Journal of Applied Analysis
Vol. 2, No. 1 (1996), pp. 79–104

# INITIAL – BOUNDARY VALUE PROBLEM FOR EQUATIONS OF GENERALIZED NEWTONIAN INCOMPRESSIBLE FLUID

E. ZADRZYŃSKA

*Abstract.* In the paper the existence theorem for a initial – boundary problem with the Neumann type boundary condition to equations of the motion of a generalized newtonian incompressible fluid is proved. Moreover, for a special case of a non-newtonian fluid uniqueness of a solution is proved, as well.

**1. Introduction.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with boundary $S$ of class $C^1$. Denote $\Omega^T \equiv \Omega \times (0,T)$ and $S^T \equiv S \times (0,T)$, where $T > 0$.

In this paper we consider the motion of a non-newtonian incompressible fluid in $\Omega^T$ which is described by the equations

$$v_t + (v \cdot \bigtriangledown)v - \operatorname{div} \mathbb{T}(v,p) = f \quad \text{in } \Omega^T, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega^T, \tag{1.2}$$

with the boundary conditions

$$v \cdot \overline{n} = 0 \quad \text{on } S^T, \tag{1.3}$$

$$\overline{\tau}_\alpha \mathbb{T}(v,p)\overline{n} = g_\alpha \ (\alpha = 1,2) \quad \text{on } S^T \tag{1.4}$$

1991 *Mathematics Subject Classification.* Primary 35Q35, 76D99.
*Key words and phrases.* Non–newtonian incompressible fluid, initial–boundary value problem, local and global existence.

and with the initial condition

$$v\big|_{t=0} = v_0 \quad \text{in } \Omega \tag{1.5}$$

where $\mathbb{T} = \mathbb{T}(v,p)$ is the stress tensor, $v = v(x,t)$ and $p = p(x,t)$ are the unknown velocity and the unknown pressure, respectively. Moreover, $f = f(x,t)$ is the external force field, $g_\alpha = g_\alpha(x,t)$ $(\alpha = 1,2)$ are the tangent components of the surface forces, $\overline{n}$ and $\overline{\tau}_\alpha$ $(\alpha = 1,2)$ are unit orthonormal vectors such that $\overline{n}$ is the outward normal vector and $\overline{\tau}_1, \overline{\tau}_2$ are tangent to $S$.

We assume that the stress tensor is given by

$$\mathbb{T} = \mathbb{T}(v,p) = -pI + \phi_1 \mathbb{D}, \tag{1.6}$$

where $I$ is the unit tensor, $\phi_1 = \phi_1(\overline{II}_D)$, $\overline{II}_D = \operatorname{tr}\mathbb{D}^2$ is the invariant of $\mathbb{D}, \mathbb{D} = \{D_{ij}(v)\} = \{\frac{1}{2}(v_{i,x_j} + v_{j,x_i})\}$ is the velocity deformation tensor. Fluids described by constitutive equation (1.6) are called generalized newtonian incompressible fluids and they are special cases of Reiner–Rivlin non-newtonian incompressible fluids (see[12]), constitutive equations of which are given by

$$\mathbb{T} = -pI + \phi_1 \mathbb{D} + \phi_2 \mathbb{D}^2,$$

where $\phi_1 = \phi_1(\overline{II}_D, \overline{III}_D)$, $\phi_2 = \phi_2(\overline{II}_D, \overline{III}_D)$, $\overline{III}_D = \operatorname{tr}\mathbb{D}^3$. In the case when $\phi_1(\overline{II}_D) = 2\mu$, where $\mu > 0$ is a constant, relation (1.6) is the constitutive equation for a newtonian fluid with the coefficient of viscosity $\mu$ and in this case (1.1) are Navier–Stokes equations.

The aim of the paper is to prove the existence of a solution of problem (1.1) – (1.5). System (1.1) – (1.2) was considered in some papers before, but all of them were devoted to the initial – boundary value problem for (1.1) – (1.2) with the Dirichlet boundary condition. The first papers concerned with system (1.1) – (1.2) were papers of O. Ladyzhenskaja [6]–[8]. The author considered in [7] system (1.1) – (1.2) with the stress tensor $\mathbb{T} = -pI + \widetilde{\mathbb{T}} = \{-p\delta_{ik} + \widetilde{T}_{ik}\}$ satisfying the conditions:

1. $|\widetilde{T}_{ik}(D_{jl})| \le c(1 + |\mathbb{D}|^{2\mu})|\mathbb{D}|$,
   where $\mathbb{D} = \mathbb{D}(v)$, $|\mathbb{D}| = (\sum_{i,j=1}^3 D_{ij}^2)^{1/2}$, $c > 0$ is a constant;
2. $\widetilde{T}_{ik}(D_{jl})D_{ik} \ge \nu\mathbb{D}^2(1 + \varepsilon|\mathbb{D}|^{2\mu})$,
   where $\nu$ and $\varepsilon$ are positive constants;
3. $\int_\Omega [\widetilde{T}_{ik}(D'_{jl}) - \widetilde{T}_{ik}(D''_{jl})](D'_{ik} - D''_{ik})dx \ge \nu_0 \int_\Omega \sum_{i,k=1}^3 (D'_{ik} - D''_{ik})^2 dx$
   for all vectors $v', v'' \in W_2^1(\Omega) \cap W_{2+2\mu}^1(\Omega)$ with $\operatorname{div} v' = \operatorname{div} v'' = 0$ and $v'|_S = v''|_S$, where $\nu_0 > 0$ is a constant, $D'_{ik} = D'_{ik}(v')$ , $D''_{ik} = D''_{ik}(v'')$.

Under assumptions 1–3 O. Ladyzhenskaja proved in [7] existence of a unique weak solution to system (1.1) – (1.2) with the initial condition (1.5) and with the boundary condition $v|_{S^T} = 0$ in space $\overset{\circ}{J}{}^{1,0}_{2+2\mu,2}(\Omega^T)$, which is the completion of $\{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$ in the norm

$$|v|_{\overset{\circ}{J}{}^{1,0}_{2+2\mu,2}(\Omega^T)} = \sup_{0\le t\le T} ||v||_{L^2(\Omega)} + ||v_x||_{L^2(\Omega^T)} + \big|\big||\mathbb{D}|\big|\big|_{L^{2+2\mu}(\Omega^T)}.$$

In [8] the following systems are considered:

$$\left\{\begin{array}{rcl} v_t - \frac{\partial}{\partial x_k}\Big[A(v_x)v_{x_k}\Big] + v_k v_{x_k} & = & -\bigtriangledown p + f(x,t), \\ \operatorname{div} v & = & 0; \end{array}\right. \tag{1.7}$$

$$\left\{\begin{array}{rcl} v_t + \operatorname{rot}\big[(\nu_0 + \nu_1|\operatorname{rot} v|^{2\mu})\big]\operatorname{rot} v + v_k v_{x_k} & = & -\bigtriangledown p + f(x,t), \\ \operatorname{div} v & = & 0; \end{array}\right. \tag{1.8}$$

or

$$\left\{\begin{array}{rcl} v_t - \nu(v_x)\triangle v + v_k v_{x_k} & = & -\bigtriangledown p + f(x,t), \\ \operatorname{div} v & = & 0; \end{array}\right. \tag{1.9}$$

where $A(v_x) = \nu_0 + \nu_1|v_x|^{2\mu}$; $\nu_0$ and $\nu_1$ are positive constants;

$$\nu(v_x) = \nu_0 + \nu_1\int_\Omega v_x^2(x,t)dx \quad \text{or} \quad \nu(v_x) = \nu_0 + \nu_1\int_\Omega \operatorname{rot}^2 v(x,t)dx;$$

$$v = (v_1, v_2, v_3), \;\; v^2 = |v|^2 = \sum_{i=1}^{3} v_i^3, \;\; v_x^2 = |v_x|^2 = \sum_{i,j=1}^{3} v_{ix_j}^2;$$

moreover, in (1.7) – (1.9) the summation over repeated indices is assumed.

O. Ladyzhenskaja proved in [8] the existence of weak solutions of initial – boundary problems to the systems both (1.7) and (1.8) with the initial condition (1.5) and with the Dirichlet boundary condition. She obtained the existence of solutions in the space $\overset{\circ}{J}{}^{1,1}_{2+2\mu,2}(\Omega^T)$ (where $\mu \ge 1/5$) which is the completion of $\{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$ in the norm:

$$|v|_{\overset{\circ}{J}{}^{1,1}_{2+2\mu,2}(\Omega^T)} = \big|\big|v_x\big|\big|_{L^{2+2\mu}(\Omega^T)} + \big|\big|v_t\big|\big|_{L^2(\Omega^T)}.$$

Moreover, she proved uniqueness of solutions if $\mu \ge \frac{1}{4}$. Next, O. Ladyzhenskaja proved in [8] existence of a unique solution of the initial – boundary value problem with $v|_{S^T} = 0$ to system (1.9) in the space

$$\mathfrak{M} = \Big\{ v : \operatorname{ess\,sup}_{0\le t\le T} ||v_t(x,t)||_{L^2(\Omega)} + ||v_{tx}||_{L^2(\Omega^T)} < \infty \Big\}$$

(see also [6]) and she formulated the classical existence theorem for the above problem.

The initial – boundary value problem for system (1.7) is also examined in [4], where the authors prove existence and uniqueness of a global in time weak solution of (1.7) with condition (1.7) and $v|_{S^T} = 0$ if $\mu \geq \frac{1}{10}$.

Moreover, to the system (1.1) – (1.2) are devoted the papers [2] and [10] and the book [9].

In [9] different types of initial – boundary value problems for system (1.1) – (1.2) are considered. The author obtain existence of weak solutions assuming that the function $\phi_1$ occuring in (1.6) satisfies:

$$\forall \xi \in [0,\infty) \quad a_1 \leq \phi_1 \leq a_2,$$

where $a_1, a_2 > 0$ are constants. Moreover, some additional conditions are imposed on $\phi_1$.

The next paper concerned with system (1.1) – (1.2) is [10], where the authors prove the existence theorem for the initial – boundary value problem to system (1.1) – (1.2) in the case when $\Omega = (0,L)^n$ $(n = 2$ or $3,\ L > 0)$ and with the boundary conditions:

$$\begin{aligned} u|_{\Gamma_j} &= u|_{\Gamma_{j+n}} \quad p|_{\Gamma_j} = p|_{\Gamma_{j+n}}; \\ \frac{\partial u}{\partial x_k}\Big|_{\Gamma_j} &= \frac{\partial u}{\partial x_k}\Big|_{\Gamma_{j+n}} \quad \forall_{j,k=1,2,\ldots,n}, \end{aligned}$$

where $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$, $\Gamma_{j+n} = \partial\Omega \cap \{x_j = L\}$.

Finally, H. Amann considers in [2] problem (1.1) – (1.2), (1.5) with $v|_{S^T} = 0$. Using the theory of semigroups he proves existence and uniqueness of a solution $v \in C(\mathbb{R}^1_+; W^2_q(\Omega)) \cap C^1(\mathbb{R}^1_+; L^q(\Omega))$, $p \in C(\mathbb{R}^1_+; W^1_q(\Omega))$ (where $q \in (3,\infty)$) without assuming any coerciveness conditions.

In this paper we prove existence and uniqueness of a weak solution of problem (1.1) – (1.5) in $\Omega^{T^*}$, where $T^*$ is depending on $f$, $g$ and $v_0$ (see Theorem 3.1). In the case $g_\alpha = 0$ $(\alpha = 1,2)$ we obtain a global in time solution (see Theorem 3.2). In the proofs of Theorems 3.1 and 3.2 we use the methods of H.W. Alt and S. Luckhaus (see [1]) and J. Filo and J. Kačur (see [5]). We also use the Korn type inequality from [3] which is presented in this paper in Lemma 2.4.

In Section 4 we consider the case when $\mathbb{T} = \mathbb{T}(v,p) = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D}$. In this case by using the methods of [8] we prove existence of a solution $v$ of (1.1) – (1.5) such that

$$||v_t||^2_{L^2(\Omega^{T^*})} + \operatorname*{ess\,sup}_{0\leq t\leq T^*} ||v(t)||^{2q}_{W^1_{2q}(\Omega)} < \infty.$$

Finally, in Section 5 we prove existence and uniqueness of a solution $v \in L^\infty(0,T; W^1_{2q}(\Omega))$ with $v_t \in L^2(\Omega^T)$ of problem (1.1) – (1.5) in the case when

$$\mathbb{T} = \mathbb{T}(v,p) = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D} + c_2\mathbb{D}$$

(where $c_1$ and $c_2$ are positive constants).

**2. Notation and auxiliary lemmas.** In this paper we denote by $W^1_s(\Omega)$ $(1 \le s \le \infty)$ the usual Sobolev space of functions $v = v(x)$ such that

$$||v||^s_{L^s(\Omega)} + \sum_{i,j=1}^n \Big|\Big|\frac{\partial v_i}{\partial x_j}\Big|\Big|^s_{L^s(\Omega)} < \infty,$$

where $v = (v_1, \dots, v_n)$, $||v||^s_{L^s(\Omega)} = \sum_{i=1}^n \int_\Omega |v_i|^s dx$, $\frac{\partial v_i}{\partial x_j}$ $(j = 1, \dots, n)$ are distributional derivatives of $v_i$ $(i = 1, \dots, n)$. Let $X$ be a Banach space. By $C(0,T;X)$ we denote the space of defined and continuous on $[0,T]$ functions $v$ with values in $X$ equipped with the norm

$$\sup_{0\le t\le T} ||v(t)||_X.$$

By $L^2(0,T;\ X)$ and $L^\infty(0,T;\ X)$ we denote the spaces of functions $v$ such that

$$\int_0^T ||v(t)||^s_X dt < \infty \quad \text{and} \quad \operatorname{ess}\sup_{0\le t\le T} ||v(t)||_X < \infty,$$

respectively. Next, by $|\cdot|_{s,\Omega}$ we denote the norm in $L^s(\Omega)$. By $D_{ij}(v)$ we denote the components of the deformation tensor $\mathbb{D} = \mathbb{D}(v)$, i.e.

$$D_{ij}(v) = \frac{1}{2}\Big(v_{ix_j} + v_{jx_i}\Big), \quad \text{where} \quad v_{ix_j} = \frac{\partial v_i}{\partial x_j} \quad \text{and} \quad |\mathbb{D}| = \Big(\sum_{i,j=1}^3 D^2_{ij}\Big)^{1/2}.$$

In the paper the summation convention over the repeated indices is assumed. Moreover, by $C$ we denote different positive constants occuring in the paper.

In the next section we use the following lemmas.

**Lemma 2.1.** *(see [5], Proposition 1)*
*Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$ be a bounded $C^1$-domain. Assume that*

$$0 < m \le p < \frac{r(n+m+1)+m+1}{n}, \quad r > 0 \tag{2.1}$$

*and let $v$ be any function in $W^1_{r+1}(\Omega) \cap L^m(\Omega)$. Then*

$$\int_\Omega |v|^{p+1} dx \le \eta ||v||^{r+1}_{W^1_{r+1}(\Omega)} + C\eta^{-\sigma}\Big(\int_\Omega |v|^{m+1}dx\Big)^{1+\gamma}, \tag{2.2}$$

*for any $0 < \eta < \infty$, where*

$$\gamma = \frac{(r+1)(p-m)}{r(n+m+1)+m+1-np} \quad \text{and} \quad \sigma = \frac{n(p-m)}{r(n+m+1)+m+1-np}.$$

*The positive constant $C$ depends on $\Omega$, $n$, $p$, $r$, $m$ and does not depend on $v$ and $\eta$.*

**Lemma 2.2.** *Assume that $v_\mu \to v$ weakly in $L^{r+1}(0,T;W^1_{r+1}(\Omega))$, where $0 < r < \infty$.*

*Moreover, assume that the following estimates hold*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega (v_\mu(t+h)-v_\mu(t))^2\,dx\,dt \le C$$

*and*

$$\int_\Omega v_\mu^2(t)dx \le C \quad \text{for} \quad 0<t<T,$$

*where $C > 0$ is a constant independent of $v_\mu$. Then $v_\mu \to v$ in $L^1(\Omega^T)$ for a subsequence.*

*Proof.* The proof is similar to the proof of Lemma 1.9 of [1]. □

**Lemma 2.3.** *(see [5], Lemma 2)*
*Let $0 < m, r < \infty$. Suppose that*

$$\{v_\mu\}_{\mu=1}^\infty \subset L^{r+1}(0,T;W^1_{r+1}(\Omega)) \cap L^\infty(0,T;L^{m+1}(\Omega))$$

*and*

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_\Omega |v_\mu(t)|^{m+1}dx + \int_0^T ||v_\mu||^{r+1}_{W^1_{r+1}(\Omega)}dt \le C.$$

*Moreover, let $v_\mu \to v$ almost everywhere on $\Omega^T$. Then*

$$v_\mu \to v \quad \text{strongly in} \quad L^{q+1}(\Omega^T)$$

*provided*

$$0 \le q \le \max\{m, (r(m+4)+m+1)/3\}.$$

The above lemma is proved in [5].

**Lemma 2.4.** *Let $v = (v_1,\, v_2,\, v_3)$ and let $v \in W^1_{2q}(\Omega)$, $q \geq 1$. Then*

$$\int_\Omega \sum_{k,l=1}^{3} [D^{2q}_{kl}(v)]^{2q} dx + |v|^{2q}_{2q,\Omega} \geq \widetilde{c}||v||^{2q}_{W^1_{2q}(\Omega)}, \tag{2.3}$$

*where $|v|_{2q,\Omega} = ||v||_{L^{2q}(\Omega)}$, $D_{kl}(v) = \frac{1}{2}(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k})$, $\widetilde{c} > 0$ is a constant.*

*Proof.* Consider the following system of differential operators:

$$P_j(D)v = \sum_{s=1}^{3} \sum_{|\beta|=1} c_{js\beta} D^\beta v_s = \sum_{s=1}^{3} P_{js}(D) v_s \quad (j = 1, \dots, 6),$$

where $\beta$ is a multiindex, i.e. $\beta = (\beta_1, \beta_2, \beta_3)$, $\beta_1 + \beta_2 + \beta_3 = 1$; $c_{11(1,0,0)} = c_{42(0,1,0)} = c_{63(0,0,1)} = 1$, $c_{21(0,1,0)} = c_{22(1,0,0)} = c_{31(0,0,1)} = c_{33(1,0,0)} = c_{52(0,0,1)} = c_{53(0,1,0)} = \frac{1}{2}$ and the other $c_{js\beta}$'s are equal to 0; $P_{11}(D)v_1 = \frac{\partial v_1}{\partial x_1}$, $P_{42}(D)v_2 = \frac{\partial v_2}{\partial x_2}$, $P_{63}(D)v_3 = \frac{\partial v_3}{\partial x_3}$, $P_{21}(D)v_1 = \frac{1}{2}\frac{\partial v_1}{\partial x_2}$, $P_{22}(D)v_2 = \frac{1}{2}\frac{\partial v_2}{\partial x_1}$, $P_{31}(D)v_1 = \frac{1}{2}\frac{\partial v_1}{\partial x_3}$, $P_{33}(D)v_3 = \frac{1}{2}\frac{\partial v_3}{\partial x_1}$, $P_{52}(D)v_2 = \frac{1}{2}\frac{\partial v_2}{\partial x_3}$, $P_{53}(D)v_3 = \frac{1}{2}\frac{\partial v_3}{\partial x_2}$, $P_{12}(D)v_2 = P_{13}(D)v_3 = P_{23}(D)v_3 = P_{32}(D)v_2 = P_{41}(D)v_1 = P_{42}(D)v_3 = P_{51}(D)v_1 = P_{61}(D)v_1 = P_{62}(D)v_2 = 0$.

Next, introduce the matrix $\{P_{js}(\xi)\}$ defined by

$$P_{js}(\xi) = \sum_{|\beta|=1} c_{js\beta}\, \xi_1^{\beta_1}\, \xi_2^{\beta_2}\, \xi_3^{\beta_3},$$

where $\beta_1 + \beta_2 + \beta_3 = |\beta| = 1$, i.e. $\{P_{js}(\xi)\}$ has the form

$$\{P_{js}(\xi)\} = \begin{bmatrix} \xi_1 & 0 & 0 \\ \frac{1}{2}\xi_2 & \frac{1}{2}\xi_1 & 0 \\ \frac{1}{2}\xi_3 & 0 & \frac{1}{2}\xi_1 \\ 0 & \xi_2 & 0 \\ 0 & \frac{1}{2}\xi_3 & \frac{1}{2}\xi_2 \\ 0 & 0 & \xi_3 \end{bmatrix}$$

Since rank $\{P_{js}(\xi)\} = 3$ for any complex $\xi \neq 0$, Theorem 11.3 of [3] yields inequality (2.3). □

**3. Existence of solution of problem (1.1) − (1.5).** Introduce the spaces:

$$\begin{aligned} \mathcal{V} :=\ & \{v \in C^\infty(\overline{\Omega}) :\ \operatorname{div} v = 0,\ v \cdot \overline{n} = 0 \text{ on } S\}, \\ H :=\ & \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega), \\ V :=\ & \text{the closure of } \mathcal{V} \text{ in } W^1_{2q}(\Omega)\ (q \geq 1). \end{aligned}$$

Since

$$(\bigtriangledown p, \zeta) = -(p, \operatorname{div} \zeta) = 0 \quad \forall \zeta \in \mathcal{V},$$

where $p$ is sufficiently regular we introduce the following definition of the weak solution of problem (1.1) – (1.5).

**Definition 3.1.** A vector-valued function $v \in C(0,T;H) \cap L^{2q}(0,T;V)$ is called a weak solution of (1.1) – (1.5) if

$$\frac{d}{dt}\Big(v,\zeta\Big) + a(v,\zeta) + b(v,v,\zeta) = < f,\zeta > + \sum_{\alpha=1}^{2} < g_\alpha \zeta, \tau_\alpha > \quad \forall \zeta \in V, \tag{3.1}$$

$$v(0) = v_0, \qquad \text{where} \qquad (v,\zeta) = \int_\Omega v \cdot \zeta dx,$$

$$a(v,\zeta) = \frac{1}{2}\int_\Omega \phi_1 D_{kl}(v) D_{kl}(\zeta) dx, \tag{3.2}$$

$$b(v,v,\zeta) = \int_\Omega v_l \frac{\partial v_k}{\partial x_l} \zeta_k dx, \qquad < g_\alpha \zeta, \tau_\alpha > = \int_S g_\alpha \zeta \cdot \tau_\alpha ds \tag{3.3}$$

for any $g_\alpha$ for which the right side makes sense, $< f,\zeta >$ denotes the value of a linear functional $f$ at $\zeta$ and $< f,\zeta > = \int_\Omega f \cdot \zeta dx$ for any $f$ for which the right side makes sense.

Now, we formulate the existence theorem for problem (1.1) – (1.5).

**Theorem 3.1.** *Let the following assumptions are satisfied:*

$$\phi_1(\overline{II}_D)\overline{II}_D \geq c_1 \overline{II}_D^q \tag{3.4}$$

*for any symmetric tensor* $\mathbb{D}$*;*

$$\big|\phi_1(\overline{II}_D)\big| \leq c_2 + c_3 \overline{II}_D^{q-1} \tag{3.5}$$

*for any symmetric tensor* $\mathbb{D}$*;*

$$\Big[\phi_1(\overline{II}_D) D_{kl} - \phi_1(\overline{II}_{\tilde{D}}) \widetilde{D}_{kl}\Big]\Big(D_{kl} - \widetilde{D}_{kl}\Big) \geq 0 \tag{3.6}$$

*for all symmetric tensors* $\mathbb{D} = \{D_{kl}\}$ *and* $\widetilde{\mathbb{D}} = \{\widetilde{D}_{kl}\}$*. In (3.4) – (3.6)* $\overline{II}_D = \operatorname{tr} \mathbb{D}^2$, $c_i > 0$ $(i = 1,2,3)$ *are constants; in (3.4) – (3.5)* $\frac{11}{10} \leq q < \infty$*. Moreover, let* $v_0 \in H$, $f \in L^{\frac{2q}{2q-1}}(0,T;V')$, $g_\alpha \in L^{\frac{2q}{2q-1}}(S^T)(\alpha = 1,2)$*. Then there exists* $T^* \in (0,T]$ *depending on* $f, g$ *and* $v_0$ *(satisfying (3.19)) such that problem (1.1) – (1.5) has a weak solution* $v \in C(0,T^*;H) \cap L^{2q}(0,T^*;V)$*.*

*Proof.* We shall apply a Galerkin procedure. Let us choose the sequence of functions $w_1, \dots, w_\mu, \dots$ such that: $\forall_i\ w_i \in V$; $\forall_\mu\ w_1, \dots, w_\mu$ are linearly independent; the set of all linear combinations of functions $w_i$ is dense in $V$. For any $\mu$ we define an approximate solutions of problem (1.1) – (1.5) by

$$v_\mu = \sum_{i=1}^{\mu} c_{i\mu}(t) w_i, \tag{3.7}$$

$$(v_{\mu t}, w_j) + a(v_\mu, w_j) + b(v_\mu, v_\mu, w_j) =< f, w_j > + \\ + \sum_{\alpha=1}^{2} < g_\alpha w_j, \tau_\alpha > \ \forall w_j, \tag{3.8}$$

$$v_\mu(0) = v_{0\mu}, \quad \text{where} \tag{3.9}$$

$$v_{0\mu} \to v_0 \quad \text{in } H \text{ and } |v_{0\mu}|_{2,\Omega} \le |v_0|_{2,\Omega}.$$

Putting (3.7) into (3.8) we get

$$\sum_{i=1}^{\mu} (w_i, w_j) c'_{i\mu}(t) + A_{j\mu}(c_\mu) =< f, w_j > + \sum_{\alpha=1}^{2} < g_\alpha w_j, \tau_\alpha > \\ j = 1, \dots, \mu, \tag{3.10}$$

$$c_{i\mu}(0) = (v_{0\mu}, w_i), \quad i = 1, \dots, \mu, \tag{3.11}$$

where

$$A_{j\mu}(c_\mu) = a\Big(\sum_{i=1}^{\mu} c_{i\mu}(t) w_i, w_j\Big) + \sum_{i,l=1}^{\mu} b(w_i, w_l, w_j) c_{i\mu}(t) c_{l\mu}(t).$$

Since $w_i$ for $i = 1, \dots, \mu$ are linearly independent, the determinant of the matrix $\{(w_i, w_j)\}$ is not equal to zero and therefore (3.10) can be rewritten as

$$c'_{i\mu}(t) + B_{i\mu}(c_\mu) = \sum_{j=1}^{\mu} \alpha_{ij} \Big(< f, w_j > + \sum_{\alpha=1}^{2} < g_\alpha w_j, \tau_\alpha >\Big). \tag{3.12}$$

By general results concerning nonlinear ordinary differential equations there exists a solution of (3.11) – (3.12) in an interval $[0, t_\mu]$. We shall prove that $t_\mu$ is the same for all $\mu$.

Multiplying the both sides of system (3.8) by $c_{j\mu}(t)$ and summing up from 1 to $\mu$ we obtain

$$(v_{\mu t}, v_\mu) + a(v_\mu, v_\mu) + b(v_\mu, v_\mu, v_\mu) =< f, v_\mu > + \sum_{\alpha=1}^{2} < g_\alpha v_\mu, \tau_\alpha > . \tag{3.13}$$

Since

$$b(v_\mu, v_\mu, v_\mu) = \int_\Omega v_{\mu i} \frac{\partial v_{\mu j}}{\partial x_i} v_{\mu j} dx = \frac{1}{2} \int_S v_{\mu i} v_{\mu j}^2 n_i ds - \frac{1}{2} \int_\Omega \sum_{i=1}^3 \frac{\partial v_{\mu i}}{\partial x_i} v_{\mu j}^2 = 0$$

(what follows from the definition of $V$), instead of (3.13) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|v_\mu(t)\right|_{2,\Omega}^2 &+ \frac{1}{2}\int_\Omega \phi_1 D_{kl}(v_\mu(t)) D_{kl}(v_\mu(t)) dx = \\ &= \ < f(t), v_\mu(t) > + \sum_{\alpha=1}^2 \int_S g_\alpha(t) v_\mu(t) \cdot \tau_\alpha ds, \end{aligned}$$

where $\left|v_\mu(t)\right|_{2,\Omega}^2 = \int_\Omega \left|v_\mu(t)\right|^2 dx$, $\phi_1 = \phi_1(\operatorname{tr}\mathbb{D}^2(v_\mu(t))) = \phi_1(D_{kl}(v_\mu(t)) \cdot D_{kl}(v_\mu(t)))$.
Hence, by assumption (3.4) and by the Young inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|v_\mu(t)\right|_{2,\Omega}^2 &+ c_1 \int_\Omega [D_{kl}^2(v_\mu(t))]^q dx \le c(\eta)||f(t)||_{V'}^{\frac{2q}{2q-1}} + \\ &+ \eta||v_\mu(t)||_{W_{2q}^1(\Omega)}^{2q} + c(\eta)\sum_{\alpha=1}^2 ||g_\alpha(t)||_{L^{\frac{2q}{2q-1}}(S)}^{\frac{2q}{2q-1}} \\ &+ \eta||v_\mu(t)||_{L^{2q}(S)}^{2q}. \end{aligned} \tag{3.14}$$

Integrating (3.14) with respect to $t$ in $(0, t)$ $(0 < t \le T)$ and using inequality (2.3) for sufficiently small $\eta$ we have

$$\begin{aligned} \left|v_\mu(t)\right|_{2,\Omega}^2 + c_4 \int_0^t ||v_\mu(t)||_{W_{2q}^1(\Omega)}^{2q} dt \le c(\eta)\int_0^t ||f(t)||_{V'}^{\frac{2q}{2q-1}} dt + \\ + c(\eta)\sum_{\alpha=1}^2 \int_0^t ||g_\alpha(t)||_{L^{\frac{2q}{2q-1}}(S)}^{\frac{2q}{2q-1}} dt + \left|v_0\right|_{2,\Omega}^2 + \int_0^t \left|v_\mu(t)\right|_{2q,\Omega}^{2q} dt. \end{aligned} \tag{3.15}$$

Now, we estimate $|v_\mu(t)|_{2q,\Omega}^{2q}$ using inequality (2.2) with $p = 2q-1$, $r = 2q-1$, $n = 3$, $m = 1$. We get

$$\left|v_\mu(t)\right|_{2q,\Omega}^{2q} \le \eta||v_\mu(t)||_{W_{2q}^1(\Omega)}^{2q} + c(\eta)\left(\int_\Omega \left|v_\mu(t)\right|^2 dx\right)^q. \tag{3.16}$$

Using (3.16) (with sufficiently small $\eta$) in (3.15) yields

$$
\begin{aligned}
|v_\mu(t)|^2_{2,\Omega} \quad &+ \quad c_5\int_0^t ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)}dt \le c(\eta)\int_0^t ||f(t)||^{\frac{2q}{2q-1}}_{V'}dt + \quad (3.17)\\
&+ \quad c(\eta)\sum_{\alpha=1}^{2}\int_0^t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{q-1}}(S)}dt\\
&+ \quad |v_0|^2_{2,\Omega} + c(\eta)\int_0^t\Big(\int_\Omega |v_\mu(t)|^2dx\Big)^q \quad \text{for} \quad t\le T.
\end{aligned}
$$

Denote

$$y(t) = |v_\mu(t)|^2_{2,\Omega}.$$

Then inequality (3.17) gives

$$y(t) \le c + d\int_0^t (y(s))^q ds, \quad 0 < t \le T, \tag{3.18}$$

where

$$c = c(\eta)\int_0^t ||f(t)||^{\frac{2q}{2q-1}}_{V'}dt + c(\eta)\sum_{\alpha=1}^{2}\int_0^t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)}dt + |v_0|^2_{2,\Omega}, \quad d = c(\eta).$$

As in [5] we compare a solution of inequality (3.18) (with the initial condition $y(0) = |v_{0\mu}|^2_{2,\Omega} \le |v_0|^2_{2,\Omega} \le c$) with the solution $z(t)$ of the integral inequality

$$z(t) = c + d\int_0^t (z(s))^q ds, \quad 0 < t \le T,$$

i.e. with the function

$$z(t) = \frac{c}{[1 - c^{q-1}d(q-1)t]^{\frac{1}{q-1}}} \quad \text{for} \quad t \in [0, 1/c^{q-1}d(q-1)).$$

We obtain

$$0 \le y(t) \le z(t) \quad \forall t \in [0, T^*],$$

where

$$0 < T^* = \begin{cases} T & \text{if} \quad T < \frac{1}{c^{q-1}d(q-1)},\\ < \frac{1}{c^{q-1}d(q-1)} & \text{otherwise.}\end{cases} \tag{3.19}$$

Hence, by (3.17) we get

$$\sup_{0\le t\le T^*} |v_\mu(t)|^2_{2,\Omega} + \int_0^{T^*} ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)}dt \le \widetilde{C}, \tag{3.20}$$

where $\widetilde{C} > 0$ is a constant depending on $f, g$ and $v_0$. Inequality (3.18) yields

$$v_\mu \to v \quad \text{weakly in} \quad L^{2q}(0, T^*; V) \tag{3.21}$$

for a subsequence of $(v_\mu)$ still denoted by $(v_\mu)$. Moreover,

$$v_\mu \to v \quad * \text{ - weakly in } L^\infty(0,T^*;H). \tag{3.22}$$

Now we shall prove the estimate

$$\frac{1}{h}\int_0^{T^*-h}\int_\Omega (v_\mu(t+h)-v_\mu(t))^2 dx\,dt \le C. \tag{3.23}$$

To do this notice that equality (3.8) is fulfilled for all $\zeta \in V_\mu = \operatorname{span}\{w_1,\dots,w_\mu\}$. Thus,

$$(v_{\mu t},\zeta)+a(v_\mu,\zeta)+b(v_\mu,v_\mu,\zeta) =< f,\zeta > + \sum_{\alpha=1}^2 < g_\alpha\zeta,\tau_\alpha > \ \forall\zeta\in V_\mu. \tag{3.24}$$

Integrating (3.24) with respect to $t$ in $(t,t+h)$, where $0<t<t+h<T^*$, we get

$$\begin{aligned}(v_\mu(t+h)-v_\mu(t),\zeta)+&\int_t^{t+h} a(v_\mu(t)),\zeta)dt+\int_t^{t+h} b(v_\mu(t),v_\mu(t),\zeta)dt\\ &=\int_t^{t+h} < f,\zeta > dt+\sum_{\alpha=1}^2\int_t^{t+h} < g_\alpha\zeta,\tau_\alpha > dt.\end{aligned} \tag{3.25}$$

To obtain estimate (3.23) we shall use that

$$\lim_{h\to 0}\int_t^{t+h} F(\tau)d\tau = F(t). \tag{3.26}$$

From (3.26) it follows that

$$\forall_{\varepsilon>0}\ \exists_{h>0}\ |h|<\delta \Rightarrow \int_t^{t+h} F(\tau)d\tau \le h(F(t))+\varepsilon). \tag{3.27}$$

Putting in (3.25) $\zeta = v_\mu(t+h)-v_\mu(t)$ and using (3.27) with $F(t) = a(v_\mu(t),v_\mu(t+h)-v_\mu(t))$, $b(v_\mu(t),\ v_\mu(t),\ v_\mu(t+h)-v_\mu(t))$, $< f,v_\mu(t+h)-v_\mu(t) >$, $\sum_{\alpha=1}^2 < g_\alpha(v_\mu(t+h)-v_\mu(t)),\tau_\alpha >$, respectively, we get

$$\begin{aligned}&\int_0^{T^*-h}\int_\Omega (v_\mu(t+h)-v_\mu(t))^2 dx\,dt \le \qquad (3.28)\\ &\le Ch\int_0^{T^*-h}\Big\{\int_\Omega |\phi_1 D_{kl}\ (v_\mu(t))[D_{kl}(v_\mu(t+h))-D_{kl}(v_\mu(t))]|dx\\ &+\int_\Omega\Big|v_{\mu l}(t)\frac{\partial v_{\mu k}}{\partial x_l}(t)(v_{\mu k}(t+h)-v_{\mu k}(t))\Big|dx+|< f(t),v_\mu(t+h)-v_\mu(t) >|\\ &+\sum_{\alpha=1}^2\int_S |g_\alpha(t)(v_\mu(t+h)-v_\mu(t))\cdot\tau_\alpha|ds\Big\}dt+C.\end{aligned}$$

At first estimate

$$J_1 = \int_0^{T^*-h} \Big\{ \int_\Omega |\phi_1 D_{kl}(v_\mu(t))\big[D_{kl}(v_\mu(t+h)) - D_{kl}(v_\mu(t))\big]|dx\,dt.$$

By (3.5) and (3.20) we have

$$\begin{aligned} J_1 &\leq C\Big(\int_0^{T^*-h}\int_\Omega |\phi_1||v_{\mu x}(t)||v_{\mu x}(t+h)|dx\,dt + \\ &+ \int_0^{T^*-h}\int_\Omega |\phi_1||v_{\mu x}(t)|^2 dx\,dt\Big) \leq C\Big(\int_0^{T^*-h}\int_\Omega |v_{\mu t}(t)|^2 dx\,dt \\ &+ \int_0^{T^*-h}\int_\Omega |v_{\mu x}(t+h)|^2 dx\,dt \\ &+ \int_0^{T^*-h}\int_\Omega |v_{\mu x}(t)|^{2q-1}|v_{\mu x}(t+h)|dx\,dt \\ &+ \int_0^{T^*-h}\int_\Omega |v_{\mu x}(t)|^{2q} dx\,dt\Big) \leq C\int_0^{T^*}\int_\Omega |v_{\mu x}(t)|^{2q} dx\,dt \leq \widetilde{C}, \end{aligned} \tag{3.29}$$

where by $C$ in (3.29) we denote different positive constants.

Next, estimate

$$J_2 = \int_0^{T^*-h}\int_\Omega \Big|v_{\mu l}(t)\frac{\partial v_{\mu k}}{\partial x_l}(t)\big(v_{\mu k}(t+h) - v_{\mu k}(t)\big)\Big|dx\,dt.$$

Using the Young inequality we get

$$\begin{aligned} J_2 &\leq C\Big(\int_0^{T^*-h}\int_\Omega |v_{\mu x}(t)|^{2q} dx\,dt + \int_0^{T^*-h}\int_\Omega |v_\mu(t)|^{\frac{4q}{2q-1}} dx\,dt + \\ &+ \int_0^{T^*-h}\int_\Omega |v_\mu(t+h)|^{\frac{4q}{2q-1}} dx\,dt\Big) \\ &\leq C\Big(\int_0^{T^*}\int_\Omega |v_{\mu x}(t)|^{2q} dx\,dt + \int_0^{T^*}\int_\Omega |v_\mu(t)|^{\frac{4q}{2q-1}} dx\,dt\Big). \end{aligned} \tag{3.30}$$

To estimate the second term on the right-hand side of (3.30) we use interpolation inequality (2.2) with $p = \frac{2q+1}{2q-1}$, $r = 2q-1$, $m = 1$. Hence, by inequality (2.2) we have

$$\begin{aligned} \int_0^{T^*-h}\int_\Omega |v_\mu|^{\frac{4q}{2q-1}} dx\,dt &\leq \eta\int_0^{T^*} ||v_\mu||^{2q}_{W^1_{2q}(\Omega)} dt + \\ &+ C\eta^{-\sigma}\int_0^{T^*}\Big(\int_\Omega |v_\mu|^2 dx\Big)^{1+\gamma} dt \end{aligned} \tag{3.31}$$

for $\sigma = \frac{3}{q(10q-11)}$, $\gamma = \frac{4q^2-2q+1}{q(10q-11)}$ and $q \geq \frac{11}{10}$, what follows from inequality (2.1).

Estimates (3.30), (3.31) and (3.20) yield

$$\int_0^{T^*-h}\int_\Omega \Big|v_{\mu l}(t)\frac{\partial v_{\mu k}}{\partial x_l}(t)(v_{\mu k}(t+h)-v_{\mu k}(t))\Big|dx\,dt \le C. \tag{3.32}$$

In the same way we obtain the estimate

$$\int_0^{T^*-h}\Big[|< f(t), v_\mu(t+h)-v_\mu(t) >| + \sum_{\alpha=1}^{2}\int_S |g_\alpha(t)(v_\mu(t+h)- \\ - v_\mu(t))\cdot\tau_\alpha|ds\Big]dt \le C. \tag{3.33}$$

Taking into account (3.28), (3.29), (3.32) and (3.33) we get estimate (3.23). Next, from (3.23), (3.21), (3.20) and Lemma 2.2 it follows that there exists a subsequence of $(v_\mu)$ still denoted by $(v_\mu)$ such that

$$v_\mu \to v \quad \text{strongly in} \quad L^1(\Omega^{T^*})$$

and hence

$$v_\mu \to v \quad \text{a.e. in} \quad \Omega^{T^*}. \tag{3.34}$$

From (3.34), (3.20) and Lemma 2.3 it follows

$$v_\mu \to v \quad \text{strongly in} \quad L^{2q}(\Omega^{T^*}). \tag{3.35}$$

Now, we shall prove the estimate

$$\forall_{1\le k\le 3, 1\le l\le 3}\ ||\phi_1 D_{kl}(v_\mu)||_{L^{\frac{2q}{q-1}}(\Omega^{T^*})} \le C. \tag{3.36}$$

Integrating (3.13) in $(0,t)$ (where $t \le T^*$) and using (3.2) and (3.3) we obtain

$$\begin{aligned}\int_0^t (v_{\mu t}, v_\mu)dt \quad &+ \quad \frac{1}{2}\int_0^t\int_\Omega \phi_1 D_{kl}(v_\mu)D_{kl}(v_\mu)dx\,dt = \\ &= \quad \int_0^t < f, v_\mu > dt + \sum_{\alpha=1}^{2}\int_0^t < g_\alpha v_\mu, \tau_\alpha > dt.\end{aligned}$$

Hence, by the Young inequality and (3.20) we get

$$\begin{aligned}|v_\mu(t)|^2_{2,\Omega} \quad &+ \quad \frac{1}{2}\int_0^t\int_\Omega \phi_1 D_{kl}(v_\mu)D_{kl}(v_\mu)dx\,dt + \int_0^t |< f, v_\mu >|dt + \\ &+ \quad \sum_{\alpha=1}^{2}\int_0^t |< g_\alpha v_\mu, \tau_\alpha >|dt \le C\Big(\int_0^t ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)}dt \\ &+ \quad \int_0^t ||f(t)||^{\frac{2q}{2q-1}}_{V'}dt + \sum_{\alpha=1}^{2}\int_0^t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)}dt\Big) \le C.\end{aligned}$$

Hence

$$\int_0^t \int_\Omega \phi_1 D_{kl}(v_\mu) D_{kl}(v_\mu) dx\, dt \le C, \quad 0 < t \le T^*. \tag{3.37}$$

In view of assumption (3.6) we have

$$\begin{aligned} &\int_0^{T^*} \int_\Omega \phi_1(\overline{II}_D) D_{kl}(v_\mu) \widetilde{D}_{kl} dx\, dt \le \\ &\le \int_0^{T^*} \int_\Omega \phi_1(\overline{II}_D) D_{kl}(v_\mu) D_{kl}(v_\mu) dx\, dt + \\ &+ \int_0^{T^*} \int_\Omega \phi_1(\overline{II}_{\widetilde{D}}) \widetilde{D}_{kl} \widetilde{D}_{kl} dx\, dt - \int_0^{T^*} \int_\Omega \phi_1(\overline{II}_{\widetilde{D}}) \widetilde{D}_{kl} D_{kl}(v_\mu) dx\, dt \end{aligned} \tag{3.38}$$

for any symmetric $\widetilde{\mathbb{D}} \in L^{2q}(\Omega^{T^*})$, where $\overline{II}_D = tr\mathbb{D}^2(v_\mu)$, $\overline{II}_{\widetilde{D}} = tr\widetilde{\mathbb{D}}^2$. Now, we estimate the last term on the right-hand side of (3.38). Using (3.5) and (3.20) yields

$$\begin{aligned} &\int_0^{T^*} \int_\Omega \phi_1(\overline{II}_{\widetilde{D}}) D_{kl}(v_\mu) \widetilde{D}_{kl} dx\, dt \le \\ &\le C\Big(\int_0^{T^*} \int_\Omega |\widetilde{\mathbb{D}}||v_{\mu x}| dx\, dt + \int_0^{T^*} \int_\Omega |\widetilde{\mathbb{D}}|^{2q-1} |v_{\mu x}| dx\, dt\Big) \le \\ &\le C\Big(\int_0^{T^*} \int_\Omega |\widetilde{\mathbb{D}}|^{2q} dx\, dt + \int_0^{T^*} \int_\Omega |v_{\mu x}|^{2q} dx\, dt\Big) \le C. \end{aligned} \tag{3.39}$$

Estimates (3.37) and (3.39) imply

$$\int_0^{T^*} \int_\Omega \phi_1(\overline{II}_D) D_{kl}(v_\mu) \widetilde{D}_{kl} dx\, dt \le C$$

for every $\widetilde{\mathbb{D}} \in L^{2q}(\Omega^{T^*})$. Hence (3.36) holds and

$$\forall_{1\le k\le 3, 1\le l\le 3} \quad \phi_1(\overline{II}_D) D_{kl}(v_\mu) \to \chi_{kl} \quad \text{weakly in } L^{\frac{2q}{2q-1}}(\Omega^{T^*}) \tag{3.40}$$

for a subsequence of $(\phi_1(\overline{II}_D) D_{kl}(v_\mu))$. Taking into account (3.8) – (3.9), (3.21), (3.22), (3.35) and (3.40) we conclude that $v \in L^\infty(0, T^*; H) \cap L^{2q}(0, T^*; V)$ satisfies

$$\frac{d}{dt}(v, \zeta) + (\chi, \mathbb{D}(\zeta)) + b(v, v, \zeta) =< f, \zeta > + \sum_{\alpha=1}^{2} < g_\alpha \zeta, \tau_\alpha > \quad \forall \zeta \in V$$

and $v(0) = v_0$, where

$$(\chi, \mathbb{D}(\zeta)) = \frac{1}{2} \int_\Omega \chi_{kl} D_{kl}(\zeta) dx.$$

In order to prove that $\chi = \phi_1(\overline{II}_D)\mathbb{D}(v)$ we use assumption (3.6) and apply the same argument as in [7], [8] or [9].

In order to complete the proof it suffices to show that $v \in C(0,T^*;H)$. To do this denote

$$(Bv,u) = \int_0^{T^*} b(v,v,u)dt$$

and

$$(Av,u) = \int_0^{T^*} a(v,u)dt + \sum_{\alpha=1}^{2}\int_0^{T^*} < g_\alpha u, \tau_\alpha > dt.$$

We see that $B : L^{2q}(0,T^*;V) \to L^{\frac{2q}{2q-1}}(0,T^*;V')$ and $A : L^{2q}(0,T^*;V) \to L^{\frac{2q}{2q-1}}(0,T^*;V')$. In fact,

$$\begin{aligned}
||Bv||_{L^{\frac{2q}{2q-1}}(0,T^*;V')} &= \sup_{||u||_{L^{2q}(0,T^*;V)}\le 1}\Big|\int_0^{T^*}\int_\Omega v_l\frac{\partial v_k}{\partial x_l}u_k dxdt\Big| \le \\
&\le \sup_{||u||_{L^{2q}(0,T^*;V)}\le 1} ||v_x||_{L^{2q}(\Omega^{T^*})}||v||_{L^{\frac{4q}{2q-1}}(\Omega^{T^*})}||u||_{L^{\frac{4q}{2q-1}}(\Omega^{T^*})} \\
&\le C||v||^2_{L^{2q}(0,T^*;V)}
\end{aligned}$$

and

$$\begin{aligned}
||Av||_{L^{\frac{2q}{2q-1}}(0,T^*;V')} &= \sup_{||u||_{L^{2q}(0,T^*;V)}\le 1}\Big|\frac{1}{2}\int_0^{T^*}\int_\Omega \phi_1 D_{kl}(v)D_{kl}(u)dxdt + \\
&+ \sum_{\alpha=1}^{2}\int_0^{T^*}\int_S g_\alpha u\cdot\tau_\alpha dsdt\Big| \le C \sup_{||u||_{L^{2q}(0,T^*;V)}\le 1}\Big[||u||_{L^{2q}(0,T^*;V)}\cdot \\
&\cdot \Big(||v||^{2q-1}_{L^{2q}(0,T^*;V)} + ||v||_{L^{2q}(0,T^*;V)} + \sum_{\alpha=1}^{2}||g_\alpha||_{L^{\frac{2q}{2q-1}}(S^{T^*})}\Big)\Big] \\
&\le C\Big(||v||^{2q-1}_{L^{2q}(0,T^*;V)} + ||v||_{L^{2q}(0,T^*;V)} + \sum_{\alpha=1}^{2}||g_\alpha||_{L^{\frac{2q}{2q-1}}(S^{T^*})}\Big).
\end{aligned}$$

Hence $v_t \in L^{\frac{2q}{2q-1}}(0,T^*;V')$ what yields that $v \in C(0,T^*;H)$. This completes the proof of the theorem. □

*Remark* 3.1. Since $v \in C(0,T^*;H) \cap L^{2q}(0,T^*;V)$ is a solution of (3.1) there exists a distribution $p$ such that $\nabla p = f - v_t - (v\cdot\nabla)v + \mathrm{div}(\phi_1\mathbb{D})$ in $\Omega^{T^*}$ (see for example [11]) and $\nabla p \in L^{\frac{2q}{2q-1}}(0,T^*;V')$.

In the case $g_\alpha = 0$ $(\alpha = 1,2,)$ we have.

**Theorem 3.2.** *Let assumptions (3.4) – (3.6) be satisfied. Moreover, let* $v_0 \in H$, $f \in L^2(\Omega^T)$, $g_\alpha = 0$ $(\alpha = 1, 2)$. *Then there exists a weak solution* $v \in C(0,T;H) \cap L^{2q}(0,T;V)$ *of problem (1.1) – (1.5).*

*Proof.* In this case the boundary term $\sum_{\alpha=1}^{2} < g_\alpha v_\mu, \tau_\alpha >$ in (3.13) vanishes and instead of (3.14) we get

$$\frac{1}{2}\frac{d}{dt}\left|v_\mu(t)\right|^2_{2,\Omega} + c_1 \int_\Omega \left[D^2_{kl}(v_\mu(t))\right]^q dx \leq \frac{1}{2}\left|f(t)\right|^2_{2,\Omega} + \frac{1}{2}\left|v_\mu(t)\right|^2_{2,\Omega}. \tag{3.41}$$

Integrating (3.41) with respect to $t$ in $(0,t)$ we have

$$\begin{aligned} |v_\mu(t)|^2_{2,\Omega} \quad + \quad & c_6 \int_0^t \int_\Omega \left[D^2_{kl}(v_\mu)\right]^q dx\, dt \leq \\ \leq \quad & \int_0^t |f(t)|^2_{2,\Omega} dt + \int_0^t |v_\mu(t)|^2_{2,\Omega} dt + |v_0|^2_{2,\Omega}. \end{aligned} \tag{3.42}$$

By the Gronwall inequality, (3.42) yields

$$\sup_{0\leq t\leq T} \left|v_\mu(t)\right|^2_{2,\Omega} \leq C \tag{3.43}$$

and hence also

$$\int_0^t \int_\Omega \left[D^2_{kl}(v_\mu)\right]^q dx\, dt \leq C. \tag{3.44}$$

From (3.44) and inequality (2.3) it follows

$$\int_0^t ||v_\mu||^{2q}_{W^1_{2q}(\Omega)}\, dt \leq C + \int_0^t |v_\mu|^{2q}_{2q,\Omega}. \tag{3.45}$$

Now, using in (3.45) inequality (3.16) for $\eta$ sufficiently small and (3.43) we obtain

$$\sup_{0\leq t\leq T} |v_\mu(t)|^2_{2,\Omega} + \int_0^T ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} dt \leq C. \tag{3.46}$$

Next, using (3.46) and the same argument as in Theorem 3.1 we obtain the assertion of the theorem. □

## 4. Existence theorem for problem (1.1) – (1.5) with $\mathbf{T = -pI + c_1|D|^{2q-2}D}$

In the case when $\mathbb{T} = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D}$ (where $c_1 > 0$ is a constant) we obtain the following theorem

**Theorem 4.1.** *Let in (1.1)* $\mathbb{T} = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D}$, *where* $\frac{6}{5} \leq q \leq \infty$. *Let* $v_0 \in V$, $f \in L^2(\Omega^T)$, $g_\alpha \in C(0,T;L^{\frac{2q}{2q-1}}(S))$, $g_{\alpha t} \in L^{\frac{2q}{2q-1}}(S^T)$

$(\alpha = 1, 2)$. *Then there exists* $T^* \in (0, T]$ *depending on* $f$, $g$ *and* $v_0$ *(satisfying (3.19)) such that problem (1.1) – (1.5) has a weak solution* $v \in L^\infty(0, T^*; V)$ *with* $v_t \in L^2(\Omega^{T^*})$.

*Proof.* The function $\phi_1 = c_1|\mathbb{D}|^{2q-2}$ satisfies the assumptions of Theorems 3.1 and 3.2. Our aim is to obtain an appropriate estimate for $v_\mu$ which is the solution of (3.8) – (3.9) with $\phi_1 = c_1|\mathbb{D}|^{2q-2}$ and where we assume that $v_{0\mu} \to v_0$ in $V$ and $||v_{0\mu}||_{W^1_{2q}(\Omega)} \leq ||v_0||_{W^1_{2q}(\Omega)}$. To do this multiply the both sides of system (3.8) by $\frac{dc_{i\mu}(t)}{dt}$. We get

$$\begin{aligned}(v_{\mu t}, v_{\mu t}) \quad &+ \quad \frac{1}{2}c_1 \int_\Omega \left|\mathbb{D}(v_\mu)\right|^{2q-2} D_{kl}(v_\mu) D_{kl}(v_{\mu t}) dx + b(v_\mu, v_\mu, v_{\mu t}) = \\ &= \quad < f, v_{\mu t} > + \sum_{\alpha=1}^{2} < g_\alpha v_{\mu t}, \tau_\alpha > .\end{aligned}$$

Hence, using the integration by parts we have

$$\begin{aligned}\int_0^t \int_\Omega v_{\mu t}^2 dx\, dt + \frac{c_1}{4q} \int_\Omega \left|\mathbb{D}(v_\mu)\right|^{2q} \Big|_{t=0}^{t=t} dx + \int_0^t \int_\Omega (v_\mu \cdot \nabla)\, v_\mu v_{\mu t} dx\, dt \\ = \int_0^t \int_\Omega f \cdot v_{\mu t} dx\, dt + \sum_{\alpha=1}^{2} \int_0^t \int_S g_\alpha v_{\mu t} \cdot \tau_\alpha ds\, dt. \quad (4.1)\end{aligned}$$

Consider $\int_0^t \int_S g_\alpha v_{\mu t} \cdot \tau_\alpha ds\, dt$. Integrating by parts we obtain

$$\begin{aligned}
\int_0^t \int_S g_\alpha v_{\mu t} \cdot \tau_\alpha ds\, dt &= \int_S g_\alpha(t) v_\mu(t) \cdot \tau_\alpha ds - \\
&- \int_S g_\alpha(0) v_\mu(0) \cdot \tau_\alpha ds - \int_0^t \int_S g_{\alpha t} v_\mu \cdot \tau_\alpha ds\, dt \leq \\
&\leq \frac{1}{2} \big|\big| g_\alpha(0) \big|\big|^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} + \frac{1}{2} \big|\big| v_\mu(0) \big|\big|^{2q}_{L^{2q}(S)} + c(\eta) ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} \\
&+ \eta ||v_\mu(t)||^{2q}_{L^{2q}(S)} + \frac{1}{2} \int_0^t ||g_{\alpha t}||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} dt \\
&+ \frac{1}{2} \int_0^t \big|\big| v_\mu \big|\big|^{2q}_{L^{2q}(S)} \leq \frac{1}{2} \big|\big| g_\alpha(0) \big|\big|^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} \\
&+ C ||v_0||^{2q}_{W^1_{2q}(\Omega)} + C(\eta) ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} \\
&+ \eta_1 ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} + \frac{1}{2} \int_0^t ||g_{\alpha t}||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} \\
&+ C \int_0^t ||v_\mu||^{2q}_{W^1_{2q}(\Omega)} dt,
\end{aligned} \tag{4.2}$$

where $\eta_1 > 0$ is sufficiently small. Moreover,

$$\int_0^t \int_\Omega f \cdot v_{\mu t} dx\, dt \leq C(\eta) \int_0^t \int_\Omega f^2 dx\, dt + \eta \int_0^t \int_\Omega v^2_{\mu t} dx\, dt \tag{4.3}$$

and the following estimate is derived in [8] (pp. 86-87)

$$\int_0^t \int_\Omega (v_\mu \cdot \nabla) v_\mu \cdot v_{\mu t} dx\, dt \leq \eta \int_0^t \int_\Omega v^2_{\mu t} dx\, dt + C(\eta) \int_0^t ||v_\mu||^4_{W^1_{2q}(\Omega)} dt, \tag{4.4}$$

where $q \geq \frac{6}{5}$.

Using (4.2) – (4.4) in (4.1) we obtain for sufficiently small $\eta$

$$\begin{aligned}
\int_0^t \int_\Omega v^2_{\mu t} dx\, dt + C \int_\Omega \big| \mathbb{D} v_\mu(t) \big|^{2q} dx &\leq C \Big( \int_\Omega \big| \mathbb{D} v_\mu(0) \big|^{2q} dx + \\
+ ||f||^2_{L^2(\Omega^t)} + ||g_{\alpha t}||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S^t)} &+ \sup_t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} \\
+ \eta_1 ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} &+ \int_0^t ||v_\mu||^{2q}_{W^1_{2q}(\Omega)} dt + \int_0^t ||v_\mu||^4_{W^1_{2q}(\Omega)} dt \Big).
\end{aligned} \tag{4.5}$$

Now, using to estimate $\int_\Omega |\mathbb{D} v_\mu(t)|^{2q} dx$ inequality (2.3), next inequality (3.16) and the estimate

$$\int_\Omega \big| \mathbb{D} v_\mu(0) \big|^{2q} dx \leq C \int_\Omega \big| v_{\mu x}(0) \big|^{2q} dx \leq C ||v_0||^{2q}_{W^1_{2q}(\Omega)}$$

we get instead of (4.5) for sufficiently small $\eta_1$

$$\int_0^t \int_\Omega v_{\mu t}^2 dx\, dt + C||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} \le \tag{4.6}$$

$$\le C\Big[||v_0||^{2q}_{W^1_{2q}(\Omega)} + ||f||^2_{L^2(\Omega^t)} + ||g_{\alpha t}||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S^t)} + \sup_t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)}$$

$$+ \int_0^t ||v_\mu||^{2q}_{W^1_{2q}(\Omega)} dt + \int_0^t (\int_\Omega |v_\mu(t)|^2 dx)^q dt$$

$$+ \int_0^t ||v_\mu||^4_{W^1_{2q}(\Omega)} dt\Big].$$

By estimate (3.20) (which holds for $0 \le t \le T^*$) inequality (4.6) gives

$$||v_{\mu t}||^2_{L^2(\Omega^t)} + \sup_t ||v_\mu(t)||^{2q}_{W^1_{2q(\Omega)}} \le e(t) + h \int_0^t ||v_\mu||^4_{W^1_{2q}(\Omega)} dt \tag{4.7}$$

for $0 \le t \le T^*$, where

$$e(t) = C\Big[||v_0||^{2q}_{W^1_{2q}(\Omega)} + ||f||^2_{L^2(\Omega^t)} + ||g_{\alpha t}||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S^t)} + \sup_t ||g_\alpha(t)||^{\frac{2q}{2q-1}}_{L^{\frac{2q}{2q-1}}(S)} + \widetilde{C}\Big],$$

$h > 0$ is a constant.
Now, denote

$$y(t) = ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)}.$$

Then inequality (4.7) yields

$$y(t) \le e(t) + h \int_0^t y^{\frac{2}{q}}(t) dt, \quad \text{where } \ 0 \le t \le T^*. \tag{4.8}$$

Considerations of [8] (pp. 87–88) concerning inequality (4.8) yield

$$\int_0^t y^{\frac{2}{q}}(t) dt \le k(t), \tag{4.9}$$

where $k = k(t)$ is an increasing function of $t \in [0, T^*]$.
By (4.8) and (4.9) we obtain

$$||v_{\mu t}||^2_{L^2(\Omega^{T^*})} + \sup_{0 \le t \le T^*} ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} \le C. \tag{4.10}$$

Estimate (4.10) and the same argument as that used in Theorem 3.1 complete the proof of the theorem. □

If moreover $g_\alpha = 0$ $(\alpha = 1, 2)$ the following theorem holds.

**Theorem 4.2.** *Let in (1.1)* $\mathbb{T} = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D}$, *where* $\frac{6}{5} \le q < \infty$. *Let* $v_0 \in V$, $f \in L^2(\Omega^T)$, $g_\alpha = 0$ $(\alpha = 1, 2)$. *Then there exists a weak solution* $v \in L^\infty(0, T; V)$ *with* $v_t \in L^2(\Omega^T)$ *of problem (1.1) – (1.5).*

*Proof.* The similar considerations as in Theorem 4.1 and estimate (3.46) yield

$$||v_{\mu t}||^2_{L^2(\Omega^T)} + \sup_{0\le t\le T} ||v_\mu(t)||^{2q}_{W^1_{2q}(\Omega)} \le C,$$

what completes the proof of the theorem. □

**5. Existence and uniqueness theorem for problem (1.1) – (1.5) with $\mathbf{T} = -\mathbf{pI} + \mathbf{c_1}|\mathbf{D}|^{\mathbf{2q-2}}\mathbf{D} + \mathbf{c_2D}$.** In the nondenegerate case $\mathbb{T} = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D} + c_2\mathbb{D}$, (where $c_1 > 0$, $c_2 > 0$ are constants) considered also in [4] and [6] – [7], we obtain not only the existence of a solution, but the uniqueness, as well. We prove the following theorem

**Theorem 5.1.** *Let in (1.1)* $\mathbb{T} = -pI + c_1|\mathbb{D}|^{2q-2}\mathbb{D} + c_2\mathbb{D}$, *where* $\frac{5}{4} \le q < \infty$. *Let* $v_0 \in V$, $f \in L^2(\Omega^T)$, $g_\alpha \in C(0,T;L^2(S))$, $g_{\alpha t} \in L^2(S)$ $(\alpha = 1,2)$. *Then there exists a unique weak solution* $v \in L^\infty(0,T;V)$ *with* $v_t \in L^2(\Omega^T)$ *of problem (1.1) – (1.5).*

*Proof.* In this case we define an approximate solution of problem (1.1) – (1.5) by

$$\begin{aligned}(v_{\mu t}, w_j) \quad &+ \quad \frac{c_2}{2}\int_\Omega D_{kl}(v_\mu)D_{kl}(w_j)dx + \qquad (5.1)\\ &+ \quad \frac{c_1}{2}\int_\Omega |\mathbb{D}(v_\mu)|^{2q-2} D_{kl}(v_\mu)D_{kl}(w_j)dx + b(v_\mu, v_\mu, w_j)\\ &= \quad < f, w_j > + \sum_{\alpha=1}^{2} < g_\alpha w_j, \tau_\alpha > \quad \forall w_j,\end{aligned}$$

$$v_\mu(0) = v_{0\mu},$$

$$v_{0\mu} \to v_0 \quad \text{in} \quad V \quad \text{and} \quad ||v_{0\mu}||_{W^1_{2q}(\Omega)} \le ||v_0||_{W^1_{2q}(\Omega)}.$$

Multiplying the both sides of (5.1) by $c_{j\mu}(t)$ and summing up from 1 to $\mu$ we obtain

$$\begin{aligned}(v_{\mu t}, v_\mu) \quad &+ \quad \frac{c_2}{2}\int_\Omega [D_{kl}(v_\mu)]^2 dx + \frac{c_1}{2}\int_\Omega |\mathbb{D}(v_\mu)|^{2q-2}[D_{kl}(v_\mu))]^2 dx =\\ &= \quad < f, v_\mu > + \sum_{\alpha=1}^{2} < g_\alpha v_\mu, \tau_\alpha > .\end{aligned}$$

Hence, using (2.3) with $q = 1$ and the Young inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big|v_\mu(t)\big|^2_{2,\Omega} &+ \frac{c_1}{2}\int_\Omega ||v_\mu(t)||^2_{W^1_2(\Omega)}dt + \\ &+ \frac{c_2}{2}\int_\Omega |\mathbb{D}(v_\mu)|^{2q}dx \le C(\eta)\big|f(t)\big|^2_{2,\Omega} \\ &+ C\big|v_\mu(t)\big|^2_{2,\Omega} + C(\eta)\sum_{\alpha=1}^2 \big|g_\alpha(t)\big|^2_{2,S} + \eta\big|v_\mu(t)\big|^2_{2,S}. \end{aligned} \tag{5.2}$$

Since $|v_\mu(t)|^2_{2,S} \le C|||v_\mu(t)|^2_{W^1_2(\Omega)}$, integrating (5.2) with respect to $t$ in $(0,t)$ we get for sufficiently small $\eta$

$$\begin{aligned} \big|v_\mu(t)\big|^2_{2,\Omega} &+ c_3\int_0^t ||v_\mu(t)||^2_{W^1_2(\Omega)}dt + c_4\int_0^t\int_\Omega |\mathbb{D}(v_\mu)|^{2q}dx \le \\ &\le C\Big(\int_0^t |f(t)|^2_{2,\Omega}dt + + \int_0^t |g_\alpha(t)|^2_{2,S}dt + |v_0|^2_{2,\Omega} + \int_0^t |v_\mu(t)|^2_{2,\Omega}\Big). \end{aligned}$$

Therefore, by the Gronwall inequality we obtain

$$\sup_{0\le t\le T}\big|v_\mu(t)\big|^2_{2,\Omega} + \int_0^T ||v_\mu(t)||^2_{W^1_2(\Omega)}dt + \int_0^T |\mathbb{D}(v_\mu)|^{2q}_{2q,\Omega}dt \le C, \tag{5.3}$$

where $C$ is a constant depending on $||f||_{L^2(\Omega^T)}$, $||g_\alpha||_{L^2(S^T)}$ and $|v_0|_{2,\Omega}$. Next, using to estimate $\int_0^T |\mathbb{D}(v_\mu)|^{2q}_{2q,\Omega}dt$ inequalities (2.3), (3.16) and estimate (5.3) yields

$$\sup_{0\le t\le T}\big|v_\mu(t)\big|^2_{2,\Omega} + \int_0^T ||v_\mu||^{2q}_{W^1_2(\Omega)}dt \le C. \tag{5.4}$$

Next, we shall obtain the estimate of type (4.10). To do this multiply the both sides of (5.1) by $\frac{dc_{i\mu}(t)}{dt}$. We get

$$\begin{aligned} (v_{\mu t}, v_{\mu t}) &+ \frac{c_2}{2}\int_\Omega D_{kl}(v_\mu)D_{kl}(v_{\mu t})dx + \frac{c_1}{2}\int_\Omega \big|\mathbb{D}(v_\mu)\big|^{2q-2}D_{kl}(v_\mu)D_{kl}(v_{\mu t})dx + \\ &+ b(v_\mu, v_\mu, v_{\mu t}) = < f, v_{\mu t} > + \sum_{\alpha=1}^2 < g_\alpha v_{\mu t}, \tau_\alpha > . \end{aligned}$$

Now, using the integration by parts we obtain

$$\begin{aligned}\int_0^t\int_\Omega v_{\mu t}^2 dx\,dt &+ \frac{c_2}{4}\int_\Omega \left|\mathbb{D}(v_\mu)\right|^2\Big|_{t=0}^{t=t} dx + \frac{c_1}{4q}\int_\Omega \left|\mathbb{D}(v_\mu)\right|^{2q}\Big|_{t=0}^{t=t} dx + \\ &+ \int_0^t\int_\Omega (v_\mu\cdot\bigtriangledown)v_\mu\cdot v_{\mu t}dx\,dt \\ &= \int_0^t\int_\Omega f\cdot v_{\mu t}dx\,dt + \sum_{\alpha=1}^2\int_0^t\int_S g_\alpha v_{\mu t}\cdot\tau_\alpha ds\,dt.\end{aligned} \tag{5.5}$$

Consider $\int_0^t\int_S g_\alpha v_{\mu t}\cdot\tau_\alpha ds\,dt$. Integrating by parts we obtain

$$\begin{aligned}&\int_0^t\int_S g_\alpha v_{\mu t}\cdot\tau_\alpha ds\,dt = \int_S g_\alpha(t)v_\mu(t)\cdot\tau_\alpha ds - \\ &-\int_S g_\alpha(0)v_\mu(0)\cdot\tau_\alpha ds - \int_0^t\int_S g_{\alpha t}v_\mu\cdot\tau_\alpha ds\,dt \le \\ &\le \frac{1}{2}\Big|\Big|g_\alpha(0)\Big|\Big|^2_{L^2(S)} + C||v_0||^2_{W_2^1(\Omega)} + C(\eta)||g_\alpha(t)||^2_{L^2(S)} + \eta||v_\mu(t)||^2_{W_2^1(\Omega)} \\ &+\frac{1}{2}\int_0^t ||g_{\alpha t}||^2_{L^2(S)}dt + C\int_0^t ||v_\mu||^2_{W_2^1(\Omega)}dt,\end{aligned} \tag{5.6}$$

where $\eta > 0$ is sufficiently small. Using in (5.5) estimates (5.6), (4.3), (4.4) and (2.3) (to estimate $\int_\Omega |\mathbb{D}v_\mu(t)|^2dx$ and $\int_\Omega |\mathbb{D}v_\mu(t)|^{2q}dx$) we have

$$\begin{aligned}&\int_0^t\int_\Omega v_{\mu t}^2dx\,dt + ||v_\mu(t)||^2_{W_2^1(\Omega)} + ||v_\mu(t)||^{2q}_{W_{2q}^1(\Omega)} \le \\ &\le C\Big(||v_0||^{2q}_{W_{2q}^1(\Omega)} + ||v_0||^2_{W_2^1(\Omega)} + ||f||^2_{L^2(\Omega^T)} \\ &+||g_{\alpha t}||^2_{L^2(S^T)} + \sup_t||g_\alpha(t)||^2_{L^2(S)} + \int_0^t ||v_\mu||^2_{W_2^1(\Omega)}dt \\ &+\int_0^t |v_\mu(t)|^2_{2,\Omega}dt + \int_0^t |v_\mu(t)|^q_{2,\Omega}dt + \int_0^t ||v_\mu||^4_{W_{2q}^1(\Omega)}dt\Big).\end{aligned} \tag{5.7}$$

Applying to (5.7) the same considerations as in the proof of Theorem 4.1 we get

$$||v_{\mu t}||^2_{L^2(\Omega^T)} + \sup_{0\le t\le T}||v_\mu(t)||^2_{W_2^1(\Omega)} + \sup_{0\le t\le T}||v_\mu(t)||^{2q}_{W_{2q}^1(\Omega)} \le C, \tag{5.8}$$

where $C > 0$ is a constant.
Estimates (5.4), (5.8) and the same argument as in Theorem 3.1 complete the proof of the existence.

Now, we prove the uniqueness of the solution. Assume that there exist two solutions $v'$ and $v''$ of the considered problem. Denote $u = v' - v''$. Then

$$(u_t, \zeta) + \frac{c_1}{2} \int_\Omega \Big[\big|\mathbb{D}(v')\big|^{2q-2} D_{kl}(v') - \big|\mathbb{D}(v'')\big|^{2q-2} D_{kl}(v'')\Big] D_{kl}(\zeta) dx +$$

$$+ \frac{c_2}{2} \int_\Omega D_{kl}(u) D_{kl}(\zeta) dx + \int_\Omega (w_k u_{x_k} + u_k w_{x_k}) \cdot \zeta dx = 0 \qquad \forall \zeta \in V, \quad (5.9)$$

where $w = \frac{1}{2}(v' + v'')$ and $u(0) = 0$.
Putting in (5.9) $\zeta = u(t)$ and integrating with respect to $t$ in $(0, t)$ we obtain

$$\begin{aligned}\frac{1}{2}\big|u(t)\big|^2_{2,\Omega} \quad &+ \quad \frac{c_1}{2} \int_0^t \int_\Omega \Big[\big|\mathbb{D}(v')\big|^{2q-2} D_{kl}(v') + \\ &- \quad \big|\mathbb{D}(v'')\big|^{2q-2} D_{kl}(v'')\Big] (D_{kl}(v') - D_{kl}(v'')) dx\, dt \\ &+ \quad \frac{c_2}{2} \int_0^t \int_\Omega \big[D_{kl}(u)\big]^2 dx\, dt \\ &+ \quad \int_0^t \int_\Omega (w_k u_{x_k} + u_k w_{x_k}) u\, dx\, dt = 0.\end{aligned}$$

Since $q \geq \frac{5}{4}$ and $\int_0^t \int_\Omega [|\mathbb{D}(v')|^{2q-2} D_{kl}(v') - |\mathbb{D}(v'')|^{2q-2} D_{kl}(v'')] D_{kl}(u) dx\, dt \geq 0$ using inequality (2.3) with $q = 1$ and the same argument as in [7] (pp. 139-141) we get

$$\sup_{0 \leq \tau \leq t} \big|u(\tau)\big|^2_{2,\Omega} + \int_0^t ||u||^2_{W^1_2(\Omega)} dt \leq \widetilde{c} ||w_x||_{L^{2q}(\Omega^{t_1})} \Big( \int_0^t ||u||^2_{W^1_2(\Omega)} dx +$$

$$+ \sup_{0 \leq \tau \leq t} \big|u(\tau)\big|^2_{2,\Omega} \Big) + \int_0^t \big|u(t)\big|^2_{2,\Omega} dt, \quad (5.10)$$

where $0 \leq t \leq t_1$, $t_1 \in (0, T]$, $\widetilde{c} > 0$ is a constant. Choosing $t_1$ so small that $\widetilde{c} ||w_x||_{L^{2q}(\Omega^{t_1})} < 1$ estimate (5.10) and next the Gronwall inequality yield

$$\sup_{0 \leq t \leq t_1} \big|u(t)\big|^2_{2,\Omega} + \int_0^{t_1} ||u||^2_{W^1_2(\Omega)} dt \leq 0.$$

Therefore $u = 0$ for $t \in [0, t_1]$. In the same way we can prove that $u \equiv 0$ for $t \in [t_1, t_2]$. After a finite number of steps we obtain $u \equiv 0$ for $t \in [0, T]$, what gives the uniqueness and completes the proof of the Theorem. $\square$

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INSTITUTE OF MATHEMATICS
AND OPERATION RESEARCH
MILITARY UNIVERSITY OF TECHNOLOGY
S. KALISKIEGO 2
01-489 WARSAW, POLAND