, \ldots, n\} \to \{jn+1, jn+2, \ldots, (j+1)n\}.$$

To prove Theorem 4 we use the maps f_j, $1 \leq j \leq k-1$, to construct a prime labeling of the palm tree $PT_{n,k}$. First label the linked leaves consecutively from $n+1$ to $2n$ so that the vertex labeled $n+i, 1 \leq i < n$, is adjacent to the vertex labeled $n+i+1$. Then label the centers of the stars from 1 to n so that the linked leaf labeled $n+i$, $1 \leq i \leq n$, is adjacent to the vertex labeled $f_1^{-1}(n+i)$. Finally label the remaining leaves from $2n+1$ to nk such that the vertex labeled i, $1 \leq i \leq n$, is adjacent to the leaf labeled $f_j(i)$, for $2 \leq j \leq k-1$. □

A prime labeling of the palm tree $PT_{6,4}$ is given in Figure 1. The labeling follows the construction given in the proof of Theorem 4.

3.2. Spider Colonies and Binomial Trees

In this section we define spider colonies and use Theorem 2 to prove that certain families of spider colonies, including regular spider colonies and binomial trees, are prime.

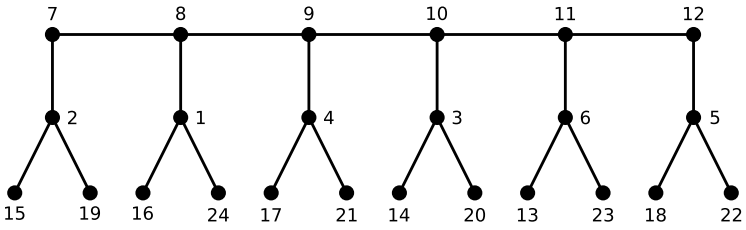

Figure 1: A prime labeling of the palm tree $PT_{6,4}$.

A *spider* is a tree with at most one vertex of degree greater than two. Some authors require that there is a vertex of degree greater than two, but we do not make this requirement so that spiders are a family of trees that includes paths and the tree consisting of a single vertex. If a spider has a vertex of degree greater than two, then we take this vertex to be the root. The paths from a spider's leaves to its root are called *legs*. A *regular spider* is a spider whose legs all have the same length.

In ecology, a group of spiders that lives together and builds its webs in a single tree is called a *spider colony*. Thus we are inclined to make the following definitions. Let T be a tree with vertex set V, $W \subseteq V$ be a set of vertices of T, and S be a spider. Then we call the tree obtained by identifying the root of a copy of the spider S with each vertex in W the *spider colony* obtained by *colonizing* W by the spider S, and we denote it by $\mathrm{Col}(T, W, S)$. In other words, if S has m legs of lengths $\ell_1, \ell_2, \ldots, \ell_m$, then $\mathrm{Col}(T, W, S)$ is the tree obtained by connecting a leaf of a copy of each of the m paths $P_{\ell_1}, P_{\ell_2}, \ldots, P_{\ell_m}$ to each of the vertices in W, where P_ℓ denotes the path on ℓ vertices. If $W = V$ we say that T has been *colonized* by the spider S, and for ease of notation write $\mathrm{Col}(T, S)$ for $\mathrm{Col}(T, V, S)$ (see Figure 2). Also, we define regular spider colonies recursively as follows. The *regular* (k, ℓ)*-spider colony* $SC_1(k, \ell)$ *of order 1* is the regular spider with k legs of length ℓ; the *regular* (k, ℓ)*-spider colony* $SC_n(k, \ell)$ *of order* n is the tree obtained by colonizing $SC_{n-1}(k, \ell)$ by $SC_1(k, \ell)$.

Below we show that binomial trees are a family of regular spider colonies. In the literature, the term "binomial tree" has two separate and unrelated meanings. For this paper, we use the following recursive definition: The *binomial tree* B_0 *of order 0* consists of a single vertex; the *binomial tree* B_n *of order* n has a root vertex whose children are the roots of the binomial trees of order $0, 1, 2, \ldots, n-1$ (see Figure 3). The name comes from the fact that B_n has height n and the number of vertices at level i is equal to the binomial coefficient $\binom{n}{i}$, *i.e.*, the coefficient of x in $(1+x)^n$. Similarly, the number of vertices at level i of the regular spider colony $SC_n(k, \ell)$ is equal to the coefficient of x in $(1 + kx + kx^2 + \cdots + kx^\ell)^n$.

Lemma 5. *For $n \geq 1$, the binomial tree B_n can be obtained from B_{n-1} by attaching a child to every vertex of B_{n-1}, that is, B_n is equal to the regular spider colony $SC_n(1, 1)$.*

Proof. We use induction on n. The assertion is immediate when $n = 1$, so assume $n \geq 2$ and the assertion holds for all binomial trees of order $m < n$. Attach a child to every vertex of B_{n-1}. Then, by the induction hypothesis and the definition of B_0, the children of the root vertex become the binomial trees of orders $0, 1, 2, \ldots, n-1$, and we get the binomial tree of order n as needed. □

At the end of this section we prove the following theorem.

Theorem 6. *Regular spider colonies are prime, i.e., $SC_n(k, \ell)$ is prime for all positive integers n, k, and ℓ. In particular, all binomial trees are prime.*

Theorem 6 is a consequence of the following lemma and its corollary, which describe larger families of spider colonies that are prime.

Lemma 7. *Let T be a prime tree with vertex set V, $N = |V|$, and $L : V \to \{1, 2, \ldots, N\}$ be a prime labeling of T. Let $n \leq N$ be a positive integer and $W = \{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(n)\}$ be the set of n vertices of T that are labeled from 1 to n. Let S be a spider with $k \geq 1$ legs of lengths $\ell_1, \ell_2, \ldots, \ell_k$. Suppose*

$$\gcd(\ell_1, N) = 1 \text{ or } \gcd(\ell_1, N + 1) = 1,$$

and, for $2 \leq i \leq k$, that either

$$\gcd(\ell_i, N + n(\ell_1 + \ell_2 + \cdots + \ell_{i-1})) = 1$$

or

$$\gcd(\ell_i, N + 1 + n(\ell_1 + \ell_2 + \cdots + \ell_{i-1})) = 1.$$

Then the spider colony $\mathrm{Col}(T, W, S)$ obtained by colonizing W by the spider S is prime. Moreover, $\mathrm{Col}(T, W, S)$ has a prime labeling that agrees with L on the vertices of T.

Proof. Let S be a spider with $k \geq 1$ legs whose lengths satisfy the conditions given in the statement of the lemma. We prove the lemma by induction on k. First suppose $k = 1$. Label the vertices of T from 1 to N according to the prime labeling L, then colonize W by S to obtain the tree $\mathrm{Col}(T, W, S)$. In this case, $\mathrm{Col}(T, W, S)$ is the tree obtained by attaching a leaf of a copy of the path P_{ℓ_1} on ℓ_1 vertices to each of the vertices of T that are labeled from 1 to n. We use Theorem 2 to construct a coprime labeling of the $n\ell_1$ vertices just added.

First suppose $\gcd(\ell_1, N + 1) = 1$. Let $A = \{N + 1 + \ell_1 t \,|\, 0 \leq t \leq n - 1\}$ be the set of n integers in arithmetic progression with leading term $N + 1$ and common difference ℓ_1. Then, by Theorem 2, there is a coprime mapping $g : \{1, 2, \ldots, n\} \to A$. Label the $n\ell_1$ vertices of the n copies of P_{ℓ_1} consecutively along each path (*i.e.,* so that adjacent vertices have consecutive labels) from $N + 1$ to $N + n\ell_1$ such that for $i = 1, 2, \ldots, n$, the vertex labeled $g(i)$ is adjacent to the vertex labeled i. This results in a prime labeling of $\mathrm{Col}(T, W, S)$.

Now suppose $\gcd(\ell_1, N) = 1$. Let $B = \{N + \ell_1 + \ell_1 t \,|\, 0 \le t \le n-1\}$ be the set of n integers in arithmetic progression with leading term $N + \ell_1$ and common difference ℓ_1. Then, by Theorem 2, there is a coprime mapping $h : \{1, 2, \ldots, n\} \to B$. Again, label the $n\ell_1$ added vertices consecutively along each path, but this time in decreasing order from $N + n\ell_1$ to $N + 1$ such that for $i = 1, 2, \ldots, n$, the vertex labeled i is adjacent to the vertex labeled $h(i)$. This results in a prime labeling of $\mathrm{Col}(T, W, S)$, and the assertion holds when $k = 1$.

For the inductive step, suppose $k \ge 2$ and that the assertion holds if W is colonized by a spider with $m < k$ legs whose lengths satisfy the conditions given in the statement of the lemma. Note that $\mathrm{Col}(T, W, S)$ has a subtree $\mathrm{Col}(T, W, S')$ obtained by colonizing W by the spider S' with $k-1$ legs of lengths $\ell_1, \ell_2, \ldots, \ell_{k-1}$. By the induction hypothesis, there is a prime labeling of $\mathrm{Col}(T, W, S')$ that agrees with L on the vertices of T. Label the subtree $\mathrm{Col}(T, W, S')$ of $\mathrm{Col}(T, W, S)$ according to this labeling. Now, $\mathrm{Col}(T, W, S)$ is obtained from $\mathrm{Col}(T, W, S')$ by attaching a copy of the path P_{ℓ_k} on ℓ_k vertices to the vertices of $\mathrm{Col}(T, W, S')$ labeled from 1 to n. Notice that $\mathrm{Col}(T, W, S')$ has $N' = N + n(\ell_1 + \ell_2 + \cdots + \ell_{k-1})$ vertices. By hypothesis, either $\gcd(\ell_k, N') = 1$ or $\gcd(\ell_k, N' + 1) = 1$, so the case $k = 1$ of the lemma applies. Thus $\mathrm{Col}(T, W, S)$ has a prime labeling that agrees with L on the vertices of T. □

Figure 2 provides an example of a spider colony with a prime labeling constructed as in the proof of Lemma 7. The tree in the figure is the spider colony $\mathrm{Col}(T, S)$, where T is the tree that consists of the larger vertices labeled from 1 to 8, and S is the spider with two legs of length two and one leg of length one.

Corollary 8. *Let T be a prime tree with N vertices and S be a regular spider whose legs have length ℓ. If $\gcd(\ell, N) = 1$ or $\gcd(\ell, N + 1) = 1$ then the tree $\mathrm{Col}(T, S)$ obtained by colonizing T by the spider S is prime. In particular, $\mathrm{Col}(T, S)$ is prime if $\ell = 1$ or ℓ is a power of a prime.*

Proof. The assertion follows from Lemma 7 with $n = N$ and $\ell_i = \ell$, $1 \le i \le k$, since for any integer t, $\gcd(\ell, N) = 1$ implies that $\gcd(\ell, N(1 + t\ell)) = 1$, and $\gcd(\ell, N + 1) = 1$ implies $\gcd(\ell, N(1 + t\ell) + 1) = 1$. □

We are now ready to prove Theorem 6.

Proof of Theorem 6. We use induction on n. The assertion holds when $n = 1$ since all spiders are prime. So assume $n \ge 2$ and that the assertion holds for $SC_{n-1}(k, \ell)$. The number of vertices of $SC_{n-1}(k, \ell)$ is $(k\ell + 1)^{n-1}$, which is coprime to ℓ. Thus, by Corollary 8, the tree obtained by colonizing $SC_{n-1}(k, \ell)$ by the spider $SC_1(k, \ell)$, *i.e.*, the regular spider colony $SC_n(k, \ell)$, is prime. □

A prime labeling of the binomial tree B_4 is given in Figure 3. The labeling follows the construction given above. Namely, it arises by labeling the single vertex on the

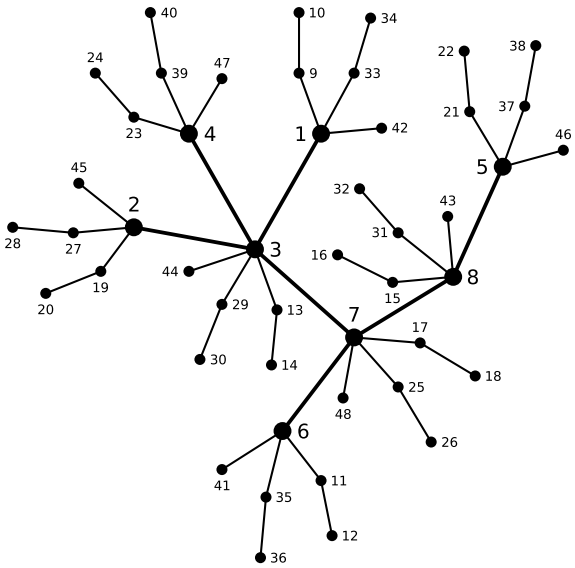

Figure 2: A prime labeling of a spider colony.

binomial tree B_0 with 1, and, for i from 1 to 4, constructing B_i by colonizing B_{i-1} by the spider consisting of a path of length one and labeling B_i following the construction given in the proof of Lemma 7.

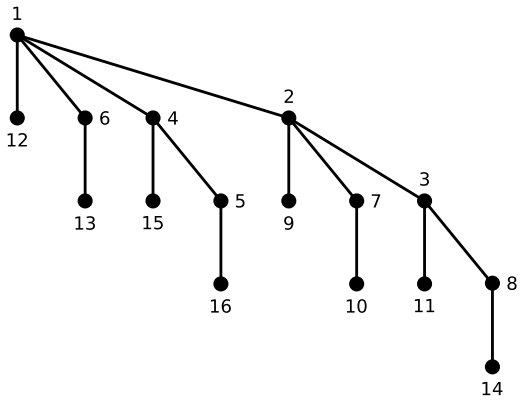

Figure 3: A prime labeling of the binomial tree B_4.

3.3. Banana Trees

A banana tree is a tree obtained by joining one leaf of each of any number of stars to a new root vertex [2, 5] (see Figure 4). If all the stars have the same number

of vertices, then the banana tree is said to be *regular*. In this section we prove the following theorem.

Theorem 9. *Banana trees are prime.*

Our proof of Theorem 9 relies of the following extension of Lemma 7.

Lemma 10. *Let T be a prime tree with vertex set V, $N = |V|$, and $L : V \to \{1, 2, \ldots, N\}$ be a prime labeling of T. Let n be a positive integer such that $2n-1 \leq N$, and $W = \{L^{-1}(1), L^{-1}(3), \ldots, L^{-1}(2n-1)\}$ be the set of n vertices of T that are labeled with the n odd integers from 1 to $2n-1$. Let S be a spider with $k \geq 1$ legs of lengths $\ell_1, \ell_2, \ldots, \ell_k$. Suppose*

$$\gcd(\ell_1, N) = 1 \text{ or } \gcd(\ell_1, N+1) = 1,$$

and, for $2 \leq i \leq k$, that either

$$\gcd(\ell_i, N + n(\ell_1 + \ell_2 + \cdots + \ell_{i-1})) = 1$$

or

$$\gcd(\ell_i, N + 1 + n(\ell_1 + \ell_2 + \cdots + \ell_{i-1})) = 1.$$

Then the spider colony $\mathrm{Col}(T, W, S)$ obtained by colonizing W by the spider S is prime. Moreover, $\mathrm{Col}(T, W, S)$ has a prime labeling that agrees with L on the vertices of T.

Proof. We use the induction proof given for Lemma 7 with the following three modifications: (1) attach the paths to the vertices of T labeled with the odd integers from 1 to $2n-1$ instead to those labeled with the integers from 1 to n; (2) replace the set $\{1, 2, \ldots, n\}$ with the set $\{1, 3, \ldots, 2n-1\}$; and (3) use Theorem 3 throughout instead of Theorem 2. The details are left to the reader. □

Proof of Theorem 9. Let BT be a banana tree. Then there are disjoint stars $S_{k_1}, S_{k_2}, \ldots, S_{k_n}$ on $k_1, k_2, \ldots, k_n$ vertices, respectively, such that BT is obtained by joining one leaf of each of $S_{k_1}, S_{k_2}, \ldots, S_{k_n}$ to a new root vertex r. Without loss of generality assume $k_1 \geq k_2 \geq \cdots \geq k_n \geq 1$. For $i = 1, 2, \ldots, n$, let c_i denote the vertex of BT that is the center of the star S_{k_i}.

If $k_1 \leq 3$, then BT is a spider, which is known to be prime. If $k_1 \geq 4$ but $k_2 \leq 3$, then label the root r of BT with 1 and the vertices on S_{k_1} from 2 to $k_1 + 1$ such that the center c_1 is labeled with the largest prime not exceeding $k_1 + 1$. In this case, the stars $S_{k_1}, S_{k_2}, \ldots, S_{k_n}$ are all paths, so we label the remaining vertices of BT from $k_1 + 2$ to $k_1 + k_2 + \cdots + k_n + 1$ consecutively along each path. One easily checks that this is a prime labeling of BT, so BT is prime in this case as well. Thus to prove the theorem it remains to construct a prime labeling of BT in the case where $k_2 \geq 4$.

Assume $k_2 \geq 4$. Let $t \leq n$ be the largest integer such that $k_t \geq 4$. Then the regular banana tree $BT_{t,4}$, obtained by joining one leaf of t copies of the star with 4 vertices to a new root vertex, is a subtree of BT, with the centers of the joined stars of $BT_{t,4}$ being the vertices $c_1, c_2, \ldots, c_t$ of BT. To describe a prime labeling of BT we first label the vertices of its subtree $BT_{t,4}$ from 1 to $4t+1$ as follows. Label the root vertex r with 2 and, for $1 \leq i \leq t$, label c_i with the odd integer $2i-1$. Label the vertex of the star S_{k_t} that is on the first level of $BT_{t,4}$ with $2t+1$, and one of the vertices of S_{k_t} on the third level with $2t+3$. Now, by Theorem 3, there is a coprime mapping

$$g_1 : \{1, 3, \ldots, 2t-3\} \to \{2t+5, 2t+7, \ldots, 4t+1\}.$$

Label the remaining vertices on the first level of $BT_{t,4}$ with the odd integers from $2t+5$ to $4t+1$ such that for $i = 1, 3, \ldots, 2t-3$, the vertex on the second level labeled i is adjacent to the vertex on the first level labeled $g_1(i)$ (*i.e.*, the vertex on the first level that is a leaf of S_{k_i} is labeled $g_1(i)$). Similarly, by Theorem 3, there are coprime mappings

$$g_2 : \{1, 3, \ldots, 2t-3\} \to \{4, 6, \ldots, 2t\}$$

and

$$g_3 : \{1, 3, \ldots, 2t-1\} \to \{2t+2, 2t+4, \ldots, 4t\}.$$

Thus we can label the remaining vertices on the third level of $BT_{t,4}$ with the even integers $4, 6, \ldots, 4t$ such that for $i = 1, 3, \ldots, 2t-3$, the two vertices on the third level that are adjacent to the vertex on the second level labeled i, are labeled $g_2(i)$ and $g_3(i)$, and the remaining vertex on the third level is labeled $g_3(2t-1)$. This completes a prime labeling of $BT_{t,4}$.

Now let BT' be the banana tree obtained by attaching one leaf of each of the t stars $S_{k_1}, S_{k_2}, \ldots, S_{k_t}$ to a root vertex. Then, $BT_{t,4}$ is a subtree of BT', which in turn is a subtree of BT. We now construct a prime labeling of BT', and so complete a prime labeling of the vertices on BT that lie on $S_{k_1}, S_{k_2}, \ldots, S_{k_t}$. First, label $BT_{t,4}$ as described above. Then, for $1 \leq i \leq t$, the center vertex c_i has been labeled with the odd integer $2i-1$. Colonize the set of center vertices $\{c_1, c_2, \ldots, c_t\}$ of $BT_{t,4}$ by the regular spider with $k_t - 4$ legs of length 1. By Lemma 10, the resulting tree is prime and the added vertices can be labeled accordingly, finishing a prime labeling of S_{k_t}. Next colonize $\{c_1, c_2, \ldots, c_{t-1}\}$ on this new tree by the regular spider with $k_{t-1} - k_t$ legs of length 1. Again, by Lemma 10, the resulting tree is prime and the added vertices can be labeled accordingly, finishing a prime labeling of $S_{k_{t-1}}$. Continue in this fashion until the set $\{c_1\}$ has been colonized by the regular spider with $k_1 - k_2$ legs of length 1. This results in a prime labeling from 1 to $k_1 + k_2 + \cdots + k_t + 1$ of the subtree BT' of BT.

If $t = n$ we have found a prime labeling of BT as needed. Otherwise, it remains to label the vertices of BT on the stars $S_{k_{t+1}}, S_{k_{t+2}}, \ldots, S_{k_n}$. In this case,

let s be the number of stars among $S_{k_{t+1}}, S_{k_{t+2}}, \ldots, S_{k_n}$ that consist of exactly one vertex (the others consist of exactly two or three vertices). If $s > 0$, then the s stars $S_{k_{n-s+1}}, \ldots, S_{k_n}$ consist only of the s center vertices $c_{n-s+1}, \ldots, c_n$. To finish the coprime labeling of BT, we first relabel the root vertex of BT and, if $s > 0$, label the s center vertices $c_{n-s+1}, \ldots, c_n$. Let $k = k_1 + k_2 + \cdots + k_t + 1$ be the number of previously labeled vertices of BT. Let p be the largest prime not exceeding $k + s$. Relabel the root r of BT with the prime p. Recall that the root vertex was previously labeled with 2, so we need to reassign the label 2 and, if $p \leq k$, relabel the vertex that was previously labeled with p. If $p \leq k$ then, since $k \geq 4t + 1$, it follows from Bertrand's Postulate that $p \geq 2t + 1$, so there is a vertex on the first or third level of BT that was previously labeled p. All the vertices adjacent to this vertex have odd labels, so relabel it with 2. If $s > 0$, also label the vertices $c_{n-s+1}, \ldots, c_n$ from $k + 1$ to $k + s$. This is a coprime labeling since the vertices $c_{n-s+1}, \ldots, c_n$ are only adjacent to the root r, which was assigned the label p, and the integers from $k + 1$ to $k + s$ are coprime to p by Bertrand's Postulate and our choice of p. If $p \geq k + 1$ then the label p was not previously assigned. In this case, $s \geq 1$ and we label one of the s vertices $c_{n-s+1}, \ldots, c_n$ with 2 and the others with the $s - 1$ unassigned integers from $k + 1$ to $k + s$, since again these labels are all coprime to p.

It remains to label the vertices on the stars $S_{k_{t+1}}, S_{k_{t+2}}, \ldots, S_{k_{n-s}}$ from $k+s+1$ to $k + s + k_{t+1} + k_{t+2} + \cdots + k_{n-s}$. Each of these stars is a path on two or three vertices, a leaf of which is connected to the root r that is labeled with the prime $p \geq 3$. The stars can thus be labeled consecutively along each path, in decreasing order where necessary, since at least one of the endpoints of every interval of two or three consecutive integers is coprime to p. This completes a prime labeling of BT. □

A prime labeling of the banana tree obtained by joining seven stars with 7, 5, 4, 3, 3, 1, 1 vertices, respectively, is given in Figure 4. The labeling follows the construction given in the proof of Theorem 9.

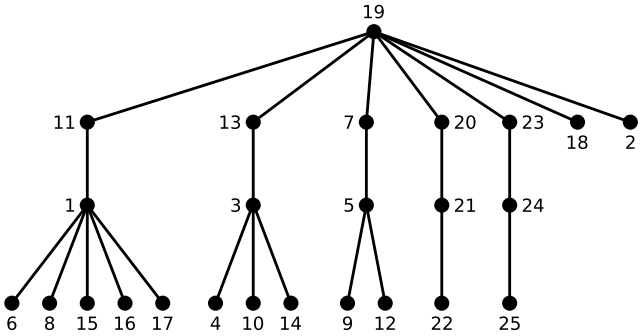

Figure 4: A prime labeling of a banana tree.

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