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FORCED OSCILLATIONS OF FIRST ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. First order forced nonlinear neutral differential equations are studied and sufficient conditions are derived for all solutions to be oscillatory.

1. Introduction. We shall be concerned with the oscillatory behavior of solutions of the forced nonlinear neutral differential equation

$$\frac{d}{dt}\left[x(t)+\sum_{i=1}^{k}P_i(t)x(\tau_i(t))\right]+\sum_{j=1}^{m}Q_j(t)f_j(x(\sigma_j(t)))=R(t), \tag{1.1}$$

where k and m are some positive integers, $P_i(t)$, $Q_j(t)$, $R(t)\in C((0,\infty),\mathbb{R})$ $(i=1,2,\dots,k;\ j=1,2,\dots,m)$, $f_j(u)\in C(\mathbb{R},\mathbb{R})$, $uf_j(u)>0$ for $u\neq 0$ $(j=1,2,\dots,m)$, $\tau_i(t),\sigma_j(t)\in C((0,\infty),\mathbb{R})$ $(i=1,2,\dots,k; j=1,2,\dots,m)$, $\lim_{t\to\infty}\tau_i(t)=\infty$, and $\lim_{t\to\infty}\sigma_j(t)=\infty$.

By a *solution* of (1.1), we mean a function $x:[T_x,\infty)\longrightarrow\mathbb{R}$ such that $x(t)$ is continuous, $x(t)+\sum_{i=1}^{k}P_i(t)x(\tau_i(t))$ is continuously differentiable and satisfies (1.1) for all sufficiently large $t>T_x$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros.

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Oscillation properties of first order neutral differential equations have been investigated by many authors. We refer the reader to Graef, Grammatikopoulos and Spikes [2], Grammatikopoulos, Ladas and Sficas [3], and Jaroš and Kusano [4, 5] for neutral differential equations without forcing term. In particular, the oscillation of neutral differential equations with forcing term was studied by Grace and Lalli [1]. However, it seems that very little is known about the forced oscillations of neutral differential equations.

The purpose of this paper is to present conditions which imply that every solution of (1.1) is oscillatory. In Section 2 we consider the case where the coefficients are oscillatory. In Section 3 we consider the case where the coefficients are nonnegative.

2. Equations with oscillating coefficients. In this section we derive sufficient conditions for no solution of the neutral differential inequality

$$\frac{d}{dt}\left[x(t)+\sum_{i=1}^{k}P_i(t)x(\tau_i(t))\right]+\sum_{j=1}^{m}Q_j(t)f_j(x(\sigma_j(t)))\le R(t) \tag{2.1}$$

to be eventually positive, and then we obtain the oscillation results for (1.1). Our method is an extended adaptation of that used by Kusano and Yoshida [7], and Yoshida [9].

Theorem 2.1. *Suppose that the following hypotheses hold:*

(H2.1) $f_\ell(u)$ *is nondecreasing on* R *for some* $\ell\in\{1,2,\dots,m\}$;

(H2.2) $\sigma_\ell(t)\le t$ *and* $\sigma_\ell(t)$ *is nondecreasing.*

Assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ *such that:*

$$\lim_{n\to\infty} t_n=\infty; \tag{2.2}$$

$$P_i(t)\le 0 \quad \text{on } [\sigma_\ell^2(t_n),\ \sigma_\ell(t_n)],\ \ P_i(\sigma_\ell(t_n))=0 \text{ and } P_i(t_n)\ge 0 \quad \text{for } i\in I_k; \tag{2.3}$$

$$Q_j(t)\ge 0 \quad \text{on } [\sigma_\ell^2(t_n),\ t_n] \quad \text{for } j\in I_m; \tag{2.4}$$

$$\int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s)f_\ell\left(c-\int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)}R(r)\,dr\right)ds-\int_{\sigma_\ell(t_n)}^{t_n}R(s)\,ds\ge c \quad \text{for any } c>0, \tag{2.5}$$

where $\sigma_\ell^2(t)=\sigma_\ell(\sigma_\ell(t))$ *and* $I_N=\{1,2,\dots,N\}$ $(N=1,2,\dots)$. *Then* (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence* $\{\widetilde{t}_n\}_{n=1}^{\infty}$ *such that:*

$$\lim_{n\to\infty}\widetilde{t}_n=\infty; \tag{2.6}$$

$$P_i(t) \le 0 \quad on\ [\sigma_\ell^2(\widetilde{t}_n),\ \sigma_\ell(\widetilde{t}_n)],\ P_i(\sigma_\ell(\widetilde{t}_n)) = 0\ and\ P_i(\widetilde{t}_n) \ge 0 \quad for\ i \in I_k; \tag{2.7}$$

$$Q_j(t) \ge 0 \quad on\ [\sigma_\ell^2(\widetilde{t}_n),\ \widetilde{t}_n] \quad for\ j \in I_m; \tag{2.8}$$

$$-\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) f_\ell\left(-c - \int_{\sigma_\ell(s)}^{\sigma_\ell(\widetilde{t}_n)} R(r)\,dr\right) ds + \int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds \ge c \quad for\ any\ c > 0. \tag{2.9}$$

Then every solution of (1.1) *is oscillatory.*

Proof. First we show that (2.1) has no eventually positive solution. Suppose that $x(t)$ is eventually positive solution of (2.1). In view of the fact that $\lim_{t\to\infty} \tau_i(t) = \infty$ and $\lim_{t\to\infty} \sigma_j(t) = \infty$, we find that $x(\tau_i(t))$ $(i \in I_k)$ and $x(\sigma_j(t))$ $(j \in I_m)$ are eventually positive. We set

$$y(t) = x(t) + \sum_{i=1}^{k} P_i(t) x(\tau_i(t)). \tag{2.10}$$

Then inequality (2.1) and condition (2.4) imply that

$$y'(t) + Q_\ell(t) f_\ell(x(\sigma_\ell(t))) \le R(t), \quad t \in [\sigma_\ell^2(t_n),\ t_n] \tag{2.11}$$

and

$$y'(t) \le R(t), \quad t \in [\sigma_\ell^2(t_n),\ t_n] \tag{2.12}$$

for all sufficiently large n. It follows from (2.3) that

$$x(\sigma_\ell(s)) \ge x(\sigma_\ell(s)) + \sum_{i=1}^{k} P_i(\sigma_\ell(s)) x(\tau_i \circ \sigma_\ell(s)) = y(\sigma_\ell(s)), \quad s \in [\sigma_\ell(t_n),\ t_n], \tag{2.13}$$

$$y(t_n) = x(t_n) + \sum_{i=1}^{k} P_i(t_n) x(\tau_i(t_n)) \ge x(t_n) > 0,$$

and

$$y(\sigma_\ell(t_n)) = x(\sigma_\ell(t_n)) + \sum_{i=1}^{k} P_i(\sigma_\ell(t_n)) x(\tau_i \circ \sigma_\ell(t_n)) = x(\sigma_\ell(t_n)) > 0,$$

where $\tau_i \circ \sigma_\ell(t) = \tau_i(\sigma_\ell(t))$. Integration of (2.12) over $[\sigma_\ell(s),\ \sigma_\ell(t_n)]$ yields

$$y(\sigma_\ell(s)) \ge y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\,dr, \quad s \in [\sigma_\ell(t_n),\ t_n]. \tag{2.14}$$

Integrating (2.11) over $[\sigma_\ell(t_n),\ t_n]$, we have

$$y(t_n) - y(\sigma_\ell(t_n)) + \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) f_\ell(x(\sigma_\ell(s)))\, ds \le \int_{\sigma_\ell(t_n)}^{t_n} R(s)\, ds. \tag{2.15}$$

Combining (2.13)–(2.15), we obtain

$$y(t_n) \le y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) f_\ell\left(y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\, dr \right) ds + \int_{\sigma_\ell(t_n)}^{t_n} R(s)\, ds.$$

The left hand side of the above inequality is positive. However, (2.5) implies that the right hand side of the above inequality is nonpositive. This is a contradiction. Hence, (2.1) has no eventually positive solution. We now prove that every solution of (1.1) is oscillatory. Since (1.1) is included in (2.1), (1.1) has no eventually positive solution. Suppose that $x(t)$ is eventually negative solution of (1.1). Then, $z(t) \equiv -x(t)$ satisfies

$$\frac{d}{dt}\left[z(t) + \sum_{i=1}^{k} P_i(t) z(\tau_i(t)) \right] + \sum_{j=1}^{m} Q_j(t)[-f_j(-z(\sigma_j(t)))] = -R(t). \tag{2.16}$$

Proceeding as in the case where $x(t)$ is eventually positive, we conclude that (2.16) has no eventually positive solution. This contradicts that $z(t)$ is eventually positive. The proof is complete. $\square$

Corollary 2.1. *Suppose that* (H2.2) *holds, and that the following hypothesis* (H2.3) *holds*:

(H2.3) *There exists a constant $\beta > 0$ such that*

$$\frac{f_\ell(u)}{u} \ge \beta, \quad \textit{for all } u \ne 0.$$

Assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ which satisfies (2.2)–(2.4) *and the following*:

$$\int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s)\, ds \ge \frac{1}{\beta}; \tag{2.17}$$

$$\int_{\sigma_\ell(t_n)}^{t_n} R(s)\, ds + \beta \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\, dr\, ds \le 0. \tag{2.18}$$

Then (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence $\{\widetilde{t}_n\}_{n=1}^{\infty}$ which satisfies* (2.6)–(2.8) *and the following*:

$$\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s)\, ds \ge \frac{1}{\beta};$$

$$\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds + \beta \int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) \int_{\sigma_\ell(s)}^{\sigma_\ell(\widetilde{t}_n)} R(r)\,dr\,ds \geq 0.$$

Then every solution of (1.1) *is oscillatory.*

Proof. It suffices to show that (2.1) has no eventually positive solution. Let $x(t)$ be an eventually positive solution of (2.1) and define $y(t)$ by (2.10). Then, inequality (2.1), condition (2.4) and hypothesis (H2.3) imply that

$$y'(t) + \beta Q_\ell(t) x(\sigma_\ell(t)) \leq R(t), \quad s \in [\sigma_\ell^2(t_n),\ t_n]$$

for all sufficiently large n. By the same arguments as in the proof of Theorem 2.1, we obtain $y(t_n) > 0,\ y(\sigma_\ell(t_n)) > 0$ and

$$y(t_n) \leq y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(t_n)}^{t_n} \beta Q_\ell(s) \left(y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\,dr \right) ds + \int_{\sigma_\ell(t_n)}^{t_n} R(s)\,ds.$$

It follows from (2.17) that

$$y(t_n) \leq \beta \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\,dr\,ds + \int_{\sigma_\ell(t_n)}^{t_n} R(s)\,ds.$$

This contradicts (2.18) and completes the proof. □

Theorem 2.2. *Suppose that* (H2.1) *and* (H2.2) *hold, and that the following hypothesis* (H2.4) *holds*:

(H2.4) $\qquad \tau_i(t) \geq t$ *and* $\tau_i(t)$ *is nondecreasing for* $i \in I_k$.

Assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ *which satisfies* (2.2) *and the following*:

$$P_i(t) \geq 0 \quad on \quad \bigcup_{i \in I_k} [\tau_i \circ \sigma_\ell^2(t_n),\ \tau_i \circ \sigma_\ell(t_n)] \cup [\sigma_\ell^2(t_n),\ \sigma_\ell(t_n)] \cup \{t_n\} \quad for\ i \in I_k; \tag{2.19}$$

$$Q_j(t) \geq 0 \quad on\ [\sigma_\ell^2(t_n),\ \max\{t_n,\ \tau^* \circ \sigma_\ell(t_n)\}] \quad for\ j \in I_m; \tag{2.20}$$

$$R(t) \leq 0 \quad on\ [\sigma_\ell^2(t_n),\ \tau^* \circ \sigma_\ell(t_n)]; \tag{2.21}$$

$$\int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) f_\ell \left(\left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \right]_+ \left[c - \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\,dr \right] \right) ds - \int_{\sigma_\ell(t_n)}^{t_n} R(s)\,ds \geq c \quad for\ any\ c > 0, \tag{2.22}$$

where $\tau^*(t) = \max\{\tau_i(t) \mid i \in I_k\}$ *and* $[\varphi(t)]_+ = \max\{\varphi(t),\ 0\}$. *Then* (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence* $\{\widetilde{t}_n\}_{n=1}^{\infty}$ *which satisfies* (2.6) *and the following*:

$$P_i(t) \geq 0 \quad \textit{on} \ \bigcup_{i\in I_k} [\tau_i \circ \sigma_\ell^2(\widetilde{t}_n),\ \tau_i \circ \sigma_\ell(\widetilde{t}_n)] \cup [\sigma_\ell^2(\widetilde{t}_n),\ \sigma_\ell(\widetilde{t}_n)] \cup \{\widetilde{t}_n\} \quad \textit{for } i \in I_k; \tag{2.23}$$

$$Q_j(t) \ \geq \ 0 \quad \textit{on} \ [\sigma_\ell^2(\widetilde{t}_n),\ \max\{\widetilde{t}_n,\ \tau^* \circ \sigma_\ell(\widetilde{t}_n)\}] \quad \textit{for } j \in I_m; \tag{2.24}$$

$$R(t) \geq 0 \quad \textit{on} \ [\sigma_\ell^2(\widetilde{t}_n),\ \tau^* \circ \sigma_\ell(\widetilde{t}_n)]; \tag{2.25}$$

$$-\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) f_\ell\left(-\left[1-\sum_{i=1}^{k} P_i(\sigma_\ell(s))\right]_+ \left[c + \int_{\sigma_\ell(s)}^{\sigma_\ell(\widetilde{t}_n)} R(r)\,dr\right]\right) ds + \int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds \geq c \quad \textit{for any } c > 0. \tag{2.26}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (2.1) and define $y(t)$ by (2.10). From (2.1), (2.20) and (2.21) we have

$$y'(t) + Q_\ell(t) f_\ell(x(\sigma_\ell(t))) \leq R(t), \quad t \in \ [\sigma_\ell^2(t_n),\ \max\{t_n,\ \tau^* \circ \sigma_\ell(t_n)\}] \tag{2.27}$$

and

$$y'(t) \leq R(t) \leq 0, \quad t \in [\sigma_\ell^2(t_n),\ \tau^* \circ \sigma_\ell(t_n)] \tag{2.28}$$

for all sufficiently large n. Integration of (2.27) over $[\sigma_\ell(t_n),\ t_n]$ and integration of (2.28) over $[\sigma_\ell(s),\ \sigma_\ell(t_n)]$ yield (2.15) and (2.14) respectively. Since $y(t)$ is nonincreasing on $[\sigma_\ell^2(t_n),\ \tau^* \circ \sigma_\ell(t_n)]$, we find that

$$y(\tau_i \circ \sigma_\ell(t)) \leq y(\sigma_\ell(t)), \quad t \in [\sigma_\ell(t_n),\ t_n] \quad \text{for} \ \ i \in I_k.$$

Therefore, from (2.19) we observe that

$$\begin{aligned}
x(\sigma_\ell(s)) &= y(\sigma_\ell(s)) - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) x(\tau_i \circ \sigma_\ell(s)) \\
&\geq y(\sigma_\ell(s)) - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \Big[x(\tau_i \circ \sigma_\ell(s)) + \sum_{v=1}^{k} P_v(\tau_i \circ \sigma_\ell(s)) x(\tau_v \circ \tau_i \circ \sigma_\ell(s)) \Big] \\
&= y(\sigma_\ell(s)) - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) y(\tau_i \circ \sigma_\ell(s)) \\
&\geq \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \right] y(\sigma_\ell(s)), \quad s \in [\sigma_\ell(t_n),\ t_n].
\end{aligned}$$

Since $x(t)$ is eventually positive and

$$y(\sigma_\ell(s)) = x(\sigma_\ell(s)) + \sum_{i=1}^{k} P_i(\sigma_\ell(s)) x(\tau_i \circ \sigma_\ell(s)) > 0, \quad s \in [\sigma_\ell(t_n),\ t_n]$$

by (2.19), we see that

$$x(\sigma_\ell(s)) \geq \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \right]_+ y(\sigma_\ell(s)), \quad s \in [\sigma_\ell(t_n),\ t_n]. \tag{2.29}$$

Combining (2.14), (2.15) and (2.29), we obtain

$$\begin{aligned}
y(t_n) \leq y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) f_\ell \Bigg(& \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \right]_+ \\
& \cdot \left[y(\sigma_\ell(t_n)) - \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\, dr \right] \Bigg) ds + \int_{\sigma_\ell(t_n)}^{t_n} R(s)\, ds.
\end{aligned}$$

The condition (2.19) implies that $y(t_n) > 0$, and hence, the left hand side of the above inequality is positive. However, it follows from (2.22) that the right hand side of the above inequality is nonpositive. This is a contradiction. We have thus proved the theorem. □

Corollary 2.2. *Suppose that* (H2.2)–(H2.4) *hold, and that there exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *which satisfies* (2.2), (2.19)–(2.21) *and the following*:

$$\int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s)) \right]_+ ds \geq \frac{1}{\beta};$$

$$\int_{\sigma_\ell(t_n)}^{t_n} R(s)\,ds + \beta \int_{\sigma_\ell(t_n)}^{t_n} Q_\ell(s) \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s))\right]_+ \int_{\sigma_\ell(s)}^{\sigma_\ell(t_n)} R(r)\,dr\,ds \le 0. \quad (2.30)$$

Then (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence* $\{\widetilde{t}_n\}_{n=1}^{\infty}$ *which satisfies* (2.6), (2.23)–(2.25) *and the following*:

$$\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s))\right]_+ ds \ge \frac{1}{\beta};$$

$$\int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds + \beta \int_{\sigma_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) \left[1 - \sum_{i=1}^{k} P_i(\sigma_\ell(s))\right]_+ \int_{\sigma_\ell(s)}^{\sigma_\ell(\widetilde{t}_n)} R(r)\,dr\,ds \ge 0. \quad (2.31)$$

Then every solution of (1.1) *is oscillatory.*

The proof follows by using same arguments as in Corollary 2.1 and will be omitted.

Now we assume that the following hypothesis (H2.5) holds:

(H2.5) $P_h(t) > 0$ and $\tau_h(t)$ is strictly increasing for some $h \in I_k$.

We introduce the following notation:

$$\overline{\tau}_i(t) = \begin{cases} \tau_h^{-1}(t), & i = h, \\ \tau_h^{-1} \circ \tau_i(t), & i \ne h, \end{cases}$$

$$\overline{\sigma}_j(t) = \tau_h^{-1} \circ \sigma_j(t) \quad \text{for} \quad j \in I_m,$$

$$\overline{P}_i(t) = \begin{cases} \dfrac{1}{P_h(\tau_h^{-1}(t))}, & i = h, \\[2ex] \dfrac{P_i(t)}{P_h(\tau_h^{-1} \circ \tau_i(t))}, & i \ne h. \end{cases}$$

Let $x(t)$ be a solution of (2.1). We set $w(t) = P_h(t)x(\tau_h(t))$. Then we find that $w(t)$ satisfies

$$\frac{d}{dt}\left[w(t) + \sum_{i=1}^{k} \overline{P}_i(t) w(\overline{\tau}_i(t))\right] + \sum_{j=1}^{m} Q_j(t) f_j\left(\overline{P}_h(\sigma_j(t)) w(\overline{\sigma}_j(t))\right) \le R(t). \quad (2.32)$$

Hence, if (2.32) has no eventually positive solution, then (2.1) has no eventually positive solution. Using the same arguments as in the proof of Theorem 2.2, we obtain the following theorem.

Theorem 2.3. *Suppose that* (H2.1) *and* (H2.5) *hold, and that the following hypothesis* (H2.6) *holds*:

(H2.6) $\overline{\sigma}_\ell(t)$ *and* $\overline{\tau}_i(t)$ *are nondecreasing and* $\overline{\sigma}_\ell(t) \le t \le \overline{\tau}_i(t)$ *for* $i \in I_k$.

Assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ *which satisfies* (2.2) *and the following*:

$$\overline{P}_i(t) \ge 0 \quad \text{on} \quad \bigcup_{i\in I_k} [\overline{\tau}_i \circ \overline{\sigma}_\ell^2(t_n),\ \overline{\tau}_i \circ \overline{\sigma}_\ell(t_n)] \cup [\overline{\sigma}_\ell^2(t_n),\ \overline{\sigma}_\ell(t_n)] \cup \{t_n\} \quad \text{for } i \in I_k; \tag{2.33}$$

$$Q_j(t) \ge 0 \quad \text{on } [\overline{\sigma}_\ell^2(t_n),\ \max\{t_n,\ \overline{\tau}^* \circ \overline{\sigma}_\ell(t_n)\}] \quad \text{for } j \in I_m; \tag{2.34}$$

$$R(t) \le 0 \quad \text{on } [\overline{\sigma}_\ell^2(t_n),\ \overline{\tau}^* \circ \overline{\sigma}_\ell(t_n)]; \tag{2.35}$$

$$\int_{\overline{\sigma}_\ell(t_n)}^{t_n} Q_\ell(s) f_\ell\left(\overline{P}_h(\sigma_\ell(s))\left[1-\sum_{i=1}^{k}\overline{P}_i(\overline{\sigma}_\ell(s))\right]_+\left[c-\int_{\overline{\sigma}_\ell(s)}^{\overline{\sigma}_\ell(t_n)} R(r)\,dr\right]\right) ds - \int_{\overline{\sigma}_\ell(t_n)}^{t_n} R(s)\,ds \ge c \quad \text{for any } c>0,$$

where $\overline{\tau}^*(t) = \max\{\overline{\tau}_i(t) \mid i \in I_k\}$. *Then* (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence* $\{\widetilde{t}_n\}_{n=1}^{\infty}$ *which satisfies* (2.6) *and the following*:

$$\overline{P}_i(t) \ge 0 \quad \text{on} \quad \bigcup_{i\in I_k} [\overline{\tau}_i \circ \overline{\sigma}_\ell^2(\widetilde{t}_n),\ \overline{\tau}_i \circ \overline{\sigma}_\ell(\widetilde{t}_n)] \cup [\overline{\sigma}_\ell^2(\widetilde{t}_n),\ \overline{\sigma}_\ell(\widetilde{t}_n)] \cup \{\widetilde{t}_n\} \quad \text{for } i \in I_k; \tag{2.36}$$

$$Q_j(t) \ge 0 \quad \text{on } [\overline{\sigma}_\ell^2(\widetilde{t}_n),\ \max\{\widetilde{t}_n,\ \overline{\tau}^* \circ \overline{\sigma}_\ell(\widetilde{t}_n)\}] \quad \text{for } j \in I_m; \tag{2.37}$$

$$R(t) \ge 0 \quad \text{on } [\overline{\sigma}_\ell^2(\widetilde{t}_n),\ \overline{\tau}^* \circ \overline{\sigma}_\ell(\widetilde{t}_n)]; \tag{2.38}$$

$$-\int_{\overline{\sigma}_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s) f_\ell\left(-\overline{P}_h(\sigma_\ell(s))\left[1-\sum_{i=1}^{k}\overline{P}_i(\overline{\sigma}_\ell(s))\right]_+\left[c+\int_{\overline{\sigma}_\ell(s)}^{\overline{\sigma}_\ell(\widetilde{t}_n)} R(r)\,dr\right]\right) ds + \int_{\overline{\sigma}_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds \ge c \quad \text{for any } c>0.$$

Then every solution of (1.1) *is oscillatory.*

By the same arguments as in the proof of Corollary 2.1, we obtain the following corollary of Theorem 2.3.

Corollary 2.3. *Assume that* (H2.3), (H2.5) *and* (H2.6) *hold, and that there exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *which satisfies* (2.2), (2.33)–(2.35) *and the following*:

$$\int_{\overline{\sigma}_\ell(t_n)}^{t_n} Q_\ell(s)\overline{P}_h(\sigma_\ell(s)) \left[1 - \sum_{i=1}^{k} \overline{P}_i(\overline{\sigma}_\ell(s))\right]_+ ds \geq \frac{1}{\beta};$$

$$\int_{\overline{\sigma}_\ell(t_n)}^{t_n} R(s)\,ds + \beta \int_{\overline{\sigma}_\ell(t_n)}^{t_n} Q_\ell(s)\overline{P}_h(\sigma_\ell(s)) \left[1 - \sum_{i=1}^{k} \overline{P}_i(\overline{\sigma}_\ell(s))\right]_+ \int_{\overline{\sigma}_\ell(s)}^{\overline{\sigma}_\ell(t_n)} R(r)\,dr\,ds \leq 0. \tag{2.39}$$

Then (2.1) *has no eventually positive solution. Assume, moreover, that there exists a sequence* $\{\widetilde{t}_n\}_{n=1}^{\infty}$ *which satisfies* (2.6), (2.36)–(2.38) *and the following*:

$$\int_{\overline{\sigma}_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s)\overline{P}_h(\sigma_\ell(s)) \left[1 - \sum_{i=1}^{k} \overline{P}_i(\overline{\sigma}_\ell(s))\right]_+ ds \geq \frac{1}{\beta};$$

$$\int_{\overline{\sigma}_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} R(s)\,ds + \beta \int_{\overline{\sigma}_\ell(\widetilde{t}_n)}^{\widetilde{t}_n} Q_\ell(s)\overline{P}_h(\sigma_\ell(s)) \left[1 - \sum_{i=1}^{k} \overline{P}_i(\overline{\sigma}_\ell(s))\right]_+ \int_{\overline{\sigma}_\ell(s)}^{\overline{\sigma}_\ell(\widetilde{t}_n)} R(r)\,dr\,ds \geq 0. \tag{2.40}$$

Then every solution of (1.1) *is oscillatory.*

Remark 2.1. Corollary 2.1 is a generalization of the results of Kusano and Yoshida [7, Theorem 3] and Yoshida [9, Theorem 1 and Corollary 1], and Corollary 2.2 is a generalization of the result of Yoshida [9, Theorem 1 and Corollary 1].

Remark 2.2. Every result in this section is true when $Q_j(t) \geq 0$ for $t > 0$ and $j \in I_m$, and Theorems 2.2, 2.3, Corollaries 2.2 and 2.3 are also true in the case where $P_i(t) \geq 0$ for $t > 0$ and $i \in I_k$. However, in Theorem 2.1 and Corollary 2.1, the oscillation properties of $P_i(t)$ for $i \in I_k$ are necessary.

Remark 2.3. If $\tau^* \circ \sigma_\ell(t) \geq t$, then (2.21) and (2.25) imply that $R(t) \leq 0$ on $[\sigma_\ell^2(t_n), t_n]$ and $R(t) \geq 0$ on $[\sigma_\ell^2(\widetilde{t}_n),\ \widetilde{t}_n]$. Hence, in Corollary 2.2 the conditions (2.30) and (2.31) are unnecessary when $\tau^* \circ \sigma_\ell(t) \geq t$. In the case where $\overline{\tau}^* \circ \overline{\sigma}_\ell(t) \geq t$, the conditions (2.39) and (2.40) are unnecessary analogously.

Remark 2.4. In the case where the hypothesis (H2.5) holds, the hypothesis (H2.6) holds if and only if $\tau_i(t)$ $(i \neq h)$ and $\sigma_\ell(t)$ are nondecreasing, $\sigma_\ell(t) \leq \tau_h(t) \leq t$ and $\tau_h(t) \leq \tau_i(t)$ $(i \neq h)$.

Example 2.1. Let us consider the equation

$$\frac{d}{dt}\Big[x(t) - \cos 2t\, x(t-\pi)\Big] + 6x(t-\tfrac{1}{2}\pi)^3 + 4x(t-\tfrac{3}{2}\pi) + x(t-\tfrac{3}{4}\pi) = -\tfrac{\sqrt{2}}{2}(\sin t + \cos t). \quad (2.41)$$

Here, $k = 1$, $m = 3$, $P_1(t) = -\cos 2t$, $Q_1(t) = 6$, $Q_2(t) = 4$, $Q_3(t) = 1$, $R(t) = -\frac{\sqrt{2}}{2}(\sin t + \cos t) \equiv \sin(t - \frac{3}{4}\pi)$, $f_1(u) = u^3$, $f_2(u) = f_3(u) = u$, $\tau_1(t) = t - \pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $\sigma_2(t) = t - \frac{3}{2}\pi$, and $\sigma_3(t) = t - \frac{3}{4}\pi$. We choose $\ell = 1$. The hypotheses (H2.1) and (H2.2) are fulfilled. Setting $t_n = (2n + \frac{3}{4})\pi$ and $\widetilde{t}_n = (2n + \frac{7}{4})\pi$, we easily see that conditions (2.2)–(2.4) and (2.6)–(2.8) hold. An easy calculation shows that

$$\begin{aligned}
&\int_{(2n+\frac{3}{4})\pi-\frac{1}{2}\pi}^{(2n+\frac{3}{4})\pi} 6\left(c - \int_{s-\frac{1}{2}\pi}^{(2n+\frac{3}{4})\pi-\frac{1}{2}\pi} \sin(r-\tfrac{3}{4}\pi)\,dr\right)^3 ds \\
&- \int_{(2n+\frac{3}{4})\pi-\frac{1}{2}\pi}^{(2n+\frac{3}{4})\pi} \sin(s-\tfrac{3}{4}\pi)\,ds - c \\
&\geq \int_{(2n+\frac{1}{4})\pi}^{(2n+\frac{3}{4})\pi} 6c^3\,ds - \int_{(2n+\frac{1}{4})\pi}^{(2n+\frac{3}{4})\pi} \sin(s-\tfrac{3}{4}\pi)\,ds - c \\
&= 3\pi c^3 - c + 1 > 0
\end{aligned}$$

for any $c > 0$. Thus, condition (2.5) holds. In a similar fashion, we find that (2.9) holds. The hypotheses of Theorem 2.1 are satisfied. Hence, every solution of (2.41) is oscillatory. One such solution is $x(t) = \sin t$.

Example 2.2. We consider the equation

$$\frac{d}{dt}\Big[x(t) + \tfrac{1}{2}x(t+\tfrac{1}{2}\pi)\Big] + \tfrac{16}{3\pi}x(t-\tfrac{1}{2}\pi)^3 + \tfrac{16}{3\pi}x(t-\tfrac{3}{2}\pi)^3 + \tfrac{1}{2}x(t-2\pi) = -\sin t. \quad (2.42)$$

Here, $k = 1$, $m = 3$, $P_1(t) = \frac{1}{2}$, $Q_1(t) = Q_2(t) = \frac{16}{3\pi}$, $Q_3 = \frac{1}{2}$, $R(t) = -\sin t$, $f_1(u) = f_2(u) = u^3$, $f_3(u) = u$, $\tau_1(t) = t + \frac{1}{2}\pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $\sigma_2(t) = t - \frac{3}{2}\pi$, and $\sigma_3(t) = t - 2\pi$. We choose $\ell = 1$. The hypotheses (H2.1), (H2.2) and (H2.4) are fulfilled. We set $t_n = (2n+1)\pi$ and $\widetilde{t}_n = 2n\pi$. Then, we find that conditions (2.2), (2.6), (2.19)–(2.21) and (2.23)–(2.25) hold. We easily observe that

$$\begin{aligned}
&\int_{(2n+1)\pi-\frac{\pi}{2}}^{(2n+1)\pi} \frac{16}{3\pi} \cdot \left(\left[1 - \frac{1}{2}\right]_+ \left[c - \int_{s-\frac{\pi}{2}}^{(2n+1)\pi-\frac{\pi}{2}} (-\sin r)\, dr\right]\right)^3 ds \\
&- \int_{(2n+1)\pi-\frac{\pi}{2}}^{(2n+1)\pi} (-\sin s)\, ds - c \\
\geq{}& \int_{(2n+1)\pi-\frac{\pi}{2}}^{(2n+1)\pi} \frac{2}{3\pi} c^3\, ds + \int_{(2n+1)\pi-\frac{\pi}{2}}^{(2n+1)\pi} \sin s\, ds - c \\
={}& \frac{1}{3}c^3 - c + 1 > 0
\end{aligned}$$

for any $c > 0$. Thus, condition (2.22) holds. Analogously we conclude that (2.26) holds. Therefore, Theorem 2.2 implies that every solution of (2.42) is oscillatory. For example, $x(t) = \cos t$ is such a solution.

3. Equations with nonnegative coefficients. In this section we assume that:

(H3.1) $P_i(t) \geq 0,\ Q_j(t) \geq 0$ for $i \in I_k,\ j \in I_m$;

(H3.2) there exist the nondecreasing functions $f(u) \in C(\mathbb{R}, \mathbb{R})$ and $g(u) \in C([0,\infty),\ [0,\infty))$ such that:

$$\begin{gathered}
f_\ell(u) \geq f(u),\ u > 0 \text{ and } f_\ell(u) \leq f(u),\ u < 0 \quad \text{for some } \ell \in I_m, \\
f(u+v) \leq f(u) + f(v), \quad u > 0,\ v > 0, \\
f(u+v) \geq f(u) + f(v), \quad u < 0,\ v < 0, \\
g(0) = 0,\ g(u) > 0, \quad u > 0, \\
f(cu) \leq g(c) f(u), \quad c > 0,\ u > 0, \\
f(cu) \geq g(c) f(u), \quad c > 0,\ u < 0;
\end{gathered}$$

(H3.3) $\tau_i(t) \in C^1((0,\infty), \mathbb{R}),\ \tau_i'(t) > 0$ for $i \in I_k,\ t > 0$.

The following notation will be used:

$$Q_*(t) = \min\left\{ Q_\ell(t),\ \min\left\{ \frac{Q_\ell(\tau_i(t)) \cdot \tau_i'(t)}{g(P_i(\sigma_\ell(t)))} \;\middle|\; i \in \{j \in I_k \mid P_j(\sigma_\ell(t)) > 0\} \right\} \right\},$$

$$A = \{t \in (0,\infty) \mid \sigma_\ell(t) \leq t,\ \sigma_\ell(t) \leq \tau_i(t) \text{ for } i \in I_k\},$$

where we assume that $\min \emptyset = \infty$.

Our method is an extended adaptation of that used by Kitamura and Kusano [6].

Theorem 3.1. *Assume that* (H3.1)–(H3.3) *hold, and that the following hypotheses hold*:

(H3.4) *there exists an oscillatory function* $\theta(t) \in C^1((0,\infty),\mathbb{R})$ *such that* $\theta'(t) = R(t)$ *for* $t > 0$;

(H3.5) $\tau_i \circ \sigma_\ell(t) = \sigma_\ell \circ \tau_i(t)$ *for* $i \in I_k,\ t > 0$.

If

$$\int_{+0}^{1} \frac{du}{f(u)} < \infty, \quad \int_{-0}^{-1} \frac{du}{f(u)} < \infty, \tag{3.1}$$

$$\int_{A\cap B_+} Q_*(t)dt = \infty \tag{3.2}$$

and

$$\int_{A\cap B_-} Q_*(t)dt = \infty, \tag{3.3}$$

then every solution of (1.1) *is oscillatory, where*

$$B_\pm = \{t \in (0,\infty) \mid \pm\,\theta(\sigma_\ell(t)) \ge 0\}.$$

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then there exists a number $t_0 > 0$ such that $x(t) > 0,\ x(\tau_i(t)) > 0\ (i \in I_k),\ x(\sigma_j(t)) > 0\ (j \in I_m)$ in $[t_0,\infty)$. Setting

$$z(t) = x(t) + \sum_{i=1}^{k} P_i(t)x(\tau_i(t)) - \theta(t), \tag{3.4}$$

we have

$$z'(t) = -\sum_{j=1}^{m} Q_j(t)f_j(x(\sigma_j(t))) \le -Q_\ell(t)f(x(\sigma_\ell(t))) \le 0, \quad t \ge t_0. \tag{3.5}$$

Hence, $z(t)$ is nonincreasing in $[t_0,\infty)$. In view of the fact that $\theta(t)$ is oscillatory and $P_i(t)$ is nonnegative, we see that $z(t) > 0$ in $[t_1,\infty)$ for some $t_1 \ge t_0$. Dividing (3.5) by $f(z(t))$ and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} \frac{Q_\ell(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \le \int_{t_1}^{t} \frac{-z'(s)}{f(z(s))}ds = \int_{z(t)}^{z(t_1)} \frac{du}{f(u)} < \infty, \quad t \ge t_1. \tag{3.6}$$

Since $Q_\ell(t) \ge Q_*(t)$ and $Q_\ell(\tau_i(t))\tau_i'(t) \ge Q_*(t)g(P_i(\sigma_\ell(t)))$ for $i \in I_k$, we observe that

$$\begin{aligned}
&\int_{t_1}^{t} \frac{Q_\ell(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&= \frac{1}{k+1}\int_{t_1}^{t} \frac{Q_\ell(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&\quad + \frac{1}{k+1}\sum_{i=1}^{k}\int_{\tau_i^{-1}(t_1)}^{\tau_i^{-1}(t)} \frac{Q_\ell(\tau_i(s))f(x(\sigma_\ell \circ \tau_i(s)))\tau_i'(s)}{f(z(\tau_i(s)))}ds \\
&\ge \frac{1}{k+1}\int_{t_1}^{t} \frac{Q_*(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&\quad + \frac{1}{k+1}\sum_{i=1}^{k}\int_{\tau_i^{-1}(t_1)}^{\tau_i^{-1}(t)} \frac{Q_*(s)g(P_i(\sigma_\ell(s)))f(x(\tau_i \circ \sigma_\ell(s)))}{f(z(\tau_i(s)))}ds \\
&\ge \frac{1}{k+1}\int_{t_2}^{\alpha(t)} Q_*(s)\frac{f(z(\sigma_\ell(s)) + \theta(\sigma_\ell(s)))}{f(z(\tau_*(s)))}ds \\
&\ge \frac{1}{k+1}\int_{[t_2,\alpha(t)]\cap A\cap B_+} Q_*(s)ds
\end{aligned}$$

for some $t_2 \ge t_1$, where

$$\alpha(t) = \min\{t,\ \tau_i^{-1}(t) \mid i \in I_k\} \tag{3.7}$$

and

$$\tau_*(t) = \min\{t,\ \tau_i(t) \mid i \in I_k\}.$$

It follows form (3.6) that

$$\int_{[t_2,\alpha(t)]\cap A\cap B_+} Q_*(s)ds < \infty, \quad t \ge t_1.$$

This contradicts (3.2), and (1.1) has no eventually positive solution. In exactly the same way, we conclude that (1.1) has no eventually negative solution. We have thus proved the theorem. □

Corollary 3.1. *Assume that* (H3.1)–(H3.3) *and* (H3.5) *hold, and following hypothesis* (H3.6) *holds*:

(H3.6) *there exist a function* $\theta(t) \in C^1((0,\infty),\mathbb{R})$ *and sequences* $\{t_n\}_{n=1}^\infty$, $\{\widetilde{t}_n\}_{n=1}^\infty$ *for which* $\theta'(t) = R(t)$ *for* $t > 0$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \widetilde{t}_n = \infty$, $\theta(t_n) = \underline{\theta}$, $\theta(\widetilde{t}_n) = \overline{\theta}$ *and* $\underline{\theta} \le \theta(t) \le \overline{\theta}$ *for* $t > 0$.

If (3.1) *holds and*

$$\int_A Q_*(t)dt = \infty, \tag{3.8}$$

then every solution of (1.1) *is oscillatory.*

Proof. Letting $\widehat{\theta}(t) \equiv \theta(t) - \underline{\theta}$, we see that $\widehat{\theta}(t)$ is oscillatory, $\widehat{\theta}(t)' = R(t)$ and $\theta(t) \geq 0$ for $t > 0$. Using the same arguments as in the proof of Theorem 3.1, we conclude that (1.1) has no eventually positive solution. Similarly, it follows that (1.1) has no eventually negative solution. The proof is complete. □

Theorem 3.2. *Assume that* (H3.1)–(H3.5) *hold. If*

$$\int^{\infty} Q_*(t) f([\theta(\sigma_\ell(t))]_+) dt = \infty \tag{3.9}$$

and

$$\int^{\infty} Q_*(t) f([\theta(\sigma_\ell(t))]_-) dt = \infty, \tag{3.10}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1) and define $z(t)$ by (3.4). Integration of (3.5) over $[t_1, t]$ yields

$$\int_{t_1}^{t} Q_\ell(s) f(x(\sigma_\ell(s))) ds \leq -z(t) + z(t_1) \leq z(t_1), \quad t \geq t_1, \tag{3.11}$$

where t_1 is the same number appearing in the proof of Theorem 3.1. Using the same arguments as in the proof of Theorem 3.1, we obtain

$$\int_{t_1}^{t} Q_\ell(s) f(x(\sigma_\ell(s))) ds \geq \frac{1}{k+1} \int_{t_2}^{\alpha(t)} Q_*(s) f(z(\sigma_\ell(s)) + \theta(\sigma_\ell(s))) ds \tag{3.12}$$

for some $t_2 \geq t_1$, where $\alpha(t)$ is defined by (3.7). Since

$$z(\sigma_\ell(t)) + \theta(\sigma_\ell(t)) = x(t) + \sum_{i=1}^{k} P_i(t) x(\tau_i(t)) > 0$$

and

$$z(\sigma_\ell(t)) + \theta(\sigma_\ell(t)) > \theta(\sigma_\ell(t))$$

for all sufficiently large $t > 0$, we see that

$$z(\sigma_\ell(t)) + \theta(\sigma_\ell(t)) > [\theta(\sigma_\ell(t))]_+.$$

Hence, (3.11) and (3.12) imply that

$$\int_{t_2}^{\alpha(t)} Q_*(s) f([\theta(\sigma_\ell(s))]_+) ds < \infty.$$

This contradicts the hypothesis and the proof is complete. □

Now we assume that:

(H3.7) $\sigma_\ell(t) \in C^1((0,\infty),\mathbb{R}),\ \sigma_\ell'(t) > 0$ for $t > 0$.

We introduce the following notation:

$$\widetilde{Q}_*(t) = \min\left\{Q_\ell(t),\ \min\left\{\frac{Q_\ell(\rho_i(t))\cdot\rho_i'(t)}{g(P_i(\sigma_\ell(t)))}\ \middle|\ i \in \{j \in I_k \mid P_j(\sigma_\ell(t)) > 0\}\right\}\right\},$$

$$\widetilde{A} = \{t \in (0,\infty) \mid \sigma_\ell(t) \le t,\ \sigma_\ell(t) \le \rho_i(t) \ \text{ for } \ i \in I_k\},$$

where

$$\rho_i(t) = \sigma_\ell^{-1} \circ \tau_i \circ \sigma_\ell(t).$$

Theorem 3.3. *Assume that* (H3.1)–(H3.4) *and* (H3.7) *hold. If*

$$\int_{\widetilde{A}\cap B_+} \widetilde{Q}_*(t)dt = \infty, \tag{3.13}$$

$$\int_{\widetilde{A}\cap B_-} \widetilde{Q}_*(t)dt = \infty \tag{3.14}$$

and (3.1) *holds, then every solution of* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1) and define $z(t)$ by (3.4). Proceeding as in the proof of Theorem 3.1, we see that (3.6) holds and $z(t)$ is nonincreasing and positive in $[t_1,\infty)$ for some $t_1 > 0$. We observe that

$$\begin{aligned}
&\int_{t_1}^{t} \frac{Q_\ell(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&= \frac{1}{k+1}\int_{t_1}^{t} \frac{Q_\ell(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&\quad + \frac{1}{k+1}\sum_{i=1}^{k}\int_{\rho_i^{-1}(t_1)}^{\rho_i^{-1}(t)} \frac{Q_\ell(\rho_i(s))f(x(\tau_i\circ\sigma_\ell(s)))\rho_i'(s)}{f(z(\rho_i(s)))}ds \\
&\ge \frac{1}{k+1}\int_{t_1}^{t} \frac{\widetilde{Q}_*(s)f(x(\sigma_\ell(s)))}{f(z(s))}ds \\
&\quad + \frac{1}{k+1}\sum_{i=1}^{k}\int_{\rho_i^{-1}(t_1)}^{\rho_i^{-1}(t)} \frac{\widetilde{Q}_*(s)g(P_i(\sigma_\ell(s)))f(x(\tau_i\circ\sigma_\ell(s)))}{f(z(\rho_i(s)))}ds \\
&\ge \frac{1}{k+1}\int_{t_2}^{\widetilde{\alpha}(t)} \widetilde{Q}_*(s)\frac{f(z(\sigma_\ell(s)) + \theta(\sigma_\ell(s)))}{f(z(\rho_*(s)))}ds \\
&\ge \frac{1}{k+1}\int_{[t_2,\widetilde{\alpha}(t)]\cap\widetilde{A}\cap B_+} \widetilde{Q}_*(s)ds
\end{aligned}$$

for some $t_2 \ge t_1$, where

$$\widetilde{\alpha}(t) = \min\{t,\ \rho_i^{-1}(t) \mid i \in I_k\} \quad \text{and} \quad \rho_*(t) = \min\{t,\ \rho_i(t) \mid i \in I_k\}.$$

This contradicts (3.6) and the proof is complete. $\square$

Corollary 3.2. *Assume that* (H3.1)–(H3.3), (H3.6) *and* (H3.7) *hold. If* (3.1) *holds and*

$$\int_{\widetilde{A}} \widetilde{Q}_*(t)dt = \infty, \tag{3.15}$$

then every solution of (1.1) *is oscillatory.*

Proof. The proof is quite similar to that of Corollary 3.1 and hence will be omitted. $\square$

Theorem 3.4. *Assume that* (H3.1)–(H3.4) *and* (H3.7) *hold. If*

$$\int^{\infty} \widetilde{Q}_*(t) f([\theta(\sigma_\ell(t))]_+)dt = \infty \tag{3.16}$$

and

$$\int^{\infty} \widetilde{Q}_*(t) f([\theta(\sigma_\ell(t))]_-)dt = \infty, \tag{3.17}$$

then every solution of (1.1) *is oscillatory.*

Proof. The conclusion follows by the same arguments as in the proofs of Theorems 3.2 and 3.3. $\square$

Remark 3.1. In the case where $f_\ell(u) = |u|^\gamma \operatorname{sgn} u$ $(\gamma > 0)$, we can choose that $f(u) = \min\{1, 2^{1-\gamma}\}|u|^\gamma \operatorname{sgn} u$ and $g(u) = u^\gamma$.

Remark 3.2. It is easy to see that $\rho_i^{-1}(t) = \sigma_\ell^{-1} \circ \tau_i^{-1} \circ \sigma_\ell(t)$ and

$$\rho_i'(t) = \frac{\tau_i' \circ \sigma_\ell(t) \cdot \sigma_\ell'(t)}{\sigma_\ell' \circ \sigma_\ell^{-1} \circ \tau_i \circ \sigma_\ell(t)}.$$

Example 3.1. Let us consider the equation

$$\frac{d}{dt}\Big[x(t) + x(t-\pi)\Big] + |x(t - \tfrac{5}{2}\pi)|^{\frac{1}{3}} \operatorname{sgn} x(t - \tfrac{5}{2}\pi) = \cos t \tag{3.18}$$

Here, $k = 1$, $m = 1$, $P_1(t) = Q_1(t) = 1$, $R(t) = \cos t$, $f_1(u) = |u|^{\frac{1}{3}} \operatorname{sgn} u$, $\tau_1(t) = t - \pi$, and $\sigma_1(t) = t - \frac{5}{2}\pi$. We choose $f(u) = g(u) = |u|^{\frac{1}{3}} \operatorname{sgn} u$. The hypotheses (H3.1)–(H3.3), (H3.5) and condition (3.1) are fulfilled. Letting $\theta(t) = \sin t$, we conclude that $\theta(t)$ satisfies (H3.6). It is easily checked that $Q_*(t) = 1$, and hence (3.8) hold. It follows from Corollary 3.1 that every solution of (3.18) is oscillatory. For example, $x(t) = -\sin^3 t$ is such a solution.

Example 3.2. We consider the equation

$$\frac{d}{dt}\Big[x(t) + x(t - \tfrac{\pi}{2})\Big] + x(t - \pi) = \cos t \tag{3.19}$$

Here, $k = 1$, $m = 1$, $P_1(t) = Q_1(t) = 1$, $R(t) = \cos t$, $f_1(u) = u$, $\tau_1(t) = t - \frac{\pi}{2}$, and $\sigma_1(t) = t - \pi$. We choose $f(u) = g(u) = u$. The hypotheses (H3.1)–(H3.3) and (H3.5) are fulfilled. We put $\theta(t) = \sin t$. Then $\theta(t)$ satisfies (H3.4). It is easy to verify that $Q_*(t) = 1$, and hence (3.9) and (3.10) hold. Therefore, Theorem 3.2 implies that every solution of (3.19) is oscillatory. One such solution is $x(t) = \sin t$.

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