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STATIONARY SOLUTIONS FOR HEAT EQUATION PERTURBED BY GENERAL ADDITIVE NOISE

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Abstract. Using the semigroup approach we prove that the heat equation on a bounded domain in R^d driven by general noise has a stationary solution iff a certain functional of the jump part of the noise has a finite expectation.

1. Introduction. Let $G \subset R^d$ be a bounded domain with a regular boundary δG. We consider the heat equation on G with zero boundary conditions, perturbed by random sources of heat:

$$\begin{cases} \frac{\partial u(t,x,\omega)}{\partial t} = \Delta_{(x)} u(t,x,\omega) + f(t,x,\omega), \\ u(t_0,x,\omega) = a(x,\omega) \text{ (initial condition)}, \\ u(t,x,\omega) = 0 \text{ for } x \in \delta G \text{ (boundary condition)}, \\ t \geq t_0, \quad x \in G, \quad \omega \in \Omega, \text{ where } (\Omega, F, P) \\ \qquad\qquad\qquad\qquad \text{is a given probability space.} \end{cases} \tag{1}$$

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In equation (1) f stands for a space dependent noice which in our model is more general then white noise — we also admit jumps. Formally writing,

$$f dt = dZ_t,$$

where Z_t is a process with independent homogeneous increments, taking values in a proper function space.

By a solution of (1) we understand the so called mild solution (e.g. [4]) of the stochastic evolution equation $(*)$ corresponding to (1) — see Section 2. We are concerned with the existence and uniqueness of solution to (1) that is a strictly stationary process. Such a problem is natural and important from the view point of physics.

Treating of (1) by the semigroup approach enables us to apply general results on stationary measures for stochastic evolution equations from [3]. The main result of this note is Theorem 1 and Corollary 1, where an explicit condition for the existence of stationary solution to (1) is given. This condition is similar to the criterion obtained for finite dimensional stochastic linear equations in [10] and [6].

2. Stochastic evolution equation. The following operator A acting in the space $H = L^2(G)$ corresponds to equation (1):

$$\begin{aligned} &A = \Delta_x, \text{ with the domain} \\ &\mathcal{D}(A) = H_0^1(G) \cap H^2(G). \end{aligned} \tag{2}$$

It is well known (see for instance [9]) that A defined by (2) has the properties:

(A1) A is a self–adjoint strictly negative operator with compact resolvent.

Consequently, A has a purely point spectrum $\{-\lambda_k\}_{k=1}^{\infty}$, where

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots ; \quad \lim \lambda_k = \infty \tag{3}$$

and there is an orthonormal basis $\{g_k\}_{k=1}^{\infty}$ in H consisting of the eigenvectors of A corresponding to the eigenvalues $-\lambda_k$. In the simplest case when G is the unit cube

$$\{x \in R^d : 0 < x_i < 1, \ i = 1, 2, \ldots, d\},$$

the eigenfunctions g_k are of the form (see for instance p. 200 in [1])

$$(\sqrt{2})^d \prod_{i=1}^{d} sin k_i \pi x_i, \quad k_i = 1, 2, \ldots \tag{4}$$

and the corresponding eigenvalues $(-\lambda_k)$ are

$$-\pi^2(k_1^2 + \ldots + k_d^2). \tag{5}$$

For a general bounded domain G with a regular boundary we have

$$c_1 k^{2/d} \leq \lambda_k \leq c_2 k^{2/d} \tag{6}$$

for some constants $c_2 \geq c_1 > 0$, $k_0 \geq 1$ and all $k \geq k_0$ (see Theorem 5 p. 190 in [9]).

Then (A1) implies that A is a generator of the strongly continuous self–adjoint semigroup $(S_t)_{t\geq 0}$ of bounded linear operators on H, given by the formula:

$$S_t h = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle h, g_k \rangle g_k \quad \text{for} \quad h \in H \tag{7}$$

and S_t is compact for $t > 0$ (e.g. [1]).

Hence stochastic partial differential equation (1) can be written as the following Ito equation in a real separable Hilbert space H:

$$\begin{cases} dX_t &= AX_t dt + dZ_t \\ X_{t_0} &= Y \end{cases} \tag{$*$}$$

where A is a linear operator on H (in general unbounded), generating a strongly continuous semigroup $(S_t)_{t\geq 0}$ of bounded linear operators on H, $(Z_t)_{-\infty<t<\infty}$ is an H–valued process with independent homogeneous increments, defined on a given probability space (Ω, F, P) and Y is an H–valued random variable independent of $\{(Z_t - Z_{t_0})_{t>t_0}\}$.

Additionally we assume that (Z_t) is continuous in probability, cadlag and $Z_0 \equiv 0$. Therefore Z_t can be represented in the form (see [5]):

$$Z_t = at + W_t + \xi_t,$$

where $a \in H$, W_t is an H–valued Wiener process with nuclear covariance operator R and ξ_t is the jump part independent of the process W_t. For $t \geq 0$, ξ_t has the representation:

$$\xi_t = \int_{\|x\|<1} x[\eta(t, dx) - tM(dx)] + \int_{\|x\|>1} x\eta(t, dx)$$

where $\eta(t, dx)$ is a random Poisson measure and M is the Lévy spectral measure of the random variable ξ_1. For an arbitrary Borel set B separated from 0, $\eta(t, B)$ is a number of jumps of the process Z_t which occured from time 0 to t and belonged to B. Moreover $E(\eta(t, B)) = tM(B)$. Recall that the characteristic function of Z_t, $t \geq 0$, has the form:

$$\begin{aligned} E\exp(i\langle y, Z_t\rangle) = \exp t\Big\{ & i\langle a, y\rangle - \frac{1}{2}\langle Ry, y\rangle \\ & + \int_H (e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle \mathbf{1}_{\{\|x\|<1\}}(x))M(dx)\Big\}, \quad \text{for } y \in H. \end{aligned}$$

(Usually the process Z_t is defined for $t \in [0, +\infty)$ but it can be extended in a standard manner for $t \in (-\infty, +\infty)$ — see e.g. Remark 0.4 in [3].)

The unique *mild solution of* $(*)$ is given by the formula (see [3]):

$$X_t = S_{t-t_0} Y + \int_{t_0}^{t} S_{t-s} dZ_s, \qquad t \leq t_0 \tag{$**$}$$

and it is well known that X_t is a Markov process.

A process $(X_t)_{t\in R}$ is a *mild solution* of $(*)$ on $(-\infty, +\infty)$ iff for any $t_0 \in R$ and $t \geq t_0$, the equality $(**)$ holds with Y replaced by X_{t_0}. We say that X_t is a *stationary solution* of $(*)$, if in addition it is a strictly stationary process.

Because one–dimensional distributions $\mathcal{L}(X_t)$ determine the law of the Markov process X_t, the equality $\mathcal{L}(X_t) = \mathcal{L}(X_s)$ for all t, s implies that the process X_t is stationary.

3. Existence of stationary solution to $(*)$. We assume that the generator A of the semigroup S_t satisfies (A1) and this case covers model (1). Then from (7) we have the estimate with $\delta = \lambda_1 > 0$:

$$\|S_t\| \leq e^{-\delta t} \qquad \text{for } t \geq 0, \tag{8}$$

that means the semigroup S_t is exponentially (exp.) stable.

Remark 1. Let S_t be an arbitrary exp. stable semigroup. If $Z_t = at + W_t$, then a stationary solution of $(*)$ always exists (e.g. [4]). Therefore the jump part ξ_t of the noise is only significant.

For $H = R^d$, there exists a stationary solution to $(*)$ if and only if

$$E(\log^+ \|Z_1\|) < \infty \qquad\qquad \text{(see [10], [6]).} \tag{9}$$

The same equivalence is also true in an arbitrary Hilbert space when the generator A is bounded [7], [3]. It turns out that for an arbitrary exp. stable semigroup S_t condition (9) is too strong — it is sufficient for the existence of stationary solution to $(*)$ but it need not be necessary [3].

However, for a class of compact semigroups containing the semigroup corresponding to (1) we obtain in Theorem 1 below a simple criterion (*iii*) that is similar to (9). In the proof of Theorem 1 we will use a general result from [3] where we have given conditions of another type and rather difficult to verify. We recall this theorem for the convenience of the reader.

Theorem A [3]. *Let S_t be exp. stable. Then there exists a stationary solution of $(*)$ iff the following conditions hold:*

$$\int_0^\infty \int_H (\|S_s x\|^2) \wedge 1)\, M(dx)\, ds < +\infty \tag{v}$$

and

there exists

$$\lim_{t\to\infty} \int_0^t [\int_{\|S_s x\|\le 1 \wedge \|x\|>1} S_s x\, M(dx) - \int_{\|S_s x\|>1 \wedge \|x\|\le 1} S_s x\, M(dx)]\, ds. \tag{vv}$$

Theorem 1. *Assume that* $\dim H = \infty$ *and the operator* A *satisfies* (A1) *and*

$$\sum_{k=1}^{\infty} \lambda_k^{-1} \exp(-\lambda_k T) < +\infty \tag{A2}$$

for some $T > 0$, *where* λ_k *and* g_k *are defined in* (3).

Then the following conditions are equivalent:

$$\text{There exists a stationary solution of } (*) \tag{i}$$

$$\int_H \sup_k (\lambda_k^{-1} \log^+ |\langle x, g_k\rangle|) M(dx) < +\infty \tag{ii}$$

$$E \sup_k (\lambda_k^{-1} \log^+ |\langle Z_1, g_k\rangle|) < +\infty. \tag{iii}$$

Proof. Let $x_k := \langle x, g_k\rangle$ and let π_n mean the orthogonal projection on $\mathrm{lin}\{g_1, \dots, g_n\}$.

1. $(i) \Rightarrow (ii)$.
From (v) and (7) we have:

$$\infty > \int_0^\infty \int_H (\|S_t \pi_n x\|^2 \wedge 1) M(dx) dt \ge \int_0^\infty M(\{x : \|S_t \pi_n x\| > 1\}) dt := J. \tag{10}$$

The inclusion

$$\{x : \sum_{k=1}^n \|S_t x_k g_k\|^2 > 1\} \supset \bigcup_{k=1}^n \{x : \|S_t x_k g_k\| > 1\} = \bigcup_{k=1}^n \{x : |x_k| > e^{\lambda_k t}\} = \{x : \max_{1\le k\le n} \frac{\log^+ |x_k|}{\lambda_k} > t\}$$

implies the inequality:

$$J \geq \int_0^\infty M(\{x : \max_{1\leq k\leq n} (\lambda_k^{-1} \log^+ |x_k|) > t\})dt = \int_H \max_{1\leq k\leq n} (\lambda_k^{-1} \log^+ |x_k|)M(dx) \quad (11)$$

where the last equality follows from Appendix II, in [2]. Combining (10) with (11) and applying the Lebesgue monotone convergence theorem we obtain (ii).

2. $(ii) \Rightarrow (i)$ We will show that (v) and (vv) are satisfied. Consider (v). Under the notations:

$$B = \{x : \|x\| < 1\},$$
$$C_k = \{x : |x_k| < exp(\lambda_k t/2)\}$$

we have the estimate:

$$\begin{aligned} L_n &:= \int_0^\infty \int_H (\|S_t\pi_n x\|^2 \wedge 1)M(dx)dt \quad (12)\\ &\leq \int_0^\infty \int_B \|S_t\pi_n x\|^2\ M(dx)dt \\ &+ \int_0^\infty \int_{B'\cap C_1\cap\ldots\cap C_n} (\|S_t\pi_n x\|^2 \wedge 1)M(dx)dt \\ &+ \int_0^\infty M(C_1'\cup\ldots\cup C_n')dt := I_1 + I_2 + I_3. \end{aligned}$$

By the same method as in the first part of the proof we obtain:

$$I_3 \leq 2\int_H \sup_k(\lambda_k^{-1} \log^+ |x_k|)M(dx). \quad (13)$$

From (8) we have:

$$I_1 \leq \int_0^\infty \int_B \|S_t x\|^2 M(dx)dt \leq \int_0^\infty e^{-2\delta t}dt \int_B \|x\|^2 M(dx). \quad (14)$$

By the properties of the Lévy spectral measure M (e.g. [5]) the above expression is finite and so is $M(B')$ in (15) below:

$$I_2 \leq T\ M(B')\ +\ \int_T^\infty \int_{B'\cap C_1\cap\ldots\cap C_n} \|S_t\pi_n x\|^2 M(dx)dt, \quad (15)$$

where T is given in (A2).
The last term in (15), denoted by I_{2a} , can be estimated as follows:

$$I_{2a} \leq \int_T^\infty \int_{B'} \sum_{k=1}^n e^{-\lambda_k t} M(dx)dt \leq M(B') \sum_{k=1}^\infty \lambda_k^{-1} e^{-\lambda_k T} < +\infty. \quad (16)$$

The inequalities (12)–(16) show that the expressions L_n in (12) are bounded from above by a constant independent of n. Then, by the Fatou lemma,

(v) follows. Consider (vv): by (8) the second integral is equal to 0. For the first one we have:

$$\int_0^\infty \int_{B'\cap\{x:\|S_t x\|\le 1\}} \|S_t\pi_n x\| M(dx) \le$$

$$\int_0^\infty \int_{B'\cap C_1\cap\ldots\cap C_n} (\|S_t\pi_n x\|^2 \wedge 1) M(dx) + I_3.$$

Thus (vv) follows by the same method as (v).

3. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 2 in [8]. Indeed, note that the inequalities

$$\log^+ t \le \log(1+t) \le \log 2 \ + \log^+ t\ , \quad t > 0,$$

imply that (ii) holds iff $\int_{\|x\|\ge 1} \phi(x) M(dx) < +\infty$, and that ($iii$) holds iff $E\phi(Z_1) < +\infty$, with $\phi(x) := \sup_k [\lambda_k^{-1} \log(1+|\langle x, g_k\rangle|)]$ being a subadditive function. $\square$

4. Stationary solution for heat equation (1).

Corollary 1. *There exists a stationary solution of* (1) *iff* (iii) *holds. Moreover a stationary solution on* $(-\infty,+\infty)$ *is unique up to modification and given by the formula*

$$X_t = \int_{-\infty}^t S_{t-s} dZ_s. \tag{17}$$

Proof. It follows from (5) and (6) that the eigenvalues of the operator A defined in (2) have the properities:
for $d = 1$, $\sum_{k=1}^\infty \lambda_k^{-1} < +\infty$
for $d \ge 2$, $\sum_{k=1}^\infty \lambda_k^{-p} < +\infty$, if $p > \frac{d}{2}$.
Then (A2) holds and the first part of the corollary follows from Theorem 1. To prove the uniqueness suppose that X_t^1 and X_t^2 are such solutions and $\mathcal{L}(X_t^1) = \mathcal{L}(X_t^2) = \mu$, $-\infty < t < \infty$. Fix t. Since S_t is stable, $S_{t-s}\mu$ converges weakly to δ_0 as s tends to $-\infty$. Then $S_{t-s}X_s^i$ converges to 0 in probability, $i = 1, 2$. Hence taking $s \to -\infty$ in the equality

$$X_t^1 - X_t^2 = S_{t-s}X_s^1 - S_{t-s}X_s^2\ ,$$

we get $X_t^1 = X_t^2$. The formula (17) follows from the proof and ($**$). $\square$

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