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ON THE UNIQUENESS OF LEBESGUE AND BOREL MEASURES

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Abstract. We consider the uniqueness property for various invariant measures. Primarily, we discuss this property for the standard Lebesgue measure on the n–dimensional Euclidean space $\mathbf{R}^n$ (sphere $\mathbf{S}^n$) and for the standard Borel measure on the same space (sphere), which is the restriction of the Lebesgue measure to the Borel σ–algebra of $\mathbf{R}^n$ ($\mathbf{S}^n$). The main goal of the paper is to show an application of the well known theorems of Ulam and Ershov to the uniqueness property of Lebesgue and Borel measures.

In the present paper, we are concerned with the uniqueness property of classical Lebesgue and Borel measures on a finite–dimensional Euclidean space (sphere). Dealing with this property, we essentially exploit the following two results:

1) the classical theorem of S. Ulam [10] stating that the first uncountable cardinal number ω_1 does not admit a nonzero σ–finite diffused measure defined on the family of all subsets of ω_1 (in other words, ω_1 is not a real–valued measurable cardinal);

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2) the measure extension theorem due to M. Ershov [1], stating that a probability measure defined on a countably generated σ–subalgebra of the Borel σ–algebra of a Polish topological space E can always be extended to a Borel measure on E.

We apply the first result in order to obtain some necessary and sufficient conditions for the uniqueness property of the standard Lebesgue measure.

Also, we apply the second result (with several simple facts from the theory of Haar measure on a locally compact topological group), in order to present some natural sufficient conditions for the uniqueness of the standard Borel measure. We want to mention that, at the present time, no necessary and sufficient conditions for the uniqueness property of the standard Borel measure are known. Hence, the problem of finding such conditions remains open and seems to be interesting from the point of view of general theory of invariant measures.

First of all we wish to recall some notions concerning the uniqueness property of an invariant measure which is given in an abstract space equipped with a transformation group.

Let E be a nonempty basic set, G be a group of transformations of E and let S be a G–invariant σ–algebra of subsets of E. Suppose also that μ is a σ–finite measure defined on S. We recall that

(1) μ is a G–invariant measure if $\mu(g(X)) = \mu(X)$, for each set $X \in S$ and for each transformation $g \in G$;

(2) μ is a G–quasiinvariant measure if

$$\mu(X) = 0 \Leftrightarrow \mu(g(X)) = 0,$$

for each set $X \in S$ and for each transformation $g \in G$.

We also recall that a G–invariant measure μ has the uniqueness property if, for every σ–finite G–invariant measure ν defined on $\mathrm{dom}(\mu)$, there exists a real coefficient $t = t(\nu)$ satisfying the equality $\nu = t\mu$.

The uniqueness property of an invariant measure plays an essential role in various questions of abstract harmonic analysis (see, for instance, [3]). Namely, the uniqueness of a Haar measure on a locally compact topological group implies important consequences and has a number of applications in modern analysis.

The uniqueness property for the classical Lebesgue measure was investigated in many works. In particular, let us mention paper [6] in which some necessary and sufficient conditions for this property were announced. We begin our consideration with the proof of the corresponding result obtained in [6]. But first we need to establish several auxiliary facts.

Lemma 1. *Let E be a finite–dimensional Euclidean space and let G be a group of affine transformations of E such that*

$$\mathrm{card}(G(x)) > \omega,$$

for each point $x \in E$. Further, let μ be a σ–finite measure on E satisfying the following relations:

1) $\mathrm{dom}(\mu)$ *contains the σ–ideal of subsets of E, generated by the family of all affine hyperplanes in E;*

2) μ is a G–quasiinvariant measure.

Then we have $\mu(V) = 0$, for every affine hyperplane V in E.

Proof. We may assume, without loss of generality, that μ is a probability measure. From the assumption of our lemma it immediately follows that all affine linear manifolds in E belong to $\mathrm{dom}(\mu)$. Let L denote an arbitrary affine linear manifold in E with $\dim(L) < \dim(E)$. We are going to prove that $\mu(L) = 0$. Naturally, in order to establish this fact, we use the induction on $\dim(L)$.

If $\dim(L) = 0$, then L can be represented in the form $L = \{y\}$, where y is some point of E. Since $\mathrm{card}(G(y)) > \omega$ and our measure μ is G–quasiinvariant and satisfies the countable chain condition, we must have the equality $\mu(L) = 0$.

Suppose now that, for an affine linear manifold $L \subset E$ with $\dim(L) < \dim(E)$, the measure μ vanishes on all affine linear manifolds in E whose dimensions are strictly less than $\dim(L)$. Let us show that $\mu(L) = 0$ as well. Only two cases are possible.

1. The family of manifolds $\{g(L) : g \in G\}$ is uncountable.

In this case, one can easily check that the above–mentioned family is almost disjoint with respect to μ. In other words, the μ–measure of the intersection of any two distinct manifolds from this family is equal to zero (since the intersection is an affine linear manifold in E with dimension strictly less than $\dim(L)$). Applying the G–quasiinvariance of μ and the countable chain condition, we get the equalities

$$\mu(g(L)) = 0 \ (g \in G).$$

In particular, we have $\mu(L) = 0$.

2. The family of manifolds $\{g(L) : g \in G\}$ is countable.

In this case, taking into account the fact that all G–orbits are uncountable, it is not difficult to verify that there exists a subgroup G_1 of G satisfying the next two relations:

a) L is invariant with respect to G_1, i.e. $g(L) = L$ for all $g \in G_1$;

b) $\mathrm{card}(G_1|_L) > \omega$, where $G_1|_L$ denotes the family of all restrictions of transformations from G_1 to the manifold L.

Evidently, we can regard $G_1|_L$ as a certain uncountable group of affine transformations of L. For the sake of simplicity, we denote this group by the same symbol G_1. Also, we may assume, without loss of generality, that $\mathrm{card}(G_1) = \omega_1$. Obviously, the group G_1 (acting in L) generates the partition of L into G_1–orbits of points from L. Let Z be a selector of this partition. Then we can write

$$\cup\{g(Z) \ : \ g \in G_1\} = L.$$

Furthermore, taking into account the inductive assumption, we have

$$\mu(g(Z) \cap h(Z)) = 0,$$

for any two distinct transformations $g \in G_1$ and $h \in G_1$ (because the set $g(Z) \cap h(Z)$ is contained in an affine linear manifold with dimension strictly less than $\dim(L)$). Since $Z \in \mathrm{dom}(\mu)$, we conclude, using the G_1–quasiinvariance of μ and the countable chain condition, that $\mu(Z) = 0$. Consequently, the family of μ–measure zero sets $\{g(Z) \ : \ g \in G_1\}$ is an ω_1–covering of L and, in addition,

$$P(L) \subseteq \mathrm{dom}(\mu),$$

where $P(L)$ denotes, as usual, the collection of all subsets of L.

If we suppose, for a moment, that $\mu(L) > 0$, then a simple argument shows us that there exists a nonzero finite diffused measure ν defined on the family of all subsets of the group G_1. Indeed, we can consider a disjoint family of sets $\{Z_g \ : \ g \in G_1\}$ satisfying the relations:

(a) $Z_g \subseteq g(Z)$, for all $g \in G_1$;

(b) $\cup\{Z_g \ : \ g \in G_1\} = \cup\{g(Z) \ : \ g \in G_1\} = L$.

Now, for each subset H of G_1, we put

$$\nu(H) = \mu(\{Z_h \ : \ h \in H\}).$$

Then ν is a nonzero finite diffused measure such that $\mathrm{dom}(\nu) = P(G_1)$, where $P(G_1)$ denotes the collection of all subsets of G_1. But the existence of such a measure ν immediately yields a contradiction with the above–mentioned result of Ulam stating that $\omega_1 = \mathrm{card}(G_1)$ is not a real–valued measurable cardinal number. This contradiction finishes the proof of our lemma. □

We also need the following auxiliary proposition.

Lemma 2. *Let E be a finite–dimensional Euclidean space, G be a group of affine transformations of E such that*

$$\mathrm{card}(G(x)) > \omega,$$

for each point $x \in E$, and let μ be a σ–finite G–quasiinvariant measure defined on the σ–algebra of all Lebesgue measurable subsets of E. Then μ is absolutely continuous with respect to the standard Lebesgue measure in E.

Proof. We may assume, without loss of generality, that μ is a probability measure. We denote by λ' the standard Lebesgue measure in E. Let us take an arbitrary set X satisfying the relations

$$X \in \mathrm{dom}(\lambda'), \ \lambda'(X) = 0,$$

and let us show that $\mu(X) = 0$. Suppose otherwise: $\mu(X) > 0$. Since G is an uncountable group, we can take a subgroup G_1 of G such that $\mathrm{card}(G_1) = \omega_1$. Let $\{Z_i \ : \ i \in I\}$ be an injective family of all those G_1–orbits in E which have a nonempty intersection with the set X. Further, let Z be a selector of the family $\{Z_i \cap X \ : \ i \in I\}$. For each transformation $g \in G_1$, we put

$$X_g = g(Z) \cap X.$$

Evidently, the equality

$$\cup\{X_g \ : \ g \in G_1\} = X$$

is fulfilled. Also, it can easily be checked that if $g \in G_1, \ h \in G_1, \ g \neq h$, then the set $g(Z) \cap h(Z)$ is contained in some affine hyperplane of the space E. According to the previous lemma, we have

$$\mu(g(Z) \cap h(Z)) = 0.$$

Hence, the family of sets $\{X_g \ : \ g \in G_1\}$ is almost disjoint with respect to μ. Consequently, the G_1–quasiinvariance of μ and the countable chain condition yield the equalities

$$\mu(X_g) = 0 \ (g \in G_1).$$

Now, we can consider a disjoint family of sets $\{Y_g \ : \ g \in G_1\}$ satisfying the inclusions

$$Y_g \subseteq X_g \ (g \in G_1)$$

and the equalities

$$\cup\{Y_g \ : \ g \in G_1\} = \cup\{X_g \ : \ g \in G_1\} = X.$$

Since the Lebesgue measure λ' is complete and $\lambda'(X) = 0$, we have the inclusion

$$P(X) \subseteq \mathrm{dom}(\mu),$$

where $P(X)$ denotes the collection of all subsets of X. In particular, the union of an arbitrary subfamily of $\{Y_g \ : \ g \in G_1\}$ belongs to $\mathrm{dom}(\mu)$. Finally, for each subset H of G_1, let us define

$$\nu(H) = \mu(\cup\{Y_h \ : \ h \in H\}).$$

Then ν is a nonzero finite diffused measure defined on the family of all subsets of G_1. But this contradicts again the Ulam theorem stating that $\omega_1 = \mathrm{card}(G_1)$ is not a real–valued measurable cardinal. The obtained contradiction ends the proof of our lemma. $\square$

Remark 1. It is easy to see that Lemmas 1 and 2 have the corresponding analogues for a finite–dimensional Euclidean sphere E equipped with a group G of its isometric transformations. In order to formulate the corresponding results, we only need to replace affine linear manifolds (in those lemmas) by spheres in E.

Lemma 3. *Let (E, G) be a space with a transformation group, μ be a σ–finite G–invariant measure defined on some σ–algebra of subsets of E, and let ν be another σ–finite G–invariant measure defined on* $\mathrm{dom}(\mu)$*. Suppose also that*

1) ν is absolutely continuous with respect to μ;

2) μ is metrically transitive with respect to G, i.e. for each μ–measurable set X with $\mu(X) > 0$, there exists a countable family $\{g_n : n < \omega\}$ of transformations from G such that

$$\mu(E \setminus \cup\{g_n(X) \ : \ n < \omega\}) = 0.$$

Then the measure ν is proportional to μ; in other words, there exists a real coefficient $t = t(\nu)$ such that $\nu = t\mu$.

This lemma is well known (for the proof, see e.g. [2] or [4]).

Let E be a finite–dimensional Euclidean space (sphere) and let G be a subgroup of the group of all isometric transformations of E. Suppose also that there exists a point $e \in E$ such that

$$\mathrm{card}(G(e)) \leq \omega.$$

Then it can easily be checked that there exists a measure μ on E satisfying the next three relations:

a) $\mathrm{dom}(\mu) = P(E)$;

b) μ is a nonzero σ–finite G–invariant measure;

c) μ is concentrated on the countable set $G(e)$.

It immediately follows from these relations that the Lebesgue measure λ' on E (considered only as a G–invariant measure) does not have the uniqueness property.

Let E be again a finite–dimensional Euclidean space (sphere) and let G be a subgroup of the group of all isometric transformations of E. Let us consider again the Lebesgue measure λ' on E as a G–invariant measure. It is not difficult to show, applying the classical Lebesgue theorem on density points of Lebesgue measurable sets, that the next two assertions are equivalent:

(a) λ' is metrically transitive (with respect to G);

(b) for each point $x \in E$, the orbit $G(x)$ is dense everywhere in E.

Now, we are able to formulate and prove the following statement concerning the uniqueness property of the standard Lebesgue measure.

Theorem 1. *Let E be either the n–dimensional Euclidean space or the n–dimensional Euclidean sphere, where $n > 0$, let G be a subgroup of the group of all isometric transformations of E, and let λ' be the standard n–dimensional Lebesgue measure on E. Then the next two assertions are equivalent:*

a) the measure λ' (considered as a G–invariant measure) has the uniqueness property;

b) for each point $x \in E$, the orbit $G(x)$ is uncountable and dense everywhere in E.

In particular, if the group G acts transitively in E, then the measure λ' (considered as a G–invariant measure) has the uniqueness property.

Proof. First of all let us remind that the uniqueness property of an invariant measure implies the metrical transitivity of this measure. Hence, taking into account the preceding remarks, we see that assertion b) follows from assertion a). Conversely, suppose that b) is fulfilled, and let μ be an arbitrary σ–finite G–invariant measure defined on dom(λ'). Then, according to Lemma 2, the measure μ is absolutely continuous with respect to λ'. Also, λ' is metrically transitive with respect to G. It remains to apply Lemma 3, in order to establish the existence of a real coefficient $t = t(\mu)$ such that $\mu = t\lambda'$. Thus, assertion b) implies assertion a), and the proof of Theorem 1 is complete. $\square$

Using a similar argument, one can establish a corresponding result for the completion of a Haar measure on an uncountable σ–compact locally compact topological group (in this connection, see [5]). Namely, let E be a σ–compact locally compact topological group, G be a subgroup of E and let μ' be the completion of the left invariant Haar measure on E. Then the next two assertions are equivalent:

(a) the measure μ' (considered as a left G–invariant measure) has the uniqueness property;

(b) the group G is uncountable and dense everywhere in E.

For a detailed proof of the equivalence of assertions (a) and (b), see [5]. Here we notice only that the proof is essentially based on the above–mentioned Ulam theorem and on the so called Steinhaus property of a σ–finite Haar measure.

Thus, dealing with the classical Lebesgue measure or with the completion of a σ–finite Haar measure, we have necessary and sufficient conditions (formulated in purely group–theoretical and topological terms) for the uniqueness property.

Example 1. Let E be the n–dimensional Euclidean space, where $n > 2$. It is not difficult to show that there exists a group G of isometric transformations of E such that

1) G is a free group;

2) G acts transitively in E (in particular, card(G) is equal to the cardinality continuum);

3) if $g \in G$ is a translation of E, then g is the identity transformation of E (i.e. G does not contain nontrivial translations of E).

According to Theorem 1, the Lebesgue measure in E (considered as a G–invariant measure) has the uniqueness property.

On the other hand, let us emphasize that Theorem 1 yields nothing for the uniqueness property of the classical Borel measure on a finite–dimensional Euclidean space (sphere). This can be illustrated by the following fact. Let E be the n-dimensional Euclidean space, where $n > 1$. Let us denote by λ the n–dimensional Borel measure on E (i.e. λ is the restriction of λ' to the Borel σ–algebra of E). It can easily be proved that there exists a subgroup G of the additive group of E, satisfying the next three relations:

(1) card(G) is equal to the cardinality continuum;

(2) G is dense everywhere in E;

(3) the G–invariant measure λ does not have the uniqueness property.

Indeed, let us represent our space E in the form of a direct sum $E = E_1 + \mathbf{R}$, where E_1 is an $(n-1)$–dimensional vector subspace of E. Then we can put $G = E_1 + \mathbf{Q}$, where $\mathbf{Q}$ denotes the subgroup of $\mathbf{R}$ consisting of all rational numbers.

The results presented above show an essential difference between two classical measures, from the point of view of the uniqueness property. In this connection, it is reasonable to formulate the following problem.

PROBLEM. Let E be either the n–dimensional Euclidean space or the n–dimensional Euclidean sphere. Find a characterization of all those groups G of isometric transformations of E, for which the standard Borel measure λ in E (considered only as a G–invariant measure) has the uniqueness property.

Clearly, we mean here a characterization of groups G in those terms which do not belong to measure theory (for example, topological terms, group–theoretical terms, etc.).

The problem posed above remains open and seems to be a nontrivial one. In the present paper, we are going to discuss a simple sufficient condition on a group G of isometric transformations of E, under which the G–invariant Borel measure λ in E has the uniqueness property. However, the formulation of this condition is not purely topological and group–theoretical.

First we need some auxiliary notions, facts and propositions.

Suppose that (E, S, μ) is a space equipped with a σ–finite measure μ. Let X and Y be any two subsets of the basic set E. We say that X is thick in Y, with respect to the measure μ, if $\mu_*(Y \setminus X) = 0$, where μ_* denotes, as usual, the inner measure associated with μ.

In particular, we say that X is a thick subset of E, with respect to μ, if $\mu_*(E \setminus X) = 0$.

Let G be an arbitrary group, μ_1 and μ_2 be some σ–finite measures given on G. We recall that μ_1 is a right G–quasiinvariant measure if $\mathrm{dom}(\mu_1)$ is a right G–invariant class of subsets of G and

$$\mu_1(Xg) = 0 \Leftrightarrow \mu_1(X) = 0,$$

for all sets $X \in \mathrm{dom}(\mu_1)$ and for all elements $g \in G$.

Analogously, μ_2 is a left G–quasiinvariant measure if $\mathrm{dom}(\mu_2)$ is a left G–invariant class of subsets of G and

$$\mu_2(gX) = 0 \Leftrightarrow \mu_2(X) = 0,$$

for all sets $X \in \mathrm{dom}(\mu_2)$ and for all elements $g \in G$.

We begin with the following result useful in many situations (cf. [2], Chapter 11).

Lemma 4. *Let μ_1 be a nonzero right G–quasiinvariant measure on G and let μ_2 be a nonzero left G–quasiinvariant measure on G. Suppose also that*

1) $\mathrm{dom}(\mu_1) = \mathrm{dom}(\mu_2)$*;*

2) for each set $X \in \mathrm{dom}(\mu_1)$, the set

$$X^* = \{(g, h) \in G \times G \ : \ gh \in X\}$$

belongs to the product σ–algebra $\mathrm{dom}(\mu_1) \otimes \mathrm{dom}(\mu_2)$.

Then the measures μ_1 and μ_2 are equivalent (i.e. each of these two measures is absolutely continuous with respect to another one).

Proof. Evidently, it is sufficient to establish that μ_2 is absolutely continuous with respect to μ_1. Let us take an arbitrary set $Y \in \mathrm{dom}(\mu_1)$ with $\mu_1(Y) = 0$, and let us consider the set

$$Y^* = \{(g,h) \in G \times G \ : \ gh \in Y\}.$$

According to condition 2), this set belongs to the σ-algebra $\mathrm{dom}(\mu_1) \otimes \mathrm{dom}(\mu_2)$. Applying the classical Fubini theorem to the product measure $\mu_1 \times \mu_2$ and to the set Y^*, we can write

$$(\mu_1 \times \mu_2)(Y^*) = \int_G \mu_1(Yh^{-1}) d\mu_2(h) = 0.$$

Applying the Fubini theorem once more, we get

$$0 = (\mu_1 \times \mu_2)(Y^*) = \int_G \mu_2(g^{-1}Y) d\mu_1(g).$$

Thus, we can conclude that the function

$$g \to \mu_2(g^{-1}Y) \ (g \in G)$$

is equivalent to zero (with respect to μ_1). In particular, there exists an element $g \in G$ such that $\mu_2(g^{-1}Y) = 0$. Consequently,

$$\mu_2(Y) = \mu_2(g(g^{-1}Y)) = 0.$$

This completes the proof of the lemma. □

Let us mention some (well–known) consequences of Lemma 4.

Suppose that G is a locally compact Polish topological group, θ is the left invariant Haar measure on G and θ' is the right invariant Haar measure on G. Then

1) θ and θ' are mutually absolutely continuous measures;
2) for a subset X of G, the following two assertions are equivalent:
   a) X is a thick set with respect to θ;
   b) X is a thick set with respect to θ';
3) for a Borel subset Y of G, the next two assertions are equivalent:
   a) Y is a set of strictly positive θ–measure;
   b) Y^{-1} is a set of strictly positive θ–measure.

Notice that assertion 1) follows directly from Lemma 4. Assertions 2) and 3) can easily be deduced from assertion 1).

Let G be a σ–compact locally compact topological group. As above, we denote the left invariant Haar measure on G by the symbol θ. Let H be an arbitrary subgroup of G and let $\mathrm{cl}(H)$ be the closure of H in G. Obviously, $\mathrm{cl}(H)$ is a closed subgroup of G. Consequently, $\mathrm{cl}(H)$ is also a σ–compact locally compact group equipped with the left invariant Haar measure which is denoted by θ_H.

We say that the group H is thick in its closure if H is a θ_H–thick subset of $\mathrm{cl}(H)$.

Remark 2. Let θ_H be the left invariant Haar measure on $\mathrm{cl}(H)$ and let θ'_H be the right invariant Haar measure on $\mathrm{cl}(H)$. It follows from 2) that the next two assertions are equivalent:

a) H is a θ_H–thick subset of $\mathrm{cl}(H)$;

b) H is a θ'_H–thick subset of $\mathrm{cl}(H)$.

Thus, we see that the definition of a thick group in its closure does not depend on the choice of a Haar measure in the closure.

Remark 3. Let G be a σ–compact locally compact topological group, θ be a Haar measure on G, and let H be a dense subgroup of G. It can be shown, applying the metrical transitivity of θ, that the next two relations are equivalent:

a) H is a θ–thick subset of G;

b) $\theta^*(H) > 0$, i.e. the outer θ–measure of H is strictly positive.

Remark 4. Let E be a finite–dimensional Euclidean space (sphere), and let G be a group of isometric transformations of E such that the closure of G acts transitively in E. We cannot assert, in general, that G is thick with respect to a Haar measure in $\mathrm{cl}(G)$. Indeed, it may happen that G is even a countable group and, consequently, is of measure zero.

The following important result is due to Ershov (see [1]). It has a number of interesting applications in measure theory and probability theory.

Lemma 5. *Let E be a Polish topological space and $B(E)$ be the Borel σ–algebra of E. Let S be a countably generated σ–subalgebra of $B(E)$ and μ be a probability measure defined on S. Then there exists a measure ν defined on $B(E)$ and extending μ.*

A detailed proof of Lemma 5 is given in [1]. Here we notice only that the proof of this lemma is essentially based on some properties of the so–called Marczewski characteristic function of a sequence of sets and on a well–known theorem concerning the existence of measurable selectors. Notice also that several statements analogous to Lemma 5, for the case where E is a projective topological space, are discussed in [7]. Of course, if we deal with projective spaces, then some additional set–theoretical axioms are needed.

Remark 5. It can easily be checked that the assertion of Lemma 5 remains true for any σ–finite measure μ defined on S.

Remark 6. Let E be a topological space for which the assertion of Lemma 5 is true, i.e. if S is an arbitrary countably generated σ–subalgebra of $B(E)$ and μ is an arbitrary probability measure on S, then there exists a measure ν on $B(E)$ extending μ (in this case we say that E satisfies Lemma 5 or E has the measure extension property). Let E' be a topological space such that there exists a Borel surjection from E onto E'. Then the assertion of Lemma 5 is true for the space E', too.

In particular, it immediately follows from the previous remark that Lemma 5 holds true in the case of an analytic topological space E'. In connection with this result, we wish to notice that Lemma 5 cannot be proved, in theory **ZFC**, even for coanalytic topological spaces (for more details, see [7]). On the other hand, this lemma is true for some "bad" topological spaces. Indeed, let E be a topological space such that all one–element subsets of E are Borel, and let E' be a universally thick set in E (i.e., for each σ–finite diffused Borel measure μ on E, we have $\mu_*(E \setminus E') = 0$). It is easy to show that if E satisfies Lemma 5, then E' also satisfies this lemma. Consequently, we obtain that every Bernstein subspace of an uncountable Polish topological space satisfies Lemma 5.

Remark 7. Let us put

E = the unit segment $[0, 1]$;

S = the σ–algebra generated by the family of all first category Borel subsets of $[0, 1]$;

μ = the probability measure on S vanishing on all first category Borel subsets of $[0, 1]$.

Applying the well–known fact from topological measure theory, stating that any σ–finite diffused Borel measure in a separable metric space is concentrated on a first category set, it is easy to check that μ cannot be extended to a Borel measure on $[0, 1]$.

Thus, we see that the assumption that S is a countably generated σ–subalgebra of $B(E)$ is essential in the formulation of Lemma 5.

Lemma 6. *Let G be a locally compact Polish topological group, μ be a Haar measure on G and let H be a subgroup of G thick with respect to μ. Further, let ν be a nonzero σ–finite left (right) H–quasiinvariant measure defined on* $\mathrm{dom}(\mu)$. *Then the measures μ and ν are equivalent.*

This lemma easily follows from Lemma 4.

In addition, applying Lemmas 3 and 6, it is not difficult to show that if H is a thick subgroup of a locally compact Polish topological group G, then the left invariant Haar measure on G (considered only as a left H–invariant measure) has the uniqueness property.

Lemma 7. *Let E denote the n-dimensional Euclidean space (sphere), where $n > 0$, let G be a subgroup of the group of all isometric transformations of E, and let μ be a probability G–quasiinvariant measure defined on the Borel σ–algebra of E. Suppose that, for some point $e \in E$, the orbit $G(e)$ is μ–thick in E. Let us define a continuous mapping*

$$\phi \ : \ G \to E$$

by the formula

$$\phi(g) = g(e) \ \ (g \in G),$$

and let us put

$$S = \{\phi^{-1}(X) \ : \ X \in B(E)\}.$$

Finally, let us define a functional ν on S by the formula

$$\nu(\phi^{-1}(X)) = \mu(X) \ \ (X \in B(E)).$$

Then the following assertions are true:

1) S is a countably generated σ–subalgebra of $B(G)$;

2) the definition of the functional ν is correct;

3) ν is a probability left G–quasiinvariant measure on S.

The proof of this lemma is not difficult. One has to check directly the validity of assertions 1), 2) and 3).

Lemma 8. *Let G be a locally compact Polish topological group, μ be a Haar measure on G, and let H be a dense subgroup of G such that $\mu^*(H) > 0$. Suppose also that ν is a probability left (right) H–quasiinvariant measure defined on the Borel σ–algebra of G. Then there exists a real–valued function*

$$p \ : \ G \to \mathbf{R}$$

satisfying the following relations:

1) $p(g) > 0$, for all $g \in G$;

2) p is a Borel function;

3) for each Borel subset Z of G, we have the equality

$$\nu(Z) = \int_Z p(g)d\mu(g).$$

Proof. From the assumption of the lemma it follows that H is a thick subgroup of G with respect to μ. According to Lemma 6, the measures μ and ν are equivalent. Thus, it remains to apply the classical Radon–Nikodym theorem, in order to find a real–valued function p with the desired properties. □

The following auxiliary proposition plays the key role in our further considerations.

Lemma 9. *Let E denote the n–dimensional Euclidean space (sphere), where $n > 0$, let G be a group of isometric transformations of E, and let* $\mathrm{cl}(G)$ *be the closure of G. We denote by θ the left invariant Haar measure on* $\mathrm{cl}(G)$. *Suppose that*

1) $\theta^*(G) > 0$;

2) the group $\mathrm{cl}(G)$ *acts transitively in E.*

Suppose also that μ is a nonzero σ–finite G–quasiinvariant measure defined on the Borel σ–algebra of E. Finally, let us fix an arbitrary point $e \in E$. Then, for each Borel subset X of E, we have

$$\mu(X) = 0 \Leftrightarrow \theta(\{g \in \mathrm{cl}(G) \ : \ g(e) \in X\}) = 0.$$

Proof. We may assume, without loss of generality, that μ is a probability measure. It follows from the assumption of our lemma that the group G is thick in its closure (with respect to the Haar measure θ). Further, let us define a surjective continuous mapping

$$\phi \ : \ \mathrm{cl}(G) \to E$$

by the formula

$$\phi(g) = g(e) \ (g \in \mathrm{cl}(G))$$

and let us put

$$S = \{\phi^{-1}(X) \ : \ X \in B(E)\}.$$

According to Lemma 7, we can define a probability measure ν on the σ–algebra S by the formula

$$\nu(\phi^{-1}(X)) = \mu(X) \ (X \in B(E)).$$

Since the original measure μ is G–quasiinvariant, the measure ν on S is left G–quasiinvariant. Applying Lemma 5, we can extend ν to a probability Borel measure ν' on the group $\mathrm{cl}(G)$. Let us denote by θ' a probability measure equivalent to the Haar measure θ. Further, for each Borel subset Z of $\mathrm{cl}(G)$, let us consider a function

$$\psi \ : \ \mathrm{cl}(G) \to \mathbf{R}$$

defined by the formula

$$\psi(g) = \nu'(gZ) \ (g \in \mathrm{cl}(G)).$$

It is not difficult to show that ψ is a Borel function (obviously, integrable with respect to the measure θ'). So, we may put

$$\nu''(Z) = \int_{\mathrm{cl}(G)} \nu'(gZ) d\theta'(g).$$

It can easily be checked that ν'' is a left $\mathrm{cl}(G)$–quasiinvariant probability Borel measure on the group $cl(G)$. In particular, ν'' is left G–quasiinvariant. Consequently, applying Lemma 6, we can assert that the measures ν'' and

θ are equivalent. Hence, by Lemma 8, there exists a strictly positive Borel function

$$p \;:\; \mathrm{cl}(G) \rightarrow \mathbf{R}$$

such that

$$\nu''(Z) = \int_Z p(g) d\theta(g),$$

for each Borel subset Z of $\mathrm{cl}(G)$. Now, it is clear that, for $X \in B(E)$, we have

$$\mu(X) = 0 \Leftrightarrow \nu(\phi^{-1}(X)) = 0.$$

At the same time, we can write

$$\nu(\phi^{-1}(X)) = 0 \Leftrightarrow \nu'(\phi^{-1}(X)) = 0 \Leftrightarrow \nu''(\phi^{-1}(X)) = 0.$$

Therefore, we have

$$\mu(X) = 0 \Leftrightarrow \theta(\phi^{-1}(X)) = 0.$$

This completes the proof of the lemma. □

Finally, we can formulate and prove the following result concerning the uniqueness property of the standard Borel measure.

Theorem 2. *Let E denote the n–dimensional Euclidean space (sphere), where $n > 0$, let G be a subgroup of the group of all isometric transformations of E, and let λ be the standard Borel measure on E. Suppose also that*

1) the closure of G acts transitively in E;

2) the group G is thick in its closure.

Then the measure λ (considered only as a G–invariant measure) has the uniqueness property.

Proof. Let ν be an arbitrary σ–finite G–invariant measure defined on the Borel σ–algebra of E. Applying Lemma 9, it is easy to see that ν is absolutely continuous with respect to λ. Since λ is metrically transitive (with respect to G), we conclude, according to Lemma 3, that ν is proportional to λ. Thus, the measure λ has the uniqueness property. □

Example 2. Let $E = \mathbf{S}^n$ be the unit n–dimensional Euclidean sphere, where $n > 1$, and let G be the group of all rotations of this sphere around its centre. Using the method of transfinite induction, it is not difficult to prove that there exists a subgroup H of G satisfying the next three relations:

1) H is of cardinality continuum;

2) H is a free group;

3) H is thick in G (with respect to the Haar measure on G).

In particular, the group H is dense everywhere in G, it differs from G, and the standard Borel measure on E (considered only as an H–invariant measure) possesses the uniqueness property. In connection with this fact, let us recall that the first example of a free group of rotations of E, having the cardinality continuum, was constructed by Sierpiński (see his paper [9]). A detailed information (concerning this topic and related questions) can be found in Chapter 6 of the well-known monograph by Wagon [11].

Let us consider especially the case when $n = 3$, i.e. let E be the unit three–dimensional Euclidean sphere and G be the group of all rotations of E around its centre. From the classical theory of quaternions it is known that there exists a subgroup G' of G satisfying the following conditions:

a) G' acts transitively in E;

b) G' is a closed subgroup of G;

c) the group of all rotations of the unit two-dimensional Euclidean sphere (around its centre) is the image of G' with respect to a homomorphism such that the preimage of each rotation is a two–element subset of G'.

In particular, we have

$$\dim(G') = \dim(E) = 3 < 6 = \dim(G).$$

We see also that there exists a subgroup H' of G' such that

(1) H' is of cardinality continuum;

(2) H' is a free group;

(3) H' is thick in G' (with respect to the Haar measure on G').

Thus, applying Theorem 2, one can conclude that the standard Borel measure in E (considered only as an H'–invariant measure) possesses the uniqueness property.

Remark 8. There are several important and interesting works devoted to the uniqueness property of the standard Lebesgue measure considered as a positive, finitely additive, invariant, and normalized functional defined on the family of all bounded Lebesgue measurable subsets of an Euclidean space (sphere). This topic is discussed in detail in Chapter 11 of monograph [11] where references to the corresponding original works of Drinfeld, Margulis, Rosenblatt and Sullivan are presented (in this connection, see also [8]).

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